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L-FUNCTIONS AND CLASS NUMBERS OF IMAGINARY QUADRATIC FIELDS AND OF QUADRATIC EXTENSIONS OF AN IMAGINARY QUADRATIC FIELD

STÉPHANE LOUBOUTIN

ABSTRACT. Starting from the analytic class number formula involving its L-function, we first give an expression for the class number of an imaginary quadratic field which, in the case of large discriminants, provides us with a much more powerful numerical technique than that of counting the number of reduced definite positive binary quadratic forms, as has been used by Buell in order to compute his class number tables. Then, using class field theory, we will construct a periodic character χ, defined on the ring of integers of a field $\mathbf{K}$ that is a quadratic extension of a principal imaginary quadratic field $\mathbf{k}$, such that the zeta function of $\mathbf{K}$ is the product of the zeta function of $\mathbf{k}$ and of the L-function $L(s, \chi)$. We will then determine an integral representation of this L-function that enables us to calculate the class number of $\mathbf{K}$ numerically, as soon as its regulator is known. It will also provide us with an upper bound for these class numbers, showing that Hua's bound for the class numbers of imaginary and real quadratic fields is not the best that one could expect. We give statistical results concerning the class numbers of the first 50000 quadratic extensions of $\mathbf{Q}(i)$ with prime relative discriminant (and with $\mathbf{K}/\mathbf{Q}$ a non-Galois quartic extension). Our analytic calculation improves the algebraic calculation used by Lakein in the same way as the analytic calculation of the class numbers of real quadratic fields made by Williams and Broere improved the algebraic calculation consisting in counting the number of cycles of reduced ideals. Finally, we give upper bounds for class numbers of $\mathbf{K}$ that is a quadratic extension of an imaginary quadratic field $\mathbf{k}$ which is no longer assumed to be of class number one.

1. CLASS NUMBERS OF IMAGINARY QUADRATIC FIELDS

Let $\mathbf{k}$ be an imaginary quadratic field with discriminant $D < 0$ and character χ. The analytic class number formula for this field is

$$h(\mathbf{k}) = \frac{\omega(\mathbf{k})\sqrt{|D|}}{2\pi} L(1, \chi).$$

Knowing the functional equations satisfied by the zeta function of $\mathbf{k}$ and the Riemann zeta function, one can easily deduce the functional equation satisfied by their quotient $L(s, \chi)$, i.e., $F(s) = F(1-s)$ with

$$F(s) \stackrel{\text{def}}{=} \left(\frac{\pi}{|D|}\right)^{-s} \Gamma(s) L(2s-1, \chi) = \int_0^\infty \alpha(t) t^s \frac{dt}{t},$$

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with

$$\alpha(t) \stackrel{\text{def}}{=} \sum_{n\geq 1} \chi(n) n e^{-\pi n^2 t/|D|}, \qquad t > 0.$$

Moreover, $L(2s-1, \chi) = \sum_{N\geq 1} a_N/N^s$ with $a_N = 0$ whenever $N \neq n^2$, and $a_N = n\chi(n)$ whenever $N = n^2$. Hence, from Ogg [17, Introduction], we get $\alpha(t) = t^{-3/2}\alpha(t^{-1})$,

$$F(s) = \int_1^\infty \alpha(t)(t^s + t^{3/2-s}) \frac{dt}{t},$$

$$L(1, \chi) = \sum_{n\geq 1} \frac{\chi(n)}{n} \exp\left(-\frac{\pi n^2}{|D|}\right) + 2\sqrt{\frac{\pi}{|D|}} \sum_{n\geq 1} \chi(n) \int_{n\sqrt{\pi/|D|}}^\infty e^{-u^2}\, du.$$

We note that this expression for $L(1, \chi)$, combined with the fact that $|\chi(n)| \leq 1$, and a comparison of series with integrals yields Hua's result quoted in [10, Chapter 2]: $L(1, \chi) \leq \mathrm{Log}(\sqrt{|D|}) + 1$ (the $\mathrm{Log}(\sqrt{|D|})$ term coming from the first sum, and the constant term from the second sum).

Then, with $S(n) = \sum_{k=1}^n \chi(k)$, we have

$$L(1, \chi) = \sum_{n\geq 1} \frac{\chi(n)}{n} \exp\left(-\frac{\pi n^2}{|D|}\right) + 2\sqrt{\frac{\pi}{|D|}} \sum_{n\geq 1} S(n) \int_{n\sqrt{\pi/|D|}}^{(n+1)\sqrt{\pi/|D|}} e^{-u^2}\, du.$$

As there exists $c \in (a, b)$ such that $\int_a^b f = (b-a)f(a) + \frac{(b-a)^2}{2} f'(c)$, we have

$$0 \leq \sqrt{\frac{\pi}{|D|}} e^{-\pi n^2/|D|} - \int_{n\sqrt{\pi/|D|}}^{(n+1)\sqrt{\pi/|D|}} e^{-u^2}\, du \leq \left(\frac{\pi}{|D|}\right)^{3/2} (n+1) e^{-\pi n^2/|D|}.$$

Since $|S(n)| \leq n$, using a comparison of series with integrals, we get

$$\begin{aligned} L(1, \chi) = {} & \sum_{n\geq 1} \frac{\chi(n)}{n} \exp\left(-\frac{\pi n^2}{|D|}\right) + 2\frac{\pi}{|D|} \sum_{n\geq 1} S(n) \exp\left(-\frac{\pi n^2}{|D|}\right) \\ & + \frac{\pi}{2\sqrt{|D|}} (1 + \varepsilon(D)). \end{aligned}$$

Thus, another comparison of series with integrals provides us with:

Theorem 1. *Let* $\mathbf{k} \neq \mathbf{Q}(\sqrt{-1})$, $\mathbf{Q}(\sqrt{-3})$ *be an imaginary quadratic field with discriminant* $D < 0$ *and character* χ. *Then, with* $M = 1 + [\sqrt{|D|}\,\mathrm{Log}(|D|)/2\pi]$ (*where* $[x]$ *denotes the greatest integer less than or equal to* x), *the class number* $h(\mathbf{k})$ *of* $\mathbf{k}$ *is the nearest integer to*

$$\tilde{h}(\mathbf{k}) \stackrel{\text{def}}{=} \frac{\sqrt{|D|}}{\pi} \sum_{1\leq n\leq M} \left(\frac{\chi(n)}{n} + \frac{2\pi S(n)}{|D|}\right) e^{-\pi n^2/|D|}$$

with the same parity as $h(\mathbf{k})$. *Indeed, we have*

$$|h(\mathbf{k}) - \tilde{h}(\mathbf{k})| \leq \frac{1}{\pi} + \frac{1}{2} + \varepsilon(D), \qquad \textit{with } \lim_{|D|\mapsto+\infty} \varepsilon(D) = 0 \ (\textit{effective}).$$

Remark. If one uses this formula to calculate the class number of $\mathbf{k}$, then $\tilde{h}(\mathbf{k})$ approximates $h(\mathbf{k})$ much more accurately than the error bound given above would suggest. Indeed, empirically, the error tends to zero as $|D|$ tends to infinity.

Numerical application. Let $\mathbf{k} = \mathbf{Q}(\sqrt{-p})$ with p prime and $p \equiv 1 \pmod 4$. The 2-Sylow subgroup of the ideal class group of $\mathbf{k}$ is $\mathbf{Z}/2^n\mathbf{Z}$ for some $n \geq 1$, such that $n = 1$ if and only if $p \equiv 5 \pmod 8$. We searched for the primes $p \equiv 1 \pmod 4$, $p \leq 8 \cdot 10^7$, such that the ideal class group of $\mathbf{k}$ is a cyclic 2-group $\mathbf{Z}/2^n\mathbf{Z}$, $n \geq 1$, i.e., such that $h(\mathbf{Q}(\sqrt{-p})) = 2^n$, $n \geq 1$. The imaginary quadratic fields of class number two were determined by Stark [20], hence $h(\mathbf{Q}(\sqrt{-p})) = 2$ if and only if $p = 5$, 13, or 37. Hence, we only sieved the primes $p \equiv 1 \pmod 8$ less than or equal to $8 \cdot 10^7$ such that the prime ideals above the smallest noninert prime in $\mathbf{k}/\mathbf{Q}$ have order 2^k in the ideal class group, for some $k \geq 0$ (thanks to Hua's upper bound $h(\mathbf{Q}(\sqrt{-p})) \leq \frac{2\sqrt{p}}{\pi}(\mathrm{Log}(\sqrt{4p}) + 1)$ we only have to check whether $\mathbf{P}^{2^i}$ is principal or not for some i such that $1 \leq 2^i \leq \frac{2\sqrt{p}}{\pi}(\mathrm{Log}(\sqrt{4p}) + 1))$. Using Theorem 1, we then calculated the class numbers of these fields to get 1997 fields for which $h(\mathbf{k}) = 2^n$, $n \geq 2$. Our computations provide us with Table 1.

TABLE 1

n	$p_{\min}(n)$	N_n	$p_{\max}(n)$
1	5	3	37
2	17	4	193
3	41	7	577
4	257	11	3217
5	521	22	16417
6	4481	22	49393
7	9521	62	340657
8	21929	87	1259017
9	72089	186	4942177
10	531977	319	19277017
11	1256009	588	75661657
12	5014169	513	79986073
13	20879129	175	79566209
14	70993529	1	70993529

$$p_{\min}(n) \stackrel{\text{def}}{=} \mathrm{Min}\{p\,;\, p \equiv 1 \pmod 4,\ h(\mathbf{k}) = 2^n,\ p \leq 8 \cdot 10^7\},$$

$$N_n \stackrel{\text{def}}{=} \sharp\{p\,;\, p \equiv 1 \pmod 4,\ h(\mathbf{k}) = 2^n,\ p \leq 8 \cdot 10^7\},$$

$$p_{\max}(n) \stackrel{\text{def}}{=} \mathrm{Max}\{p\,;\, p \equiv 1 \pmod 4,\ h(\mathbf{k}) = 2^n,\ p \leq 8 \cdot 10^7\}.$$

We decided to limit our numerical computations to $p \leq 8 \cdot 10^7$ because Theorem 1(2) of Louboutin [13] (which assumes the generalized Riemann hypothesis) provides us with the lower bound $h(\mathbf{Q}(\sqrt{-p})) > 512$, $p \geq 8 \cdot 10^7$. Hence, Table 1 is complete up through $n = 9$ (under the assumption of the generalized Riemann hypothesis).

2. Class numbers of quadratic extensions of a principal imaginary quadratic field

Let $\mathbf{k}$ be any one of the nine principal imaginary quadratic fields, and let $\mathbf{R_k}$ be its ring of integers and D its discriminant. Let $\mathbf{K}/\mathbf{k}$ be a quadratic extension with relative discriminant $\delta = \delta_{\mathbf{K}/\mathbf{k}}$, and let $\Delta = |\delta_{\mathbf{K}/\mathbf{k}}|^2$. In our first theorem below, we assume that there exists a primitive character χ defined on $\mathbf{R_k}$, with conductor the ideal $(\delta_{\mathbf{K}/\mathbf{k}})$, taking the value $+1$ on the finite units group $\mathbf{U_k}$ of $\mathbf{R_k}$ (with cardinality $w(\mathbf{k})$ equal to 2 whenever $\mathbf{k} \neq \mathbf{Q}(i)$, $\mathbf{Q}(j)$ (where $j = (-1 + i\sqrt{3})/2$), equal to 4 whenever $\mathbf{k} = \mathbf{Q}(i)$, and equal to 6 whenever $\mathbf{k} = \mathbf{Q}(j)$). We assume that the factorization $\zeta_{\mathbf{K}}(s) = \zeta_{\mathbf{k}}(s)L(s, \chi)$ is valid. We finally define

$$\alpha(t) \stackrel{\text{def}}{=} \sum_{z \in \mathbf{R_k}} \chi(z) \exp\left(-\frac{2\pi|z|^2}{\sqrt{\Delta|D|}}t\right) \quad \text{and} \quad E(x) \stackrel{\text{def}}{=} \int_x^\infty e^{-t}\frac{dt}{t}.$$

Proposition A. *There holds*

$$\left(\frac{\sqrt{\Delta|D|}}{2\pi}\right)^s \Gamma(s)L(s, \chi) = \frac{1}{w(\mathbf{k})}\int_0^\infty \alpha(t)t^s\frac{dt}{t}, \qquad \mathrm{Re}(s) > 0,$$

$$\alpha(t) = \frac{1}{t}\alpha\left(\frac{1}{t}\right),$$

$$\left(\frac{\sqrt{\Delta|D|}}{2\pi}\right)^s \Gamma(s)L(s, \chi) = \frac{1}{w(\mathbf{k})}\int_1^\infty \alpha(t)(t^s + t^{1-s})\frac{dt}{t}, \qquad s \in \mathbf{C}.$$

Theorem 2. *Let* $\mathbf{K}$ *be a quartic field that is a quadratic extension* $\mathbf{K}/\mathbf{k}$ *of a principal imaginary quadratic field* $\mathbf{k}$. *Let* $\mathbf{R}(\mathbf{K})$ *be the regulator of* $\mathbf{K}$. *We have the following integral evaluation of the class number* $h(\mathbf{K})$ *of* $\mathbf{K}$:

$$\begin{aligned} h(\mathbf{K}) &= \frac{\sqrt{\Delta|D|}}{2\pi R(\mathbf{K})}L(1, \chi) \\ &= \frac{1}{w(\mathbf{k})R(\mathbf{K})}\sum_{\substack{z \in \mathbf{R_k} \\ z \neq 0}} \chi(x)\left\{E\left(\frac{2\pi|z|^2}{\sqrt{\Delta|D|}}\right) + \frac{\exp(-2\pi|z|^2/\sqrt{\Delta|D|})}{2\pi|z|^2/\sqrt{\Delta|D|}}\right\} \end{aligned}$$

whenever $\mathbf{K} \neq \mathbf{Q}(\zeta_n)$, $n = 8$, 10, *or* 12, *and* $h(\mathbf{K}) = 1$ *whenever* $\mathbf{K} = \mathbf{Q}(\zeta_n)$, $n = 8$, 10, *or* 12.

Our main purpose of giving Theorem 2 is to provide an efficient way to calculate the class numbers of these quadratic extensions (see the numerical examples below). The regulator $R(\mathbf{K})$ can be calculated using Amara's [1] techniques; then Theorem 5 will provide us with the computation of $\chi(z)$.

Theorem 3. *With effective* $O(1)$ *and* $\varepsilon(\Delta)$ *we have*:

$$\text{(a)} \quad L(1, \chi) \leq \frac{2\pi}{w(\mathbf{k})\sqrt{|D|}}\mathrm{Log}(\sqrt{\Delta}) + O(1),$$

$$\text{(b)} \quad h(\mathbf{K})R(\mathbf{K}) \leq \frac{1}{w(\mathbf{k})}\sqrt{\Delta}\,\mathrm{Log}(\sqrt{\Delta})(1 + \varepsilon(\Delta)),$$

$$h(\mathbf{K}) \leq \frac{1}{w(\mathbf{k})}\sqrt{\Delta}(1 + \varepsilon(\Delta)) \quad \text{with} \lim_{\Delta \mapsto +\infty} \varepsilon(\Delta) = 0.$$

These upper bounds do not depend on the discriminant of $\mathbf{k}$ and greatly improve the general upper bound $h(\mathbf{K})R(\mathbf{K}) = O(\sqrt{|D_{\mathbf{K}/\mathbf{Q}}|}\operatorname{Log}^{n-1}(|D_{\mathbf{K}/\mathbf{Q}}|))$, with $\mathbf{K}$ a number field of degree n and with discriminant $D_{\mathbf{K}/\mathbf{Q}}$ (see notes of Chapter VIII in [16]).

Remarks. 1. We first note that Hua's result quoted above provides us with the upper bound $h(\mathbf{k})R(\mathbf{k}) \le \frac{1}{2}\sqrt{D}\operatorname{Log}(\sqrt{D})(1+\varepsilon(D))$, for $\mathbf{k}$ a real quadratic field with discriminant $D > 0$. Thus, as $w(\mathbf{Q}) = 2$, our result above is a generalization of this well-known upper bound for the class numbers of real quadratic fields.

2. Let $\mathbf{K} = \mathbf{k}_+\mathbf{k}_-$, $\mathbf{k}_+ = \mathbf{Q}(\sqrt{d})$, $\mathbf{k}_- = \mathbf{Q}(\sqrt{-d})$, with $d > 1$ a squarefree integer. Thus, $\mathbf{k} = \mathbf{Q}(i)$. Let χ_+, χ_-, and χ be respectively the characters of the extensions $\mathbf{k}_+/\mathbf{Q}$, $\mathbf{k}_-/\mathbf{Q}$, and $\mathbf{K}/\mathbf{Q}(i)$. Hua's result is

$$L(1, \chi_+) = O(\operatorname{Log}(d)), \qquad L(1, \chi_-) = O(\operatorname{Log}(d)),$$

while Theorem 3 gives $L(1, \chi) = O(\operatorname{Log}(\Delta)) = O(\operatorname{Log}(d))$ (as $16\Delta = |D_{\mathbf{K}/\mathbf{Q}}| = 4D_{\mathbf{k}_+/\mathbf{Q}}|D_{\mathbf{k}_-/\mathbf{Q}}|$). Now, it is well known that $L(1, \chi) = L(1, \chi_+)L(1, \chi_-)$ (see [5]), which provides us with the upper bound $L(1, \chi_+)L(1, \chi_-) = O(\operatorname{Log}(d))$. Hua's result gives only $L(1, \chi_+)L(1, \chi_-) = O(\operatorname{Log}^2(d))$ and therefore is not the best result one could expect. Indeed, under the generalized Riemann's hypothesis one has $L(1, \chi_+) = O(\operatorname{Log}\operatorname{Log}(d))$ and $L(1, \chi_-) = O(\operatorname{Log}\operatorname{Log}(d))$, thus providing us with $L(1, \chi) = O(\operatorname{Log}^2(\operatorname{Log}(d)))$.

Proof of Proposition A. The first integral representation is easily proved, using

$$L(s, \chi) = \sum_{\substack{\mathbf{I}\subseteq\mathbf{R}_\mathbf{k}\\ \mathbf{I}\neq 0}} \frac{\chi(\mathbf{I})}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})^s} = \frac{1}{w(\mathbf{k})}\sum_{\substack{z\in\mathbf{R}_\mathbf{k}\\ z\neq 0}} \frac{\chi(z)}{N_{\mathbf{k}/\mathbf{Q}}((z))^s} = \frac{1}{w(\mathbf{k})}\sum_{\substack{z\in\mathbf{R}_\mathbf{k}\\ z\neq 0}} \frac{\chi(z)}{|z|^{2s}}.$$

From the functional equations satisfied by the zeta functions $\zeta_\mathbf{K}$ and $\zeta_\mathbf{k}$, we get that $f\colon s \mapsto (2\pi/\sqrt{\Delta|D|})^{-s}\Gamma(s)L(s, \chi)$ remains unchanged by $s \mapsto 1-s$. Moreover, $f(s) = \frac{1}{w(\mathbf{k})}\sum_{n\ge 1} a_n/n^s$ with $a_n = \sum_{z\in\mathbf{R}_\mathbf{k},\, |z|^2=n}\chi(z)$. Hence, from Ogg [17, Introduction], we get that $\alpha(\frac{1}{t}) = t\alpha(t)$. □

Proof of Theorem 2. Since $\zeta_\mathbf{K}$ has $4\pi^2R(\mathbf{K})h(\mathbf{K})/w(\mathbf{K})\sqrt{|D_{\mathbf{K}/\mathbf{Q}}|}$ as residue at $s=1$ and $\zeta_\mathbf{k}$ has $2\pi h(\mathbf{k})/w(\mathbf{k})\sqrt{|D_{\mathbf{k}/\mathbf{Q}}|}$ as residue at $s = 1$, and since $\zeta_\mathbf{K}(s) = \zeta_\mathbf{k}(s)L(s, \chi)$, we get

$$h(\mathbf{K}) = \frac{1}{2\pi R(\mathbf{K})}\frac{w(\mathbf{K})}{w(\mathbf{k})}\sqrt{\left|\frac{D_{\mathbf{K}/\mathbf{Q}}}{D_{\mathbf{k}/\mathbf{Q}}}\right|}L(1, \chi).$$

Moreover, $|D_{\mathbf{K}/\mathbf{Q}}| = |\delta_{\mathbf{K}/\mathbf{k}}|^2|D_{\mathbf{k}/\mathbf{Q}}|^2 = \Delta D^2$, hence

$$h(\mathbf{K}) = \frac{\sqrt{\Delta|D|}}{2\pi R(\mathbf{K})}\frac{w(\mathbf{K})}{w(\mathbf{k})}L(1, \chi).$$

Since $w(\mathbf{K}) = w(\mathbf{k})$, except when $\mathbf{K} = \mathbf{Q}(\zeta_n)$, $n = 8$, 10, or 12 with $h(\mathbf{K}) = 1$ in these three cases, we get the desired result. □

Proof of Theorem 3. The assertions of Theorem 3 are a consequence of the following lemma.

Lemma A. *With effective error terms, we have*

$$\text{(a)}\quad \sum_{\substack{z\in\mathbf{R_k}\\ z\neq 0}} \frac{\exp(-2\pi|z|^2/\sqrt{\Delta|D|})}{2\pi|z|^2/\sqrt{\Delta|D|}} = \sqrt{\Delta}\,\mathrm{Log}(\sqrt{\Delta}) + O(\sqrt{\Delta}),$$

$$\text{(b)}\quad \sum_{\substack{z\in\mathbf{R_k}\\ z\neq 0}} E\left(\frac{2\pi|z|^2}{\sqrt{\Delta|D|}}\right) = \sqrt{\Delta} + O(\mathrm{Log}(\sqrt{\Delta})),$$

$$\text{(c)}\quad R(\mathbf{K}) \geq 2\,\mathrm{Log}\left(\frac{\Delta^{1/4} + (\Delta^{1/2}-4)^{1/2}}{2}\right).$$

Proof of Lemma A. The function $\theta(x) = \sum_{z\in\mathbf{R_k}} \exp(-x|z|^2)$ is the theta function of the lattice $\mathbf{R_k}$ with fundamental volume $V = \frac{1}{2}\sqrt{|D|}$. Thus,

$$\begin{aligned}\theta(x) &= \frac{\pi}{xV}\theta\left(\frac{\pi^2}{xV^2}\right) = \frac{2\pi}{x\sqrt{|D|}} + \frac{\pi}{xV}\left(\theta\left(\frac{\pi^2}{xV^2}\right)-1\right)\\ &= \frac{2\pi}{x\sqrt{|D|}} + O(1)\end{aligned}$$

whenever $x \mapsto 0^+$.

Now, $f(x) = \sum_{z\in\mathbf{R_k},\ z\neq 0} \exp(-x|z|^2)/x|z|^2$ is such that $(xf(x))' = 1 - \theta(x) = -2\pi/x\sqrt{|D|} + O(1)$. Thus, $xf(x) = -(2\pi/\sqrt{|D|})\,\mathrm{Log}(x) + O(1)$ and we obtain (a) by setting $x = 2\pi/\sqrt{\Delta|D|}$.

Then, $g(x) = \sum_{z\in\mathbf{R_k},\ z\neq 0} E(x|z|^2)$ is such that $g'(x) = \frac{1}{x}(1-\theta(x)) = -2\pi/x^2\sqrt{|D|} + \frac{1}{x}O(1)$. Thus, $g(x) = 2\pi/x\sqrt{|D|} + O(\mathrm{Log}(x))$, and we obtain (b) by setting $x = 2\pi/\sqrt{\Delta|D|}$.

To prove (c), let $\eta = (a+b\sqrt{\delta})/2$ be a fundamental unit of $\mathbf{K}$. Then $\eta = b\sqrt{\delta}(1+a/b\sqrt{\delta})/2$ and $N_{\mathbf{K/k}}(\eta) = (a^2-\delta b^2)/4 \in \{\pm 1, \pm i\}$, i.e., $(a/b\sqrt{\delta})^2 = 1 \pm 4i/b^2\delta$ or $(a/b\sqrt{\delta})^2 = 1 \pm 4/b^2\delta$. The regulator remains unchanged when b is turned into $-b$, so, if $\sqrt{1-z}$ is the usual holomorphic determination of the square root defined for $|z|<1$ by means of its power series, we can assume that either $a/b\sqrt{\delta} = (1\pm 4i/b^2\delta)^{1/2}$, or that $a/b\sqrt{\delta} = (1\pm 4/b^2\delta)^{1/2}$. Since $|(1+\sqrt{1-z})/2| \geq (1+\sqrt{1-|z|})/2$ and $R(\mathbf{K}) = |\,\mathrm{Log}(|\eta_{\mathbf{K}}|^2)|$, we get the result. $\square$

We now show that there exists a character χ satisfying the assumptions made immediately before Proposition A, we explain how one can calculate $\chi(z)$, $z \in \mathbf{R_k}$, and we give numerical applications.

Theorem 4. *Let* $\mathbf{k}$ *be a principal imaginary quadratic field, let* $\mathbf{K/k}$ *be a quadratic extension with relative discriminant* $\delta_{\mathbf{K/k}}$, *and let* $[\frac{\delta_{\mathbf{K/k}}}{\cdot}|$ *be the symbol defined by* $[\frac{\delta_{\mathbf{K/k}}}{\mathbf{P}}] = -1, 0,$ *or* $+1$ *according as the prime ideal* $\mathbf{P}$ *is inert, is ramified, or splits in* $\mathbf{k/Q}$. *Moreover,* $[\frac{\delta_{\mathbf{K/k}}}{\cdot}]$ *is extended multiplicatively to the* *integral ideals of* $\mathbf{k}$. *Then, with (z) being the principal ideal* $z\mathbf{R_k}$,

$$\begin{cases} \mathbf{R_k} \to \{-1, 0, +1\}, \\ z \mapsto \chi(z) = \left[\frac{\delta_{\mathbf{K/k}}}{(z)}\right] \end{cases}$$

is a primitive character defined on $\mathbf{R_k}$ *with conductor the ideal* $(\delta_{\mathbf{K/k}})$, *for which* $\zeta_{\mathbf{K}}(s) = \zeta_{\mathbf{k}}(s)L(s, \chi)$, *and for which* $\chi(\mathbf{U_k}) = \{+1\}$.

Proof. Let $\mathrm{Fr}_{\mathbf{K/k}}$ be the Frobenius automorphism of the extension $\mathbf{K/k}$. Since the Galois group $\{\mathrm{Id}, \sigma\}$ of $\mathbf{K/k}$ has order 2, and since $\mathrm{Fr}_{\mathbf{K/k}}(\mathbf{P})$, whenever $\mathbf{P}$ is a prime ideal, has order equal to the residue class degree, then $\mathrm{Fr}_{\mathbf{K/k}}(\mathbf{P}) = \mathrm{Id}$ whenever $\mathbf{P}$ splits, and $\mathrm{Fr}_{\mathbf{K/k}}(\mathbf{P}) = \sigma$ whenever $\mathbf{P}$ is inert. Hence, $\mathrm{Fr}_{\mathbf{K/k}}(\mathbf{P})$ acts by multiplication by $[\frac{\delta_{\mathbf{K/k}}}{\mathbf{P}}]$ on $\sqrt{\delta_{\mathbf{K/k}}}$. Both symbols being multiplicative, this is still true with $\mathbf{I}$ any integral ideal of $\mathbf{R_k}$ (prime to $\delta_{\mathbf{K/k}}$). Thus, $\mathrm{Fr}_{\mathbf{K/k}}(\mathbf{I}) = \mathrm{Id}$ or σ according as $[\frac{\delta_{\mathbf{K/k}}}{\mathbf{I}}] = +1$ or -1. Now, whenever $\mathbf{K/k}$ is a quadratic extension, apart from its infinite part, its conductor is equal to its discriminant (see [8, Chapter 10; 11, Chapter V, 2.2; 19]). Since, whenever $\mathbf{k}$ is an imaginary quadratic field, there is no ramification at the infinite places for $\mathbf{K/k}$, the conductor of $\mathbf{K/k}$ is equal to the ideal $(\delta_{\mathbf{K/k}})$. Thus, by class field theory, $\{(z),\ z \in \mathbf{k},\ z \equiv 1 \pmod{(\delta_{\mathbf{K/k}})}\}$ is included in the kernel of the Frobenius map. But this being equivalent to $\chi(z) = \chi(z')$, $z \equiv z' \pmod{(\delta_{\mathbf{K/k}})}$, χ is thus a multiplicative function on $\mathbf{R_k}$ with the ideal $(\delta_{\mathbf{K/k}})$ as group of period, hence a character modulo $\delta_{\mathbf{K/k}}$. Since $(\delta_{\mathbf{K/k}})$ is the conductor, this character χ is primitive. □

We will say that $z \in \mathbf{R_k}$ is odd if $N_{\mathbf{k/Q}}(z) = |z|^2$ is an odd integer, and that z is primitive if, for $n \in \mathbf{N}^*$, $n\mathbf{R_k}$ divides $z\mathbf{R_k}$ implies $n = 1$. We will then give the tools which will enable us to calculate $\chi(z)$, $z \in \mathbf{R_k}$. We first note that the symbol defined in Theorem 4 extends to a symbol $[\frac{Z}{\mathbf{P}}]$, $z \in \mathbf{R_k}$, $\mathbf{P}$ a prime ideal lying above an odd prime p, by means of $[\frac{Z}{\mathbf{P}}] = -1$, $+1$, or 0 according as $X^2 \equiv Z \pmod{\mathbf{P}}$ is not solvable in $\mathbf{R_k}$, is solvable in $\mathbf{R_k}$, or $Z \in \mathbf{P}$. Let us note that $[\frac{Z}{\mathbf{P}}]$ is well defined by the property $[\frac{Z}{\mathbf{P}}] \equiv Z^{(N_{\mathbf{k/Q}}(\mathbf{P})-1)/2} \pmod{\mathbf{P}}$. This symbol is again extended multiplicatively to the integral ideals of $\mathbf{R_k}$ with odd absolute norms.

Theorem 5. *Let* z *and* Z *be in* $\mathbf{R_k}$, *let* p *be an odd prime integer in* $\mathbf{Z}$, *and let* $(\frac{\cdot}{p})$ *be the Legendre symbol. Then,*

(a) *whenever* p *is inert in* $\mathbf{K/Q}$,

$$\left[\frac{Z}{(p)}\right] = \left(\frac{N_{\mathbf{k/Q}}(Z)}{p}\right) = \left(\frac{|Z|^2}{p}\right);$$

(b) *whenever* $p = \pi\overline{\pi}$ *splits in* $\mathbf{k/Q}$,

$$\left[\frac{Z}{(\pi)}\right] = \left(\frac{\mathrm{Tr}(\pi)\,\mathrm{Tr}(\pi\overline{Z})}{p}\right) \quad \textit{and} \quad \left[\frac{Z}{(\pi)}\right] \cdot \left[\frac{Z}{(\overline{\pi})}\right] = \left(\frac{|Z|^2}{p}\right);$$

(c) *whenever* $n \in \mathbf{N}^*$, $n \neq 1$, *is a product of odd primes inert in* $\mathbf{k/Q}$,

$$\left[\frac{Z}{(n)}\right] = \left(\frac{N_{\mathbf{k/Q}}(Z)}{n}\right) = \left(\frac{|Z|^2}{n}\right);$$

(d) *whenever* z *is primitive and odd,*

$$\left[\frac{Z}{(z)}\right] = \left(\frac{\operatorname{Tr}(z)\operatorname{Tr}(z\overline{Z})}{|z|^2}\right) \quad \textit{and} \quad \left[\frac{Z}{(z)}\right]\cdot\left[\frac{Z}{(\overline{z})}\right] = \left(\frac{|Z|^2}{|z|^2}\right).$$

Thus, whenever z *is an odd integer of* $\mathbf{R_k}$ *and* $z = nz'$ *with* z' *primitive, we get*

$$\left[\frac{Z}{(z)}\right] = \left[\frac{Z}{(n)}\right]\left[\frac{Z}{(z')}\right] = \left(\frac{|Z|^2}{n}\right)\left(\frac{\operatorname{Tr}(z')\operatorname{Tr}(z'\overline{Z})}{|z'|^2}\right).$$

Proof of Theorem 5. In order to prove (a) and (b), we first note that they are both satisfied as soon as one of the symbols they involve equals zero.

(a) Since p is inert, $\mathbf{R_k}/(p)$ is a finite field with p^2 elements. Moreover, $Z \mapsto \overline{Z}$, being a nontrivial $(\mathbf{Z}/p\mathbf{Z})$-isomorphism of this field, is its own Frobenius, i.e., $\overline{Z} \equiv Z^p \pmod{p\mathbf{R_k}}$. Thus,

$$\left(\frac{|Z|^2}{p}\right) \equiv (Z\overline{Z})^{(p-1)/2} \equiv (ZZ^p)^{(p-1)/2} \equiv Z^{(p^2-1)/2} \equiv \left[\frac{Z}{\mathbf{P}}\right] \pmod{p\mathbf{R_k}},$$

and $[\frac{Z}{\mathbf{P}}] = (\frac{|Z|^2}{p})$.

(c) This is trivial as soon as (a) is known.

(b) Since p splits, the canonical injection $\mathbf{Z}/p\mathbf{Z} \hookrightarrow \mathbf{R_k}/(\pi)$ is an isomorphism, and there exists $n \in \mathbf{Z}$ such that $Z \equiv n \pmod{\pi}$. Thus, $[\frac{Z}{(\pi)}] = [\frac{n}{(\pi)}]$ and $\operatorname{Tr}(\pi)\operatorname{Tr}(\pi\overline{Z}) \equiv n\operatorname{Tr}^2(\pi) \pmod{p}$, implying $(\operatorname{Tr}(\pi)\operatorname{Tr}(\pi\overline{Z})/p) = (\frac{n}{p})$. Now, whenever n is an integer, we have $[\frac{n}{(\pi)}] \equiv n^{(p-1)/2} \pmod{\pi}$ and $(\frac{n}{p}) \equiv n^{(p-1)/2} \pmod{p}$. Thus, $[\frac{n}{(\pi)}] = (\frac{n}{p})$.

(d) We first show that (d) is valid whenever $Z = N$ is an integer in $\mathbf{Z}$. Indeed, we have

$$\left(\frac{\operatorname{Tr}(z)\operatorname{Tr}(z\overline{N})}{|z|^2}\right) = \left(\frac{N\operatorname{Tr}^2(z)}{|z|^2}\right) = \left(\frac{N}{|z|^2}\right) = \prod_{\pi/z}\left(\frac{N}{p}\right),$$

with $p = |\pi|^2$ and where the product is taken over (not necessarily pairwise distinct) primes π dividing z. Now, by (b),

$$\left[\frac{N}{(\pi)}\right] = \left(\frac{\operatorname{Tr}(\pi)\operatorname{Tr}(\pi\overline{N})}{p}\right) = \left(\frac{N}{p}\right).$$

Hence,

$$\prod_{\pi/z}\left(\frac{N}{p}\right) = \prod_{\pi/z}\left(\frac{N}{(\pi)}\right) = \left[\frac{N}{(z)}\right].$$

Now, z being primitive, the canonical projection $p\colon \mathbf{Z} \to \mathbf{R_k}/(z)$ has kernel $|z|^2\mathbf{Z}$. Moreover, $\mathbf{Z}/|z|^2\mathbf{Z}$ and $\mathbf{R_k}/(z)$ both have order $|z|^2$, thus $\tilde{p}\colon \mathbf{Z}/|z|^2\mathbf{Z} \to \mathbf{R_k}/(z)$ is an isomorphism. Hence, there exists an integer N in $\mathbf{Z}$ such that $Z \equiv N \pmod{z\mathbf{R_k}}$. But then we have

$$\left[\frac{Z}{(z)}\right] = \left[\frac{N}{(z)}\right] = \left(\frac{\operatorname{Tr}(z)\operatorname{Tr}(z\overline{N})}{|z|^2}\right) = \left(\frac{\operatorname{Tr}(z)\operatorname{Tr}(z\overline{Z})}{|z|^2}\right),$$

because $\operatorname{Tr}(z\overline{Z}) \equiv \operatorname{Tr}(z\overline{N}) \pmod{|z|^2\mathbf{Z}}$. Hence, we get the desired result. $\square$

Remarks. In our previous paper [14], we focused on the case $\mathbf{k} = \mathbf{Q}(i)$, using the quadratic reciprocity theorem for the symbol [–] due to Dirichlet [7] (who derived it from the quadratic reciprocity theorem for the symbol (–)). We first constructed the character χ of quadratic extensions of $\mathbf{Q}(i)$. Using this character and calculating Gauss' sums, we proved that χ is primitive, and that we have an integral representation of $L(s, \chi)$ from which we deduced its functional equation. Here, we make use of strong results such as the functional equations satisfied by the zeta functions of $\mathbf{K}$ and $\mathbf{k}$, and the fundamental theorem of class field theory. This, without tedious calculations, enables us to construct our character, to show that it is primitive, and to obtain the integral representation of $L(s, \chi)$. The more concepts, the less calculation.

Statistical results in the case $\mathbf{k} = \mathbf{Q}(i)$. Now we will study the distribution of the class numbers of the quadratic extensions $\mathbf{K}/\mathbf{k}$ with relative discriminant $\delta_{\mathbf{K}/\mathbf{k}}$ a noninert prime in $\mathbf{R_k}$, i.e., $\Delta = |\delta_{\mathbf{K}/\mathbf{k}}|^2 = p$ is an odd noninert prime in $\mathbf{k}/\mathbf{Q}$. Our task will be to extend the calculations made by Lakein [12] and to compare the distribution of these class numbers with the distribution of the class numbers of the real quadratic fields $\mathbf{Q}(\sqrt{p})$, $p \equiv 1 \pmod 4$ prime. Note that if $\delta_{\mathbf{K}/\mathbf{k}}$ is a noninert prime in $\mathbf{R_k}$, then $\delta_{\mathbf{K}/\mathbf{k}}$ is primary, i.e., congruent to a square modulo $4\mathbf{R_k}$ (see Louboutin [15]), $\Delta \equiv 1 \pmod 8$, and $h(\mathbf{K})$ is odd. Hence, Hecke's quadratic reciprocity theorem gives $\chi(z) = [z/(\delta_{\mathbf{K}/\mathbf{k}})]$, z an odd integer in $\mathbf{R_k}$. Both symbols being periodic with period $\delta_{\mathbf{K}/\mathbf{k}}$, and $\delta_{\mathbf{K}/\mathbf{k}}$ being odd, this identity is still true whenever z is not odd ($z+\delta_{\mathbf{K}/\mathbf{k}}$ then being odd). Hence, $\chi(z) = (\mathrm{Tr}(\pi)\,\mathrm{Tr}(\pi\overline{Z})/p)$, $z \in \mathbf{R_k}$. This enables us to calculate $\chi(z)$, $z \in \mathbf{R_k}$, in a much more efficient way than using $\chi(z) = [\delta_{\mathbf{K}/\mathbf{k}}/(z)]$. Indeed, in order to use this last formula we have to find n such that $z = nz'$, with z' primitive. Moreover, if $\delta_{\mathbf{K}/\mathbf{k}} = \alpha + i\beta$, then α is odd, β is even, and one can check that $\chi(z) = ((\alpha x + \beta y)/p)$, for $z = x + iy$. From [14], with $M = 3 + [\Delta^{1/4}\sqrt{\mathrm{Log}(\Delta)/2\pi}]$, we have

$$|h(\mathbf{K}) - \tilde{h}(\mathbf{K})| \leq \frac{1}{R(\mathbf{K})}\left\{\frac{1}{\mathrm{Log}(\Delta)} + \frac{2\sqrt{2}}{\sqrt{\pi\sqrt{\Delta}\,\mathrm{Log}(\Delta)}}\right\},$$

with

$$\tilde{h}(\mathbf{K}) \stackrel{\mathrm{def}}{=} \frac{1}{R(\mathbf{K})}\sum_{\substack{z=a+ib\\ a>0,\ b\geq 0\\ |z|\leq M}} \chi(z)\left\{E\left(\frac{\pi|z|^2}{\sqrt{\Delta}}\right) + \frac{\exp(-\pi|z|^2/\sqrt{\Delta})}{\pi|z|^2/\sqrt{\Delta}}\right\}.$$

Thus, the nearest integer to $\tilde{h}(\mathbf{K})$ provides us with the class number. The regulator $R(\mathbf{K}) = |\mathrm{Log}(|\eta_{\mathbf{K}}|^2)|$ is calculated using Amara's technique to get the fundamental unit $\eta_{\mathbf{K}}$ of $\mathbf{K}$. Tables 2 and 3 summarize the results of our computations: we computed the class numbers of the first 50000 quadratic extensions $\mathbf{K}/\mathbf{Q}(i)$ with $\mathbf{K}/\mathbf{Q}$ a non-Galois quartic extension and with $\delta_{\mathbf{K}/\mathbf{k}}$ an odd prime in $\mathbf{Q}(i)$. The case where $\delta_{\mathbf{K}/\mathbf{k}}$ is a rational prime $p \equiv 3 \pmod 8$ was excluded, since Dirichlet's theorem applies to show that $h(\mathbf{K}) = h(p)h(-p)$, the product of the quadratic class numbers.

We note that this distribution of class numbers is very close to the one conjectured by Cohen and Martinet [5], i.e., very close to that of the real quadratic

TABLE 2. Quadratics over $\mathbf{Q}(i)$ by ten thousands

h	1st	2nd	3rd	4th	5th	Total	%
1	7928	7813	7781	7769	7764	39055	78.110
3	1073	1096	1143	1125	1121	5558	11.116
5	383	363	403	368	383	1900	3.800
7	170	206	176	165	186	903	1.806
9	126	134	112	131	129	632	1.264
11	65	60	59	70	55	309	.
13	47	63	50	54	52	266	.
15	45	50	52	57	50	254	.
17	39	32	34	35	35	175	.
19	20	30	20	28	23	121	.
21	15	26	28	34	23	126	.
23	12	16	22	10	17	77	.
25	11	19	12	18	10	70	.
27	11	15	18	13	16	73	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

TABLE 3. Extreme and mean values of class numbers

N	Δ_N	$\frac{1}{\Delta_N}\sum_{\Delta\leq\Delta_n} h(\Delta)$	$h_{\max}$	$\Delta_{\max}$	$\frac{1}{4}\sqrt{\Delta_{\max}}$
10000	482441	$4.5249..10^{-2}$	85	393161	156.76
20000	1023041	$4.5325..10^{-2}$	145	815401	225.75
30000	1588673	$4.5003..10^{-2}$	175	1538321	310.07
40000	2166457	$4.5485..10^{-2}$	175	1538321	310.07
50000	2757329	$4.5483..10^{-2}$	175	1538321	310.07

Δ_N = the N th prime discriminant (with $\Delta \equiv 1 \pmod 8$) ; $h_{\max}$ = the greatest class number of the class numbers of our fields with $17 \leq \Delta \leq \Delta_N$; $\Delta_{\max}$ = the smallest discriminant less than or equal to Δ_N such that $h(\Delta) = h_{\max}$.

case with prime discriminants (see Stephens and Williams [21] for extensive calculations in this real quadratic case).

We first note that our asymptotic upper bound $h(\mathbf{K}) \leq \frac{1}{4}\sqrt{\Delta}(1+\varepsilon(\Delta))$ given in Theorem 3(b) nicely compares with the results of these computations. Moreover, one can observe that $\frac{1}{x}\sum_{\Delta\leq x} h(\Delta)$ seems to tend to a limit as x tends to infinity, even though it is not quite clear to us which value one would conjecture for $\lim_{x\to\infty}\frac{1}{x}\sum_{\Delta\le x}h(\Delta)$. Recall that in the case of real quadratic fields with prime discriminants Cohen and Martinent [5] conjectured that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{p\le x\\ p\equiv 1\ [4]}} h(\mathbf{Q}(\sqrt{p}))=\frac{1}{8}.$$

3. Upper bounds of class numbers of quadratic extensions of an imaginary quadratic field

We no longer assume that $\mathbf{k}$ is principal. Our task is to strengthen Theorem 3(b) and prove the following result.

Theorem 6. *Let $\mathbf{k}$ be a given imaginary quadratic field of class number h and with w roots of unity. Let $\mathbf{K}$ be a quadratic extension of $\mathbf{k}$ with relative discriminant $\delta_{\mathbf{K}/\mathbf{k}}$, and let Δ be $\Delta=|\delta_{\mathbf{K}/\mathbf{k}}|^2$. Then we have the effective upper bound*

$$h(\mathbf{K})R(\mathbf{K})\le\frac{h^2}{w}\sqrt{\Delta}\operatorname{Log}(\sqrt{\Delta})(1+\varepsilon(\Delta))=O\left(\sqrt{|D_{\mathbf{K}/\mathbf{Q}}|}\operatorname{Log}(|D_{\mathbf{K}/\mathbf{Q}}|)\right).$$

As in Theorem 3, this improves the general known upper bounds of $h(\mathbf{K})R(\mathbf{K})$.

Proof. Since $f(s)\overset{\text{def}}{=}\zeta_{\mathbf{K}}(s)/\zeta_{\mathbf{k}}(s)$ is such that

$$f(1)=\frac{2\pi R(\mathbf{K})h(\mathbf{K})}{\sqrt{\Delta|D|}h(\mathbf{k})}\frac{w(\mathbf{K})}{w(\mathbf{k})},$$

we have

$$h(\mathbf{K})R(\mathbf{K})=\frac{1}{2\pi}f(1)h(\mathbf{k})\frac{w(\mathbf{K})}{w(\mathbf{k})}\sqrt{\Delta|D|}.$$

Let χ be the completely multiplicative function defined on the set of integral ideals of $\mathbf{k}$ by means of $\chi(\mathbf{P})=-1$, 0, or $+1$ according as the prime ideal $\mathbf{P}$ or $\mathbf{R_k}$ is inert, is ramified, or splits in $\mathbf{K}/\mathbf{k}$. Then,

$$\begin{aligned}f(s)&=\prod_{\mathbf{P}\text{ prime ideal of }\mathbf{k}}\frac{1}{1-\chi(\mathbf{P})/N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P})^s}\\&=\sum_{\substack{\mathbf{I}\text{ integral ideal of }\mathbf{k}\\ \mathbf{I}\neq 0}}\frac{\chi(\mathbf{I})}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})^s}=\sum_{n\ge 1}\frac{a_n}{n^s}\end{aligned}$$

with

$$a_n\overset{\text{def}}{=}\sum_{\substack{\mathbf{I}\text{ integral ideal of }\mathbf{k}\\ \text{with }N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})=n}}\chi(\mathbf{I}).$$

Let $\beta(n)$ be the number of integral ideals $\mathbf{I}$ of $\mathbf{k}$ such that $N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})=n$. Then, $|a_n|\le\beta(n)=\sum_{j/n}\chi_{\text{im}}(j)$ with χ_{im} being the character of the imaginary quadratic field $\mathbf{k}$. As in Theorem 2, we have

$$f(1)=\sum_{n\geq 1}\frac{a_n}{n}\exp\left(-\frac{2\pi n}{\sqrt{\Delta|D|}}\right)+\frac{2\pi}{\sqrt{\Delta|D|}}\sum_{n\geq 1}a_nE\left(\frac{2\pi n}{\sqrt{\Delta|D|}}\right).$$

Hence, $f(1)\leq B(C)+F(C)$ with

$$B(x)\stackrel{\text{def}}{=}\sum_{n\geq 1}\frac{\beta(n)}{n}e^{-nx},\quad F(x)\stackrel{\text{def}}{=}x\sum_{n\geq 1}\beta(n)E(nx),\quad C\stackrel{\text{def}}{=}\frac{2\pi}{\sqrt{\Delta|D|}}.$$

To complete the proof, we need the following two lemmas.

Lemma B. *There holds* $B(x)=-L(1,\chi_{\text{im}})\operatorname{Log}(x)+O(1)$, $x\mapsto 0^+$.

Proof. We have

$$B(x)=\int_x^\infty\left(\sum_{n\geq 1}\beta(n)e^{-nt}\right)dt$$

and

$$\begin{aligned}\sum_{n\geq 1}\beta(n)e^{-nt}&=\sum_{n\geq 1}\left(\sum_{m/n}\chi_{\text{im}}(m)\right)e^{-nt}\\&=\sum_{m\geq 1}\chi_{\text{im}}(m)\left(\sum_{k\geq 1}e^{-kmt}\right)=\sum_{m\geq 1}\frac{\chi_{\text{im}}(m)}{e^{mt}-1}.\end{aligned}$$

Thus,

$$\begin{aligned}B(x)+L(1,\chi_{\text{im}})\operatorname{Log}(x)=&\sum_{m\geq 1}\chi_{\text{im}}(m)\int_x^1\left(\frac{1}{e^{mt}-1}-\frac{1}{mt}\right)dt\\&+\sum_{m\geq 1}\chi_{\text{im}}(m)\int_1^\infty\frac{dt}{e^{mt}-1}.\end{aligned}$$

If $S(m)=\sum_{k=1}^m\chi_{\text{im}}(k)$, $m\geq 1$, and $S(0)=0$, then using $\chi_{\text{im}}(m)=S(m)-S(m-1)$ and using the trivial upper bound $|S(m)|\leq|D|$, we get the desired result:

$$|B(x)+L(1,\chi_{\text{im}})\operatorname{Log}(x)|\leq A|D|,$$

with

$$A=\int_0^1\left(\frac{1}{t}-\frac{1}{e^t-1}\right)dt+\int_1^\infty\frac{dt}{e^t-1},$$

i.e., with $A=2\operatorname{Log}(e/(e-1))$. □

Lemma C. *There holds* $F(x)=O(1)$.

Proof. We have

$$F(x)=x\sum_{n\geq 1}\left(\sum_{m/n}\chi_{\text{im}}(m)\right)E(nx)=x\sum_{m\geq 1}\chi_{\text{im}}(m)\left(\sum_{n\geq 1}E(nmx)\right).$$

Now,

$$\sum_{n\geq 1} E(nmx) = \sum_{n\geq 1} \int_{mx}^{\infty} e^{-nt}\frac{dt}{t} = \int_{mx}^{\infty} \frac{1}{e^t - 1}\frac{dt}{t}.$$

Thus,

$$\begin{aligned} 0 \leq F(x) &\leq \sum_{m\geq 1} |S(m)| x \int_{mx}^{(m+1)x} \frac{1}{e^t - 1}\frac{dt}{t} \\ &\leq \sum_{m\geq 1} |S(m)| \left(\frac{1}{m} - \frac{1}{m+1}\right) \leq |D|. \end{aligned}$$

Note that using

$$\lim_{x\mapsto 0^+} x \int_{mx}^{(m+1)x} \frac{1}{e^t - 1}\frac{dt}{t} = \frac{1}{m} - \frac{1}{m+1},$$

we can get the more accurate estimate

$$F(x) = L(1, \chi_{\mathrm{im}}) + \varepsilon(x).$$

The effective upper bound

$$h(\mathbf{K})R(\mathbf{K}) \leq \frac{h^2}{w}\sqrt{\Delta}\,\mathrm{Log}(\sqrt{\Delta})(1 + \varepsilon(\Delta))$$

now follows from $L(1, \chi_{\mathrm{im}}) = (2\pi/w\sqrt{|D|})h$, and $w(\mathbf{K}) = w(\mathbf{k})$ whenever $\mathbf{K} \neq \mathbf{Q}(\zeta_n)$, $n = 8$, 10, or 12. $\square$

Our proof, which also applies in the situation studied by Barrucand, Loxton, and Williams [3], would have provided them with upper bounds half as large as the ones they obtained.

Theorem 7. *Let $\mathbf{k}$ be a given imaginary quadratic field of class number h and with w roots of unity. Let $\mathbf{R}_1, \dots, \mathbf{R}_r$, $\mathbf{I}_1, \dots, \mathbf{I}_s$ be $r+s$ given prime distinct ideals of $\mathbf{k}$. Let $\mathbf{K}$ be a quadratic extension of $\mathbf{k}$ with relative discriminant $\delta_{\mathbf{K}/\mathbf{k}}$, and let Δ be $\Delta = |\delta_{\mathbf{K}/\mathbf{k}}|^2$. If each $\mathbf{R}_i$ is ramified and each $\mathbf{I}_j$ is inert in $\mathbf{K}/\mathbf{k}$, then we have the effective upper bound*

$$\begin{aligned} h(\mathbf{K})R(\mathbf{K}) \leq &\left\{\prod_{i=1}^{r}\left(1 - \frac{1}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{R}_i)}\right)\right\} \\ &\cdot \left\{\prod_{j=1}^{s}\left(\frac{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I}_j) - 1}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I}_j) + 1}\right)\right\}\frac{h^2}{w}\sqrt{\Delta}\,\mathrm{Log}(\sqrt{\Delta})(1 + \varepsilon(\Delta)). \end{aligned}$$

This improves Theorem 6.

Proof. With $\mathbf{E}_k \stackrel{\text{def}}{=} \{\mathbf{I}\,;\ \mathbf{I}$ integral ideal of $\mathbf{k}$ prime to each $\mathbf{P}_i,\ 1 \le i \le k\}$, let us define

$$a_n = \sum_{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})=n} \chi(\mathbf{I}), \qquad b_n(k) = \sum_{\mathbf{I}\in\mathbf{E}_k \text{ and } N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})=n} \chi(\mathbf{I}),$$

$$\tilde{a}_n = \sum_{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})=n} |\chi(\mathbf{I})|, \qquad \tilde{b}_n(k) = \sum_{\mathbf{I}\in\mathbf{E}_k \text{ and } N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})=n} |\chi(\mathbf{I})|.$$

Let $\mathbf{P}_i,\ 1 \le i \le k$, be k distinct prime ideals of $\mathbf{k}$. Our proof relies upon the two following results, each of which is proved by induction on k:

$$\text{(a)}\quad \left\{\prod_{i=1}^{k}\left(1-\frac{\chi(\mathbf{P}_i)}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P}_i)}\right)\right\}\left(\sum_{n\ge1}\frac{a_n}{n}e^{-nC}\right) = \sum_{n\ge1}\frac{b_n(k)}{n}e^{-nC} + O(1),$$

$$\text{(b)}\quad \left\{\prod_{i=1}^{k}\left(1-\frac{1}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P}_i)}\right)\right\}\left(\sum_{n\ge1}\frac{\tilde{a}_n}{n}e^{-nC}\right) = \sum_{n\ge1}\frac{\tilde{b}_n(k)}{n}e^{-nC} + O(1).$$

For example, the first result, where we write $N(\mathbf{I})$ instead of $N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})$ whenever $\mathbf{I}$ is an integral ideal of $\mathbf{k}$, is proved as follows. Assume it is true for k. Then we have

$$\begin{aligned}
&\left\{\prod_{i=1}^{k+1}\left(1-\frac{\chi(\mathbf{P}_i)}{N(\mathbf{P}_i)}\right)\right\}\left(\sum_{n\ge1}\frac{a_n}{n}e^{-nC}\right)\\
&\quad= \left(1-\frac{\chi(\mathbf{P}_{k+1})}{N(\mathbf{P}_{k+1})}\right)\left(\sum_{n\ge1}\frac{b_n(k)}{n}e^{-nC}\right) + O(1)\\
&\quad= \left(1-\frac{\chi(\mathbf{P}_{k+1})}{N(\mathbf{P}_{k+1})}\right)\left(\sum_{\mathbf{I}\in\mathbf{E}_k}\frac{\chi(\mathbf{I})}{N(\mathbf{I})}e^{-N(\mathbf{I})C}\right) + O(1)\\
&\quad= \sum_{\mathbf{I}\in\mathbf{E}_{k+1}}\frac{\chi(\mathbf{I})}{N(\mathbf{I})}e^{-N(\mathbf{I})C} - \sum_{\mathbf{I}\in\mathbf{E}_k}\frac{\chi(\mathbf{I}\mathbf{P}_{k+1})}{N(\mathbf{I}\mathbf{P}_{k+1})}(e^{N(\mathbf{I})C} - e^{N(\mathbf{I})N(\mathbf{P}_{k+1})C}) + O(1)\\
&\quad= \sum_{n\ge1}\frac{b_n(k+1)}{n}e^{-nC} - R + O(1)
\end{aligned}$$

with

$$\begin{aligned}
|R| &\le \frac{1}{N(\mathbf{P}_{k+1})}\sum_{n\ge1}\frac{\beta(n)}{n}(e^{-nC} - e^{-nN(\mathbf{P}_{k+1})C})\\
&= \frac{B(C) - B(N(\mathbf{P}_{k+1})C)}{N(\mathbf{P}_{k+1})}.
\end{aligned}$$

From Lemma B we thus have $R = O(1)$, and we get the result for $k+1$.

Now Theorem 7 follows from

$$\begin{aligned}
&\left\{\prod_{i=1}^{k}\left(1-\frac{\chi(\mathbf{P}_i)}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P}_i)}\right)\right\} f(1)\\
&\quad = \left\{\prod_{i=1}^{k}\left(1-\frac{\chi(\mathbf{P}_i)}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P}_i)}\right)\right\}\left(\sum_{n\geq 1}\frac{a_n}{n}e^{-nC}\right)+O(1)\\
&\quad = \sum_{n\geq 1}\frac{b_n(k)}{n}e^{-nC}+O(1)\leq \sum_{n\geq 1}\frac{\tilde{b}_n(k)}{n}e^{-nC}+O(1)\\
&\quad = \left\{\prod_{i=1}^{k}\left(1-\frac{1}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P}_i)}\right)\right\}\left(\sum_{n\geq 1}\frac{\tilde{a}_n}{n}e^{-nC}\right)+O(1)\\
&\quad \leq \left\{\prod_{i=1}^{k}\left(1-\frac{1}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P}_i)}\right)\right\}\left(\sum_{n\geq 1}\frac{\beta(n)}{n}e^{-nC}\right)+O(1)\\
&\quad = \left\{\prod_{i=1}^{k}\left(1-\frac{1}{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{P}_i)}\right)\right\} L(1\,,\,\chi_{\mathrm{im}})\,\mathrm{Log}(\sqrt{\Delta})+O(1). \quad \square
\end{aligned}$$

Numerical example. We take $\mathbf{k}=\mathbf{Q}(\sqrt{-2})$ and $\mathbf{K}=\mathbf{k}(\sqrt{d})$ with $d=m^2+2=(m-\sqrt{-2})(m+\sqrt{-2})=(a+b\sqrt{-2})^2+2=a^2-2b^2+2+2ab\sqrt{-2}$ a squarefree integer in $\mathbf{Z}[\sqrt{-2}]$ (hence a is odd). Then, $\eta_{\mathbf{K}}=(m+\sqrt{d})/\sqrt{-2}$ is a unit in $\mathbf{R}_{\mathbf{K}}$ with $N_{\mathbf{K}/\mathbf{k}}(\eta_{\mathbf{K}})=+1$, and one can easily see that it is a fundamental unit of $\mathbf{K}$, that $\mathbf{R}_{\mathbf{K}}=\mathbf{R}_{\mathbf{k}}[\eta_{\mathbf{K}}]$, and that $\delta_{\mathbf{K}/\mathbf{k}}=-2d$. Moreover, we may assume that we have $a\geq 0$, $b\geq 0$ (since $\mathbf{k}(\sqrt{\bar{d}})$ is isomorphic to $\mathbf{k}(\sqrt{d})$). Now, whenever $z\in\mathbf{R}_{\mathbf{k}}$ has even absolute norm $N_{\mathbf{k}/\mathbf{Q}}(z)$, then $\chi(z)=0$, and whenever z has odd absolute norm, then $\chi(z)=[\delta_{\mathbf{K}/\mathbf{k}}/(z)]$. Thus, we can calculate $\chi(z)$, and $h(\mathbf{K})$, with the help of Theorem 2. Indeed, we have already explained in [14] how to choose M such that the nearest integer to

$$\tilde{h}(\mathbf{K}) \overset{\text{def}}{=} \frac{1}{w(\mathbf{k})R(\mathbf{K})}\sum_{\substack{z\in\mathbf{R}_{\mathbf{k}}\\ z\neq 0\\ |z|\leq M}}\chi(z)\left\{E\left(\frac{2\pi|z|^2}{\sqrt{\Delta|D|}}\right)+\frac{\exp(-2\pi|z|^2/\sqrt{\Delta|D|})}{2\pi|z|^2/\sqrt{\Delta|D|}}\right\}$$

yields the class number $h(\mathbf{K})$, using the power series expansion $E(x)=-\gamma-\mathrm{Log}(x)-\sum_{n\geq 1}(-1)^n x^n/nn!$ (with γ being Euler's constant) to get the values $E(2\pi|z|^2/\sqrt{\Delta|D|})$. We will give the results of our computations in Table 4, i.e., the values of $h(\mathbf{K})$, provided

$$\Delta=|\delta_{\mathbf{K}/\mathbf{k}}|^2=4(a^2+2(b-1)^2)(a^2+2(b+1)^2)\leq 3\cdot 10^5.$$

The aim of these computations is to show that Theorem 7 produces good upper bounds of class numbers. Let $\mathbf{P}_3=(1+\sqrt{-2})$ and $\mathbf{P}_3'=(1-\sqrt{-2})$ be the two prime ideals of $\mathbf{R}_{\mathbf{k}}$ lying above (3). Since d is assumed to be squarefree in $\mathbf{Z}[\sqrt{-2}]$, neither $(\mathbf{P}_3)^2$ nor $(\mathbf{P}_3')^2$ divides d. Hence, 3^3 does not divide Δ.

Lemma. *Assume that d is squarefree in $\mathbf{Z}[\sqrt{-2}]$. Then,*

(a) 3 *does not divide* Δ *if and only if* 3 *divides* a *and* 3 *divides* b. *In this case,* $\mathbf{P}_3$ *and* $\mathbf{P}_3'$ *are inert in* $\mathbf{K}/\mathbf{k}$.

TABLE 4

a	b	Δ	2-rank	$h(\mathbf{K})$	$M_{r,r}(\Delta)$	$M_{r,i}(\Delta)$	$M_{i,i}(\Delta)$
1	2	$228 = 4 \cdot \mathbf{3} \cdot \mathbf{19}$	1	2		1.3	
3	1	$612 = 4 \cdot \mathbf{3}^2 \cdot 17$	2	4	2.7		
3	3	$2788 = 4 \cdot 17 \cdot 41$	2	4			3.3
5	1	$3300 = 4 \cdot \mathbf{3} \cdot 5^2 \cdot \mathbf{11}$	2	4		4.8	
1	4	$3876 = 4 \cdot \mathbf{3} \cdot 17 \cdot \mathbf{19}$	2	4		5.2	
5	3	$7524 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{11} \cdot \mathbf{19}$	3	8	9.6		
1	5	$9636 = 4 \cdot \mathbf{3} \cdot \mathbf{11} \cdot 73$	2	8		8.2	
7	1	$11172 = 4 \cdot \mathbf{3} \cdot 7^2 \cdot \mathbf{19}$	2	8		8.8	
5	4	$12900 = 4 \cdot \mathbf{3} \cdot 5^2 \cdot \mathbf{43}$	2	8		9.5	
7	2	$13668 = 4 \cdot \mathbf{3} \cdot 17 \cdot \mathbf{67}$	2	8		9.7	
5	5	$22116 = 4 \cdot \mathbf{3} \cdot \mathbf{19} \cdot 97$	2	12		12.4	
3	6	$25252 = 4 \cdot \mathbf{59} \cdot \mathbf{107}$	1	6			9.9
9	2	$32868 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{11} \cdot \mathbf{83}$	3	16	20.1		
5	6	$36900 = 4 \cdot \mathbf{3}^2 \cdot 5^2 \cdot 41$	3	16	21.3		
1	7	$37668 = 4 \cdot \mathbf{3} \cdot \mathbf{43} \cdot 73$	2	8		16.2	
9	3	$40228 = 4 \cdot 89 \cdot 113$	2	8			12.5
9	4	$51876 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{11} \cdot \mathbf{131}$	3	16	25.3		
11	1	$62436 = 4 \cdot \mathbf{3} \cdot \mathbf{11}^2 \cdot \mathbf{43}$	3	16		20.8	
11	2	$68388 = 4 \cdot \mathbf{3} \cdot 41 \cdot \mathbf{139}$	2	16		21.8	
9	5	$69156 = 4 \cdot \mathbf{3}^2 \cdot 17 \cdot 113$	3	24	29.2		
3	8	$73188 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{19} \cdot \mathbf{107}$	3	24	30.1		
5	8	$92004 = 4 \cdot \mathbf{3} \cdot \mathbf{11} \cdot 17 \cdot 41$	3	16		25.3	
9	6	$93796 = 4 \cdot \mathbf{131} \cdot \mathbf{179}$	1	14			19.1
1	9	$103716 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{43} \cdot \mathbf{67}$	3	32	35.8		
3	9	$114532 = 4 \cdot \mathbf{11} \cdot \mathbf{19} \cdot 137$	2	12			21.2
13	1	$119652 = 4 \cdot \mathbf{3} \cdot 13^2 \cdot \mathbf{59}$	2	20		28.8	
7	8	$124068 = 4 \cdot \mathbf{3} \cdot 7^2 \cdot \mathbf{211}$	2	20		29.4	
9	7	$127908 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{11} \cdot 17 \cdot \mathbf{19}$	4	32	39.7		
13	3	$142308 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{59} \cdot \mathbf{67}$	3	24	41.9		
13	4	$163812 = 4 \cdot \mathbf{3} \cdot \mathbf{11} \cdot 17 \cdot 73$	3	24		33.7	
3	10	$171684 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{19} \cdot \mathbf{251}$	3	24	46.0		
7	9	$176292 = 4 \cdot \mathbf{3}^2 \cdot \mathbf{59} \cdot \mathbf{83}$	3	32	46.7		
11	7	$192228 = 4 \cdot \mathbf{3} \cdot \mathbf{83} \cdot 193$	2	28		36.6	
13	5	$193764 = 4 \cdot \mathbf{3} \cdot \mathbf{67} \cdot 241$	2	20		36.7	

TABLE 4 (continued)

a	b	Δ	2-rank	$h(\mathbf{K})$	$M_{r,r}(\Delta)$	$M_{r,i}(\Delta)$	$M_{i,i}(\Delta)$
5	10	$199716 = 4 \cdot \mathbf{3} \cdot \mathbf{11} \cdot 17 \cdot 89$	3	24		37.2	
15	1	$209700 = 4 \cdot \mathbf{3}^2 \cdot 5^2 \cdot 233$	3	40	50.9		
13	6	$233892 = 4 \cdot \mathbf{3}^2 \cdot 73 \cdot 89$	3	40	53.7		
9	9	$234916 = 4 \cdot \mathbf{11} \cdot \mathbf{19} \cdot 281$	2	16			30.3
15	3	$239524 = 4 \cdot 233 \cdot 257$	2	24			30.6
7	10	$245604 = 4 \cdot \mathbf{3} \cdot 97 \cdot \mathbf{211}$	2	28		41.3	
11	8	$247908 = 4 \cdot \mathbf{3} \cdot 73 \cdot \mathbf{283}$	2	20		41.5	

(b) 3 *divides* Δ *and* 3^2 *does not divide* Δ *if and only if* 3 *does not divide* a *and* 3 *does not divide* b. *In this case,* $\mathbf{P}_3$ *is ramified in* $\mathbf{K}/\mathbf{k}$ *and* $\mathbf{P}'_3$ *is inert in* $\mathbf{K}/\mathbf{k}$, *or* $\mathbf{P}_3$ *is inert in* $\mathbf{K}/\mathbf{k}$ *and* $\mathbf{P}'_3$ *is ramified in* $\mathbf{K}/\mathbf{k}$.

(c) 3^2 *divides* Δ *if and only if* 3 *divides* a *and does not divide* b, *or* 3 *does not divide* a *and divides* b. *In this case,* $\mathbf{P}_3$ *and* $\mathbf{P}'_3$ *are ramified in* $\mathbf{K}/\mathbf{k}$.

Proof. We have $\Delta \equiv (a^2-1)(b^2-1) \pmod 3$ and $[\delta/\mathbf{P}_3] = (((a-b)^2-1)/3) = 0$ or -1, and $[\delta/\mathbf{P}'_3] = (((a+b)^2-1)/3) = 0$ or -1. We thus get the desired result. For example, 3^2 divides δ if and only if $\mathbf{P}_3$ and $\mathbf{P}'_3$ divide δ, i.e., if and only if $[\delta/\mathbf{P}_3] = [\delta/\mathbf{P}'_3] = 0$, i.e., if and only if 3 does not divide $a+b$ and does not divide $a-b$, i.e., if and only if 3 divides a but does not divide b, or 3 does not divide a but divides b. □

Set $M(\Delta) = \frac{1}{2}\sqrt{\Delta}$, $M_{r,r}(\Delta) = \frac{1}{9}\sqrt{\Delta}$, $M_{r,i}(\Delta) = \frac{1}{12}\sqrt{\Delta}$, and $M_{i,i}(\Delta) = \frac{1}{16}\sqrt{\Delta}$. Since the prime ideal $(\sqrt{-2})$ of $\mathbf{R_k}$ is ramified in $\mathbf{K}/\mathbf{k}$, Theorem 7 and $R(\mathbf{K}) = (1+\varepsilon(\Delta))\operatorname{Log}(\sqrt{\Delta})$ provide us with the following upper bounds:

$$h(\mathbf{K}) \leq M(\Delta)(1+\varepsilon(\Delta)),$$

(a) $h(\mathbf{K}) \leq M_{r,r}(\Delta)(1+\varepsilon(\Delta))$ whenever $\mathbf{P}_3$ and $\mathbf{P}'_3$ are ramified in $\mathbf{K}/\mathbf{k}$,

(b) $h(\mathbf{K}) \leq M_{r,i}(\Delta)(1+\varepsilon(\Delta))$ whenever one of the prime ideals of $\mathbf{R_k}$ lying above (3) is ramified in $\mathbf{K}/\mathbf{k}$ while the other is inert in $\mathbf{K}/\mathbf{k}$,

(c) $h(\mathbf{K}) \leq M_{i,i}(\Delta)(1+\varepsilon(\Delta))$ whenever $\mathbf{P}_3$ and $\mathbf{P}'_3$ are inert in $\mathbf{K}/\mathbf{k}$.

In Table 4, we give the values of these asymptotic upper bounds according as we are in case (a), (b), or (c). Observe that these upper bounds nicely compare with $h(\mathbf{K})$, and compare much better with $h(\mathbf{K})$ than the general upper bound $M(\Delta)$ of Theorem 6. Moreover, this table agrees with our previous paper [15, Corollary 10(b)], i.e., with our determination of the 2-rank of the ideal class group of quadratic extensions of principal imaginary quadratic fields: if $\delta_{\mathbf{K}/\mathbf{k}}$ has t distinct prime factors in $\mathbf{Z}[\sqrt{-2}]$, then the ideal class groups of the quadratic extensions of $\mathbf{Q}(\sqrt{-2})$ have 2-rank $t-1$ if none of the odd prime divisors of Δ is congruent to 3 modulo 8, and have rank $t-2$ otherwise (in our table, the bold prime factors of Δ are those that are congruent to 3 mod 8).

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DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE CAEN, U.F.R. SCIENCES, ESPLANADE DE LA PAIX, 14032 CAEN CEDEX, FRANCE

E-mail address: loubouti@univ-caen.fr