

ENUMERATION OF SMALL NONISOMORPHIC 1-ROTATIONAL TWOFOLD TRIPLE SYSTEMS

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ABSTRACT. In this paper, twofold triple systems of order v are enumerated for all $v \leq 19$.

1. INTRODUCTION

The existence of $\text{TS}(v, 2)$'s (all terms are defined in §2) is completely settled; the condition $v \equiv 0$ or $1 \pmod{3}$ is known to be both necessary and sufficient [4]. On the other hand, enumeration efforts have not enjoyed such success. In fact, the exact number of pairwise nonisomorphic $\text{TS}(v, 2)$'s, denoted $N(v)$, has been determined only for $v \leq 10$. In particular, we have $N(3) = N(4) = 1$ (trivial), $N(6) = 1$ [5], $N(7) = 4$ [13], $N(9) = 36$ [12, 8], and $N(10) = 960$ [1, 3]. One reason for the unavailability of such enumeration results for higher values of v is the inherent computational complexity of the problem that leads to a combinatorial explosion effect. To curb this combinatorial explosion, extra conditions are often imposed to enumerate interesting classes of designs. One such condition involves specifying automorphisms that the desired designs must possess.

Cyclic $\text{TS}(v, 2)$'s, that is, $\text{TS}(v, 2)$'s possessing an automorphism of order v , have been enumerated by Colbourn [2] for $v \leq 16$. In this paper, we completely enumerate the class of 1-rotational $\text{TS}(v, 2)$'s for all $v \leq 19$. The existence of 1-rotational $\text{TS}(v, 2)$'s is determined by Kuriki and Jimbo [7], who proved that the necessary condition $v \equiv 0$ or $1 \pmod{3}$ is also sufficient. However, they did not provide any enumeration results. The only result on the enumeration of 1-rotational t - (v, k, λ) designs for $\lambda > 1$ that we are aware of is that by Mathon and Rosa [9], who determined that there are precisely 85 nonisomorphic 1-rotational 2- $(15, 5, 4)$ designs.

2. DEFINITIONS AND NOTATIONS

A t -*design*, or more specifically t - (v, k, λ) *design*, is a combinatorial system consisting of a pair (X, \mathcal{B}) , where X is a finite set of v elements called *points*, and \mathcal{B} is a collection of k -subsets of X called *blocks*, such that every t -subset of X is contained in precisely λ blocks. We generally allow t -designs to have repeated blocks. A 2- $(v, 3, \lambda)$ design is commonly called a λ -*fold triple system* of *order* v , and is denoted $\text{TS}(v, \lambda)$.

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Two t -designs, say (X_1, \mathcal{B}_1) and (X_2, \mathcal{B}_2) , are said to be *isomorphic* if there exists a bijection $\pi: X_1 \rightarrow X_2$ such that $\{x_1, x_2, \dots, x_k\} \in \mathcal{B}_1$ if and only if $\{\pi(x_1), \pi(x_2), \dots, \pi(x_k)\} \in \mathcal{B}_2$. Such a bijection is called an *isomorphism from (X_1, \mathcal{B}_1) onto (X_2, \mathcal{B}_2)* . An *automorphism* of a t -design (X, \mathcal{B}) is an isomorphism from (X, \mathcal{B}) onto itself. The set of all automorphisms of a t -design forms a group, called the *full automorphism group* of the design, under functional composition. Any subgroup of the full automorphism group is simply called an *automorphism group*.

A t -design is called *1-rotational* if it admits a permutation with one fixed point and a cycle of length $v - 1$ as an automorphism.

3. COMPUTATIONAL DETAILS

Let G be a group acting on a finite set X of v points. Then there is a natural action of G on the 2-subsets and 3-subsets of X . Let $A(G)$ be a matrix with its rows and columns indexed by G -orbits of 2-subsets and 3-subsets of X , respectively, and define the (i, j) th entry of $A(G)$, a_{ij} , as the number of 3-subsets in the G -orbit indexing column j that contain a fixed orbit representative of the G -orbit indexing row i . The number a_{ij} is independent of the choice of the orbit representative. A more general result of Kramer and Mesner [6] implies that a $\text{TS}(v, 2)$ exists with G as an automorphism group if and only if there exists a nonnegative integral vector u satisfying the matrix equation $A(G)u = 2j$, where j is the vector of all 1's. It should be clear that the vector u determines which orbits of 3-subsets (or blocks) are to be present in the $\text{TS}(v, 2)$ in a natural way.

Let $G = \langle \alpha \rangle$, where $\alpha = (0)(12 \cdots v - 1)$. It follows from a previous discussion that if we want to construct a 1-rotational $\text{TS}(v, 2)$ on the set of points $X = \{0, 1, \dots, v - 1\}$, we need only to look for a nonnegative integral vector u such that $A(G)u = 2j$. We can compute the size of the matrix $A(G)$, that is, the number of orbits of 2-subsets and 3-subsets of X under the action of G , from the Cauchy-Frobenius-Burnside lemma:

Lemma 1. *Let G be a group acting on a finite set X , and let $\chi(\pi)$ be the set of all t -subsets of X that is invariant under the permutation $\pi \in G$. Then the number of G -orbits of t -subsets of X is*

$$\frac{1}{|G|} \sum_{\pi \in G} |\chi(\pi)|.$$

Table 1 provides information on the size of the $A(G)$ matrices for various values of v .

A branch-and-bound algorithm using depth-first search was used to find all nonnegative integral vectors u that satisfy $A(G)u = 2j$. These resulted in a set of distinct designs which are then subjected to further analysis to remove isomorphic copies. Isomorphism testing of the designs was carried out using **nauty**, the isomorphism checking algorithm of McKay [10, 11]. Let $N_{1r}(v)$ and $N_{1r}^*(v)$ denote the number of pairwise nonisomorphic 1-rotational $\text{TS}(v, 2)$'s without and with repeated blocks, respectively. A summary of the enumeration results we obtained is given in Table 2.

All the pairwise nonisomorphic 1-rotational $\text{TS}(v, 2)$'s for $v \leq 19$ are listed in the Appendix to be found on the microfiche card attached to this issue. The blocks in each $\text{TS}(v, 2)$ can be obtained by developing the given starter blocks

TABLE 1

| v | size of $A(G)$ |
|-----|----------------|
| 6 | 3×4 |
| 7 | 4×7 |
| 9 | 5×11 |
| 10 | 5×14 |
| 12 | 6×20 |
| 13 | 7×25 |
| 15 | 8×33 |
| 16 | 8×38 |
| 18 | 9×48 |
| 19 | 10×55 |

TABLE 2. Nonisomorphic $TS(v, 2)$'s

| v | $N_{1r}(v)$ | $N_{1r}^*(v)$ | Total |
|-----|-------------|---------------|-------|
| 6 | 1 | 0 | 1 |
| 7 | 1 | 1 | 2 |
| 9 | 1 | 2 | 3 |
| 10 | 1 | 0 | 1 |
| 12 | 5 | 0 | 5 |
| 13 | 8 | 5 | 13 |
| 15 | 36 | 17 | 53 |
| 16 | 48 | 5 | 53 |
| 18 | 174 | 12 | 186 |
| 19 | 366 | 99 | 465 |

of each design with the permutation $(0)(1\ 2 \cdots v - 1)$. We also give the order of the full automorphism group of each design.

4. TRANSITIVE AND AFFINE DESIGNS

A $TS(v, 2)$ (X, \mathcal{B}) is *transitive* if its full automorphism group acts transitively on the set of points X . Let X be the set of elements underlying the

finite field $\text{GF}(v)$, where v is a prime power. Then a $\text{TS}(v, 2)$ (X, \mathcal{B}) is said to be *affine* if it admits the affine group

$$\{x \mapsto ax + b \mid a \neq 0; a, b \in \text{GF}(v)\}$$

as an automorphism group. It is not difficult to see that an affine $\text{TS}(v, 2)$ is both transitive and 1-rotational.

It appears that transitive $\text{TS}(v, 2)$'s are relatively rare among the 1-rotational $\text{TS}(v, 2)$'s. In all the pairwise nonisomorphic $\text{TS}(v, 2)$'s enumerated for $v \leq 19$, only six transitive $\text{TS}(v, 2)$'s were found. These designs are a $\text{TS}(6, 2)$ (design #1), a $\text{TS}(7, 2)$ (design #1), $\text{TS}(9, 2)$ (design #3), a $\text{TS}(13, 2)$ (design #2), a $\text{TS}(16, 2)$ (design #45), and a $\text{TS}(19, 2)$ (design #189). Further analysis reveals that all of these designs are actually affine designs, except for the $\text{TS}(6, 2)$. It also follows that the number of pairwise nonisomorphic affine $\text{TS}(v, 2)$'s is precisely one for each $v \in \{7, 9, 13, 16, 19\}$. We note that the affine $\text{TS}(7, 2)$, $\text{TS}(9, 2)$, $\text{TS}(13, 2)$, and $\text{TS}(16, 2)$ must have been known to Colbourn [2] too, since every affine $\text{TS}(v, 2)$ is also cyclic.

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