MATHEMATICS OF COMPUTATION
VOLUME 60, NUMBER 202
APRIL 1993, PAGES 801-810

THE MINIMUM DISCRIMINANT OF TOTALLY REAL ALGEBRAIC NUMBER FIELDS OF DEGREE 9 WITH CUBIC SUBFIELDS

HIROYUKI FUJITA

ABSTRACT. With the help of the computer language UBASIC86, the minimum discriminant $d(K)$ of totally real algebraic number fields K of degree 9 with cubic subfields F is determined. It is given by $d(K) = 16240385609$. The defining equation for K is given by $f(x) = x^9 - x^8 - 9x^7 + 4x^6 + 26x^5 - 2x^4 - 25x^3 - x^2 + 7x + 1$, and K is uniquely determined by $d(K)$ up to $\mathbf{Q}$-isomorphism. The field K has the cubic subfield F with $d(F) = 49$ defined by the polynomial $f(x) = x^3 + x^2 - 2x - 1$.

1. INTRODUCTION

Let K be a totally real algebraic number field of degree n with discriminant $d(K)$, and r_1 be the number of real conjugate fields and $2r_2$ the number of complex conjugate fields, so that we have $n = r_1 + 2r_2$.

It is an important problem to determine the minimum discriminant $d(K)$ and the corresponding field K for each pair (r_1, r_2).

When $n < 8$, for all signatures (r_1, r_2) the field K with minimum discriminant $d(K)$ is known. In the case $n = 8$, only the totally real case (i.e., $r_2 = 0$) and the totally complex case (i.e., $r_2 = 4$) have been determined (Pohst [1, 2], Pohst, Martinet, and Diaz y Diaz [4], Diaz y Diaz [11]).

In this paper we shall treat totally real fields of degree 9 (i.e., $n = 9$, $r_2 = 0$). There are two cases. The first is when K has a cubic subfield, and the other is when K does not have any such subfields. Since it seems that the latter case is harder than the former, in this paper we shall deal only with the former case. We prove the following main theorem.

Theorem 1. *Let K be a totally real algebraic number field of degree 9 with a cubic subfield F such that the discriminant $d(K)$ satisfies*

$$d(K) \leq 16983563041.$$

Let $f(x) = x^9 + a_1x^8 + a_2x^7 + a_3x^6 + a_4x^5 + a_5x^4 + a_6x^3 + a_7x^2 + a_8x + a_9$ be a defining equation for the field K. Then the complete list of such fields K is given as follows:

Received by the editor April 30, 1991 and, in revised form, January 20, 1992 and May 19, 1992.
1991 *Mathematics Subject Classification.* Primary 11R16, 11Y40, 11R29, 11R80.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	$d(K)$	$d(F)$
-1	-9	4	26	-2	-25	-1	7	1	16240385609	49
3	-7	-14	17	18	-13	-8	3	1	16440305941	229
-1	-8	7	21	-15	-20	10	5	-1	16983563041	361

Moreover, for each discriminant $d(K)$ listed above, the field K is uniquely determined up to $\mathbf{Q}$-isomorphism.

In order to prove Theorem 1, we do the following:

(1) First we determine an upper bound $d_{\max}$ and a lower bound $d_{\min}$ for the discriminant $d(K)$.

(2) Let F be a cubic subfield of K. It is known that

$$d(K) = d(F)^3 N(D(K/F)), \tag{1}$$

where $N(\cdot)$ is the norm of $F/\mathbf{Q}$, and $D(K/F)$ is the relative discriminant of K/F. So we have

$$d(F) \leq (d_{\max})^{1/3}. \tag{2}$$

For each field F, we determine all fields K with $d(K) \leq d_{\max}$ such that K contains F as a subfield.

We must use a computer in order to calculate the discriminants, determinants, coefficients of polynomials and the other data. Since these values can be greater than 10^{14}, we cannot calculate them with commonly used computer languages (for example FORTRAN, PASCAL, C). So we use the computer language UBASIC86 running on the personal computer NEC PC-9801 series. UBASIC86 is a high-precision BASIC, which is an excellent public-domain software written by Y. Kida [9]. We use the version 8.12 (October 1990). UBASIC86 can calculate with up to 2600 digits for integers and real numbers, and up to 2600 digits total for the real and imaginary parts of complex numbers. Since it does the job faster than the familiar languages, UBASIC86 is best for our purpose.

2. Cubic subfields

Let K be a totally real algebraic number field of degree 9 with discriminant $d(K)$. We determine an upper bound and a lower bound for the discriminant $d(K)$. A. Odlyzko [8] supplied a table of triples (A, B, E) and gave the following lower bounds for $d(K)$:

$$A^{r_1} B^{2r_2} e^{-E} < d(K), \tag{3}$$

where e is Euler's constant, the base of the natural logarithms. One of them, which we will use, is given as follows:

$$A = 29.534, \quad B = 14.616, \quad E = 8.2267. \tag{4}$$

In our case $(r_1 = 9,\ r_2 = 0)$ it follows by (3) and (4) that

$$4392565450.664669 < d(K). \tag{5}$$

So we take the lower bound $d_{\min} = 4392565451$.

Let $\mathbf{Q}(\zeta_m)$ be a cyclotomic field with mth root of unity ζ_m and $\mathbf{Q}(\zeta_m)^+$ be the maximal real subfield. In the case $m = 19$, we have $[\mathbf{Q}(\zeta_{19})^+ : \mathbf{Q}] = 9$ and

TABLE 1

b_1	b_2	b_3	$d(F)$	b_1	b_2	b_3	$d(F)$	b_1	b_2	b_3	$d(F)$	b_1	b_2	b_3	$d(F)$
1	−2	−1	49	1	−7	3	788	1	−8	3	1425	1	−9	−10	1957
0	−3	1	81	0	−6	1	837	0	−11	12	1436	0	−8	1	2021
1	−3	−1	148	1	−8	10	892	1	−10	7	1489	1	−10	6	2024
1	−4	1	169	0	−7	4	940	1	−9	5	1492	0	−11	11	2057
0	−4	1	229	1	10	−8	961	1	−7	−4	1509	0	−13	4	2089
1	−4	−3	257	1	−6	−1	985	1	−7	−1	1524	1	−11	8	2101
1	−4	−2	316	1	−6	−3	993	1	−9	−11	1556	1	−8	−5	2177
1	−4	−1	321	1	−6	−2	1016	1	−7	−2	1573	1	−13	12	2213
1	−6	−7	361	0	−8	6	1076	0	−9	7	1593	0	−14	18	2228
1	−5	1	404	1	−9	−12	1101	0	−12	14	1620	1	−8	−1	2233
1	−5	−4	469	0	−7	3	1129	1	−8	2	1708	0	−9	5	2241
0	−5	1	473	1	−7	−6	1229	1	−11	−16	1765	1	−13	−1	2292
1	−5	−3	564	1	−8	−9	1257	1	−12	−8	1772	1	−14	14	2296
1	−6	2	568	0	−10	10	1300	1	−8	−7	1825	1	−8	−2	2300
0	−6	3	621	0	−11	2	1304	1	−14	8	1849	0	−12	13	2349
0	−7	5	697	0	−7	1	1345	1	−9	4	1901	1	−15	16	2429
0	10	11	733	1	−12	11	1369	1	−10	−13	1929	1	−10	5	2505
0	−6	2	756	0	−8	5	1373	1	−8	1	1937	1	−9	2	2557
1	−6	1	761	1	−10	−14	1384	0	−8	2	1940				
1	−6	−5	785	1	−7	−5	1396	0	−9	6	1944				

$d(\mathbf{Q}(\zeta_{19})^+) = 19^8 = 16983563041$. So we have the upper bound

$$d_{\max} = 19^8 = 16983563041.$$

It is sufficient to determine all fields K such that

$$d_{\min} = 4392565451 \le d(K) \le d_{\max} = 16983563041. \tag{6}$$

By (2), we need all cubic real fields F with the discriminants

$$d(F) \le [d_{\max}^{1/3}] = 2570.$$

Let $f(x) = x^3 + b_1x^2 + b_2x + b_3$ be a defining equation for the field F. Then the complete list of such fields F is given as in Table 1. The integral basis is given by 1, α, $(\alpha+\alpha^2)/2$ in the cases $d(F) = 961$, 1304, 1772, 1849, 2089, and by 1, α, $(1+\alpha^2)/2$ in the case $d(F) = 2292$, and by 1, α, α^2 in the remaining cases.

3. CUBIC EXTENSIONS OF CUBIC SUBFIELDS

In this section we determine all totally real algebraic number fields K of degree 9 satisfying (6) which are cubic extensions of the cubic fields F given in Table 1.

Let K be an algebraic number field and F be a subfield of K. Let ρ be a $\mathbf{Q}$-isomorphism of K, σ be a $\mathbf{Q}$-isomorphism of F and $\rho|_F$ be the restriction of ρ to F. Then the σ-trace is defined by

$$\mathrm{Tr}_{\sigma, K/F}(\theta) = \sum_{\substack{\rho \\ \rho|_F = 0}} \rho\theta \qquad (\theta \in K).$$

The following theorem is useful in finding a field K which is an extension of its subfield F.

Theorem 2 (Martinet [5]). *Let K be an algebraic number field of degree n, F be an algebraic number field of degree n' and K be an extension of F with degree m. Then there exists $\theta \in \mathbf{O}_K$, $\theta \notin F$, such that*

$$\sum_{i=1}^{n} |\theta^{(i)}|^2 \leq \frac{1}{m} \sum_{\sigma} |\,\mathrm{Tr}_{\sigma, K/F}(\theta)|^2 + \gamma_{n-n'} \left| \frac{d(K)}{m^{n'} d(F)} \right|^{1/(n-n')},$$

where $\theta^{(i)}$ are conjugates of θ. Further, if θ satisfies this inequality, then $\theta+\delta$, for any $\delta \in F$, satisfies it also.

Let F be a cubic field and K a cubic extension of F. Let $\{\sigma_1 = 1, \sigma_2, \sigma_3\}$ be a $\mathbf{Q}$-isomorphism of F, and $\{1, \delta^{(1)}, \varepsilon^{(1)}\}$ be an integral basis of F, and $\{\delta^{(j)}, \varepsilon^{(j)}\}$ $(j = 2, 3)$ be the conjugates. Then, using Theorem 2, we have the following inequality for some $\theta \in \mathbf{O}_K$, $\theta \notin F$:

$$\sum_{i=1}^{9} |\theta^{(i)}|^2 \leq \frac{1}{3} \sum_{j=1}^{3} (\mathrm{Tr}_{\sigma_j, K/F}(\theta))^2 + \left(\frac{19^8}{3^3 d(F)} \right)^{1/6} \times \left(\frac{64}{3} \right)^{1/6}. \tag{7}$$

We put

$$T_2 = \left(\frac{19^8}{3^3 d(F)} \right)^{1/6} \times \left(\frac{64}{3} \right)^{1/6}.$$

Hence, θ is a root of an irreducible polynomial $f(x) \in \mathbf{Z}[x]$ which decomposes in $\mathbf{O}_F[x]$ into a product of three conjugate irreducible polynomials, say, $f^{(1)}(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$. We use the following notation:

$$f(x) = x^9 + a_1x^8 + a_2x^7 + a_3x^6 + a_4x^5 + a_5x^4 + a_6x^3 + a_7x^2 + a_8x + a_9 \qquad (a_i \in \mathbf{Z},\ 1 \leq i \leq 9),$$

$$f^{(1)}(x) = x^3 + a_1^{(1)}x^2 + a_2^{(1)}x + a_3^{(1)} \qquad (a_i^{(1)} \in \mathbf{O}_F,\ 1 \leq i \leq 3),$$

$$f^{(2)}(x) = x^3 + a_1^{(2)}x^2 + a_2^{(2)}x + a_3^{(2)} \qquad (a_i^{(2)} = \sigma_2(a_i^{(1)}),\ 1 \leq i \leq 3),$$

$$f^{(3)}(x) = x^3 + a_1^{(3)}x^2 + a_2^{(3)}x + a_3^{(3)} \qquad (a_i^{(3)} = \sigma_3(a_i^{(1)}),\ 1 \leq i \leq 3).$$

Clearly, we have $\mathrm{Tr}_{\sigma_j, K/F}(\theta) = -a_1^{(j)}$ $(j = 1, 2, 3)$.

We order the roots $\theta^{(i)}$ $(1 \leq i \leq 9)$ of $f(x)$ such that $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ are the roots of $f^{(1)}(x)$, $\theta^{(4)}$, $\theta^{(5)}$, $\theta^{(6)}$ are the roots of $f^{(2)}(x)$ and $\theta^{(7)}, \theta^{(8)}$, $\theta^{(9)}$ are the roots of $f^{(3)}(x)$. For each natural number j we consider the power sums

$$S_j = \sum_{i=1}^{9} \theta^{(i)^j},$$
$$s_j^{(1)} = \theta^{(1)^j} + \theta^{(2)^j} + \theta^{(3)^j},$$
$$s_j^{(2)} = \theta^{(4)^j} + \theta^{(5)^j} + \theta^{(6)^j},$$
$$s_j^{(3)} = \theta^{(7)^j} + \theta^{(8)^j} + \theta^{(9)^j}.$$

Clearly, $s_j^{(1)}$, $s_j^{(2)}$, and $s_j^{(3)}$ are conjugates in F and satisfy

$$S_j = s_j^{(1)} + s_j^{(2)} + s_j^{(3)}, \qquad |S_j| \le |s_j^{(1)}| + |s_j^{(2)}| + |s_j^{(3)}| \le \sum_{i=1}^{9} |\theta^{(i)}|^j.$$

We determine the set of all coefficients $(a_1^{(i)}, a_2^{(i)}, a_3^{(i)})$ of $f^{(i)}(x)$ $(i = 1, 2, 3)$.

We put $a_j^{(i)} = a_{j0} + a_{j1}\delta^{(i)} + a_{j2}\varepsilon^{(i)}$ $(a_{j0}, a_{j1}, a_{j2} \in \mathbf{Z},\ i, j = 1, 2, 3)$. Since $\mathrm{Tr}_{\sigma_1, K/F}(\beta) = 3\beta$, $\beta \in F$, and changing θ to $-\theta$ or $\theta + \beta$, $\beta \in F$, we may assume without loss of generality that

$$a_{10}, a_{11}, a_{12} = 0 \text{ or } 1 \text{ or } 2. \tag{8}$$

By (7),

$$S_2 \le \frac{1}{3} \sum_{i=1}^{3} (\mathrm{Tr}_{\sigma_i, K/F}(\theta))^2 + T_2 = T_3.$$

By Siegel [6], we have $14 \le S_2$. Since $S_2 = s_2^{(1)} + s_2^{(2)} + s_2^{(3)}$, we have

$$0 \le s_2^{(1)}, s_2^{(2)}, s_2^{(3)} \le T_3.$$

So we have

$$-\frac{1}{2}(T_3 - a_1^{(i)^2}) \le a_2^{(i)} \le -\frac{1}{2}(-a_1^{(i)^2}) \qquad (i = 1, 2, 3). \tag{9}$$

By the inequality of the geometric and arithmetic means, we have

$$|a_3^{(i)}| \le \left(\frac{a_1^{(i)^2} - 2a_2^{(i)}}{3} \right)^{3/2} \qquad (i = 1, 2, 3). \tag{10}$$

We put

$$y_{2i} = -\frac{1}{2}(T_3 - a_1^{(i)^2}), \quad x_{2i} = \frac{a_1^{(i)^2}}{2}, \quad x_{3i} = \left(\frac{a_1^{(i)^2} - 2a_2^{(i)}}{3} \right)^{3/2} \qquad (i = 1, 2, 3).$$

We first treat the case that we can take $\{1, \alpha, \alpha^2\}$ as $\mathbf{Z}$-basis of $\mathbf{O}_F$, i.e.,

$$\delta^{(i)} = \alpha^{(i)}, \quad \varepsilon^{(i)} = \alpha^{(i)^2} \qquad (i = 1, 2, 3,\ \alpha^{(1)} < \alpha^{(2)} < \alpha^{(3)}).$$

From the inequalities (8)–(10), we have the following inequalities for the integers a_{10}, a_{11}, a_{12}:

$$0 \le a_{10}, a_{11}, a_{12} \le 2. \tag{11}$$

For a_{20}, a_{21}, a_{22}, we have the following inequalities:

$$\begin{aligned}(12)\qquad &\frac{1}{\alpha^{(2)}-\alpha^{(3)}}\left(\frac{y_{21}-x_{22}}{\alpha^{(1)}-\alpha^{(2)}}-\frac{x_{21}-y_{23}}{\alpha^{(1)}-\alpha^{(3)}}\right)\\ &\qquad\le a_{22}\le\frac{1}{\alpha^{(2)}-\alpha^{(3)}}\left(\frac{x_{21}-y_{22}}{\alpha^{(1)}-\alpha^{(2)}}-\frac{y_{21}-x_{23}}{\alpha^{(1)}-\alpha^{(3)}}\right),\end{aligned}$$

$$(13)\qquad \begin{cases}\dfrac{x_{21}-y_{22}}{\alpha^{(1)}-\alpha^{(3)}}-a_{22}(\alpha^{(1)}+\alpha^{(2)})\\ \qquad\le a_{21}\le\dfrac{y_{21}-x_{22}}{\alpha^{(1)}-\alpha^{(2)}}-a_{22}(\alpha^{(1)}+\alpha^{(2)}),\\ \dfrac{x_{21}-y_{23}}{\alpha^{(1)}-\alpha^{(3)}}-a_{22}(\alpha^{(1)}+\alpha^{(3)})\\ \qquad\le a_{21}\le\dfrac{y_{21}-x_{23}}{\alpha^{(1)}-\alpha^{(3)}}-a_{22}(\alpha^{(1)}+\alpha^{(3)}),\end{cases}$$

$$(14)\qquad \begin{cases}y_{21}-a_{21}\alpha^{(1)}-a_{22}\alpha^{(1)^2}\le a_{20}\le x_{21}-a_{21}\alpha^{(1)}-a_{22}\alpha^{(1)^2},\\ y_{22}-a_{21}\alpha^{(2)}-a_{22}\alpha^{(2)^2}\le a_{20}\le x_{22}-a_{21}\alpha^{(2)}-a_{22}\alpha^{(2)^2},\\ y_{23}-a_{21}\alpha^{(3)}-a_{22}\alpha^{(3)^2}\le a_{20}\le x_{23}-a_{21}\alpha^{(3)}-a_{22}\alpha^{(3)^2}.\end{cases}$$

For a_{30}, a_{31}, a_{32} we have the following inequalities:

$$(15)\qquad |a_{32}|\le\frac{1}{\alpha^{(2)}-\alpha^{(3)}}\left(\frac{x_{31}+x_{32}}{\alpha^{(1)}-\alpha^{(2)}}+\frac{x_{31}+x_{33}}{\alpha^{(1)}-\alpha^{(3)}}\right),$$

$$(16)\qquad \begin{cases}\dfrac{x_{31}+x_{32}}{\alpha^{(1)}-\alpha^{(2)}}-a_{32}(\alpha^{(1)}+\alpha^{(2)})\\ \qquad\le a_{31}\le-\dfrac{x_{31}+x_{32}}{\alpha^{(1)}-\alpha^{(2)}}-a_{32}(\alpha^{(1)}+\alpha^{(2)}),\\ \dfrac{x_{31}+x_{33}}{\alpha^{(1)}-\alpha^{(3)}}-a_{32}(\alpha^{(1)}+\alpha^{(3)})\\ \qquad\le a_{31}\le-\dfrac{x_{31}+x_{33}}{\alpha^{(1)}-\alpha^{(3)}}-a_{32}(\alpha^{(1)}+\alpha^{(3)}),\end{cases}$$

$$(17)\qquad \begin{cases}-x_{31}-a_{31}\alpha^{(1)}-a_{32}\alpha^{(1)^2}\le a_{30}\le x_{31}-a_{31}\alpha^{(1)}-a_{32}\alpha^{(1)^2},\\ -x_{32}-a_{31}\alpha^{(2)}-a_{32}\alpha^{(2)^2}\le a_{30}\le x_{32}-a_{31}\alpha^{(2)}-a_{32}\alpha^{(2)^2},\\ -x_{33}-a_{31}\alpha^{(3)}-a_{32}\alpha^{(3)^2}\le a_{30}\le x_{33}-a_{31}\alpha^{(3)}-a_{32}\alpha^{(3)^2}.\end{cases}$$

In the remaining two cases we can take $\{1, \alpha, (1+\alpha^2)/2\}$, or $\{1, \alpha, (\alpha+\alpha^2)/2\}$ as a $\mathbf{Z}$-basis of $\mathbf{O}_F$. Also in these cases, from the inequalities (8)–(10) we get the inequalities for $a_{10}, a_{11}, a_{12}, \dots, a_{32}$.

Since K is totally real, we have

$$(18)\qquad \begin{vmatrix} s_0^{(i)} & s_1^{(i)} & s_2^{(i)}\\ s_1^{(i)} & s_2^{(i)} & s_3^{(i)}\\ s_2^{(i)} & s_3^{(i)} & s_4^{(i)}\end{vmatrix}>0\qquad (i=1, 2, 3).$$

By the inequality of Newton,

$$(19)\qquad 3a_1^{(i)}a_3^{(i)}\le a_2^{(i)^2}\qquad (i=1, 2, 3).$$

The following theorem is useful to determine the discriminant $d(f)$ of a polynomial $f(x)$ and to check that $f(x)$ has only real zeros.

Theorem 3 (Pohst [2]). *Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in \mathbf{R}[x]$ be an irreducible polynomial. Then $f(x)$ has only real zeros if and only if the symmetric matrix $\mathbf{A} = (a_{ij})$ with entries*

$$a_{ij} = j(n-i)a_ia_j - n\sum_{k=1}^{[n/2]}(2k+i-j)a_{i+k}a_{j-k}$$

$$(i, j = 1, \ldots, n-1;\ i \geq j;\ a_0 = 1;\ a_m = 0 \text{ if } m > n \text{ or } m < 0)$$

is positive definite. The discriminant $d(f)$ of $f(x)$ is given by

$$d(f) = n^{2-n}\det(\mathbf{A}).$$

From Theorem 3 we have the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{pmatrix} 2a_1^{(i)^2} - 6a_2^{(i)} & a_1^{(i)}a_2^{(i)} - 9a_3^{(i)} \\ a_1^{(i)}a_2^{(i)} - 9a_3^{(i)} & 2a_2^{(i)^2} - 6a_1^{(i)}a_3^{(i)} \end{pmatrix} \qquad (i = 1, 2, 3).$$

So we have the following inequalities:

$$(20) \qquad 2a_1^{(i)^2} - 6a_2^{(i)} > 0, \quad \det(\mathbf{A}) > 0 \qquad (i = 1, 2, 3).$$

By the inequalities (11)–(20), we determine the set of $(a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{32})$ and the set of all coefficients $(a_1^{(i)}, a_2^{(i)}, a_3^{(i)})$ of $f^{(i)}(x)$ $(i = 1, 2, 3)$. Since $f(x) = f^{(1)}(x)f^{(2)}(x)f^{(3)}(x)$, we determine the set of all coefficients $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ of the polynomial $f(x)$.

We examine the irreducibility of the polynomial $f(x)$. We find the zeros $\theta^{(i)}$ $(1 \leq i \leq 9)$ of $f(x)$ by Newton's method. If $f(x)$ is reducible, then $f(x)$ is divisible by a first-degree polynomial or a cubic polynomial. Hence, if $f(x)$ does not satisfy the following two conditions (21) and (22), then $f(x)$ is irreducible.

(21) There is a $\theta^{(i)}$ such that $\theta^{(i)} \in \mathbf{Z}$.

(22) There are $\theta^{(i)}$ and $\theta^{(j)}$ and $\theta^{(k)}$ such that $\theta^{(i)} + \theta^{(j)} + \theta^{(k)}$, $\theta^{(i)}\theta^{(j)} + \theta^{(j)}\theta^{(k)} + \theta^{(k)}\theta^{(i)}$ and $\theta^{(i)}\theta^{(j)}\theta^{(k)} \in \mathbf{Z}$.

Then we determine the set of all coefficients $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ such that $f(x)$ is irreducible. We denote by $N1$ the number of such polynomials $f(x)$ for each cubic field F. Then we have Table 2.

TABLE 2

$d(F)$	$N1$	$d(F)$	$N1$	$d(F)$	$N1$	$d(F)$	$N1$	$d(F)$	$N1$
49	4237	404	20	761	6	1229	4	1620	2
81	714	469	18	785	8	1304	6	1772	2
148	194	473	8	788	4	1345	4	1929	2
169	144	564	6	892	4	1373	4	1937	2
229	78	568	4	940	2	1384	2	2101	2
257	36	621	2	993	2	1489	2	2233	2
316	28	697	2	1016	2	1524	2	2300	4
321	30	733	8	1076	4	1556	2	2349	2
361	30	756	4	1129	2	1593	6		

For all the remaining cases we have $N1 = 0$.

4. Minimum discriminant of the totally real fields of degree 9

Finally, we determine the minimum discriminant $d(K)$ of the fields K obtained in §3. Using Theorem 3, we obtain the discriminant $d(f)$ of each polynomial $f(x)$ found in §3. In general, $d(f)$ is not equal to $d(K)$; it is known that $d(f) = m^2 d(K)$ $(m > 0 \in \mathbf{Z})$. So we shall determine m. We decompose $d(f) = m_1^2 m_2 d(F)^3$, where m_2 is squarefree. If $f(x)$ satisfies the inequality

$$m_2 d(F)^3 > d_{\max}, \tag{23}$$

then we can exclude such $f(x)$. We denote by $N2$ the number of $f(x)$ such that $f(x)$ does not satisfy (23). Then we have Table 3.

Table 3

$d(F)$	$N2$	$d(F)$	$N2$	$d(F)$	$N2$	$d(F)$	$N2$
49	648	169	18	321	4	1229	2
81	60	229	16	361	6	1937	2
148	24	257	8	473	2		

For all the remaining cases we have $N2 = 0$.

In these cases we check whether $p|(d(f)/d(K))$ or $p \nmid (d(f)/d(K))$ for all primes p such that $p|m_1$. In order to do this, the following theorem is useful.

Theorem 4 (Zassenhaus [3]). *Let F be an algebraic number field, $\mathbf{O}_F$ be the ring of integers in F. Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ $(a_i \in \mathbf{O}_F,\ 1 \le i \le n)$ be an irreducible polynomial over F with the discriminant $d(f)$ and α be a zero of $f(x)$ and $K = F(\alpha)$ be an extension of F with degree n. Let the decomposition of the ideal $(d(f))$ in $\mathbf{O}_F$ be as follows:*

$$d(f) = \prod_{i=1}^{s_0} \mathbf{p}_i^{e_i} \prod_{i=s_0+1}^{s} \mathbf{p}_i \qquad (e_i > 1,\ 1 \le i \le s_0;\ \mathbf{p}_i \textit{ distinct prime ideals}).$$

Let

$$f(x) \equiv \prod f_{ij}(x)^{e_{ij}} \pmod{\mathbf{p}_i} \qquad (1 \le i \le s_0)$$

be an irreducible factorization of $f(x)$ $\bmod \mathbf{p}_i$. We define $d_i(x)$, $g_i(x)$, and $h_i(x)$ by

$$d_i(x) = \prod f_{ij}(x), \qquad g_i(x) = \prod f_{ij}(x)^{e_{ij}-1},$$
$$\pi_i h_i(x) = (d_i(x)g_i(x) - f(x))\eta_i \quad \textit{in } \mathbf{O}_F,$$

where $\pi_i \in \mathbf{p}_i$, $\pi \notin \mathbf{p}_i^2$, and $\eta_i \notin \mathbf{p}_i$. Then $\mathbf{p}_i^2 \nmid (d(f)/d(K))$ $(1 \le i \le s_0)$ if and only if G.C.D. $(d_i(x), g_i(x), h_i(x)) = 1 \pmod{\mathbf{p}_i}$, *and for $\mathbf{p}_i$ $(s_0+1 \le i \le s)$ we have $\mathbf{p}_i \nmid (d(f)/d(K))$.*

Using this theorem, we find all primes p such that $p|m_1$ and $p\nmid(d(f)/d(K))$. Then, if $p_1^{2v_{p_1}(m_1)} \cdots p_j^{2v_{p_j}(m_j)} m_2 d(F)^3 > d_{\max}$, where $p_i|m_1$, $p_i \nmid (d(f)/d(K))$

$(i = 1, \dots, j)$ and $p_i^{2v_{p_i}(m_i)}$ is the largest power of p dividing m_1, then we can exclude $f(x)$.

We apply Theorem 4 in the case $f(x) = x^9 + 2x^8 - 8x^7 - 14x^6 + 22x^5 + 30x^4 - 24x^3 - 20x^2 + 10x + 2$, $d(f) = 522901049600 = 148^3 \times 2^2 \times 5^2 \times 1613$, $d(F) = 148$, and $m_1 = 2 \times 5$. If $p = 2$, then

$$f(x) \equiv x^9 \pmod 2, \quad d(x) = x, \quad g(x) = x^8,$$
$$h(x) = -x^8 + 4x^7 + 7x^6 - 11x^5 - 15x^4 + 12x^3 + 10x^2 - 5x - 1.$$

Since G.C.D.$(d(x), g(x), h(x)) = 1 \pmod 2$, we have $2 \nmid (d(f)/d(K))$. If $p = 5$, then

$$f(x) \equiv (x^2 + x + 2)^2(x^2 + 2x - 1)(x^3 - 2x^2 + 2x + 2) \pmod 5,$$
$$d(x) = (x^2 + x + 2)(x^2 + 2x - 1)(x^3 - 2x^2 + 2x + 2),\ g(x) = (x^2 + x + 2),$$
$$h(x) = 2x^7 + 4x^6 - 3x^5 + 10x^3 + 10x^2 - 2x - 2.$$

Since G.C.D.$(d(x), g(x), h(x)) = 1$, we have $5 \nmid (d(f)/d(K))$. Consequently, $m = 1$, $d(K) = d(f) = 529010496600 > d_{\max}$ and we exclude $f(x)$. By this method we can exclude 602 cases.

Using the Dirichlet Discriminant Theorem, we have the following proposition.

Proposition 1. *Let K be an algebraic number field of degree 9 and F be its subfield of degree 3. Let p be a prime number such that p ramifies in K and (p) is a prime ideal in F. We denote by $\nu_p(m)$ the p-index of m (i.e., $p^{\nu_p(m)} | m$ and $p^{\nu_p(m)+1} \nmid m$). Then, if $p \neq 2, 3$ then $\nu_p(d(K)) = 3$ or 6, if $p = 2$ then $\nu_p(d(K)) \geq 6$, if $p = 3$ then $\nu_p(d(K)) = 3$ or ≥ 9.*

Since the proof is easy, we shall omit it.

We apply Proposition 1 in the case $f(x) = x^9 - x^8 - 11x^7 + 12x^6 + 36x^5 - 41x^4 - 31x^3 + 33x^2 + 2x - 1$, $d(f) = 119414482370560 = 2^{15} \times 5 \times 151 \times 169^3$, $d(F) = 169$, $m_1 = 128 = 2^7$, and $d(f)/m_1^2 = 7288481590$.

The ideal (2) is a prime ideal in F. Since $2 | 7288481590$, we have $2 | d(K)$ and 2 ramifies in K. We have $d(K) \geq 2^5 \times 7288481590 \geq d_{\max}$. So $f(x)$ is excluded. By this method we can exclude 6 cases.

Proposition 2 (Takeuchi [7]). *Let K be an algebraic number and F be a subfield. Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbf{O}_F[x]$ be the defining polynomial for K over F. Let $\mathbf{p}$ be a prime ideal of $\mathbf{O}_F$. If $\mathbf{p} | a_i$ $(m < i \leq n)$ and $\mathbf{p}^2 \nmid a_n$, then the ramification index of $\mathbf{p}$ for K/F is at least $n - m$.*

We apply Proposition 2 in the case $f(x) = x^9 + 5x^8 - 2x^7 - 37x^6 - 20x^5 + 78x^4 + 52x^3 - 40x^2 - 16x + 8$, $d(f) = 20181511264415744 = 2^{18} \times 65449 \times 49^3$, $d(F) = 49$, $m_1 = 512 = 2^9$, and $d(f)/m_1^2 = 7700009401$.

We know that $F = \mathbf{Q}(\alpha)$, where α is a zero of $x^3 + x^2 - 2x - 1$. Then $x^3 + (1 + \alpha^2)x^2 + (-6 + 2\alpha^2)x + 2 - 2\alpha - 2\alpha^2$ is an irreducible polynomial of K over F. Since the ideal (2) is a prime ideal in F, we see that $2 | (-6 + 2\alpha^2)$, $2 | (2 - 2\alpha - 2\alpha^2)$ and $2^2 \nmid (2 - 2\alpha - 2\alpha^2)$. So 2 ramifies in K. By Proposition 1, we have $d(K) \geq 2^6 \times 7700009401 > d_{\max}$. So $f(x)$ is excluded. By this method we can exclude 106 cases.

TABLE 4

$d(F)$	N_k
16240385609	60
16440305941	10
16983563041	6

We denote by N_K the number of $f(x)$ such that the discriminant $d(K)$ of the field K given by $f(x)$ satisfies (6). Then we have Table 4.

To determine if two fields K with the same $d(K)$, but given by different polynomials $f(x)$, are isomorphic, we use the method of Takeuchi [7]. Consequently, we see that the fields K are uniquely determined up to **Q**-isomorphism. So the proof of Theorem 1 is completed.

BIBLIOGRAPHY

1. M. Pohst, *Berechnung kleiner Diskriminanten total reeller algebraischer Zahlkörper*, J. Reine Angew. Math. **278/279** (1975), 278–300.
2. ——, *The minimum discriminant of seventh degree totally real algebraic number fields*, Number Theory and Algebra (H. Zassenhaus, ed.), Academic Press, New York 1977, pp. 235–240.
3. H. Zassenhaus, *On Hensel factorization*. II, Sympos. Math. **15** (1975), 499–513.
4. M. Pohst, J. Martinet, and F. Diaz y Diaz, *The minimum discriminant of totally real octic fields*, J. Number Theory, preprint.
5. J. Martinet, *Méthodes géométriques dans la recherche des petits discriminants*, Séminaire de Théorie des Nombres (1983–84), Paris, pp.147–179.
6. C. L. Siegel, *The trace of totally positive and real algebraic integers*, Ann. of Math. **46** (1945), 302–312.
7. K. Takeuchi, *Totally real algebraic number fields of degree* 5 *and* 6 *with small discriminant*, Saitama Math. J. **2** (1984), 21–32.
8. A. M. Odlyzko, *Unconditional bounds for discriminants*, preprint.
9. Walter D. Neumann, *UBASIC: a public-domain BASIC for mathematics*, Notices Amer. Math. Soc. **36** (1989), 557–559.
10. L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Math., vol. 83, Springer-Verlag, Berlin and New York, 1982.
11. F. Diaz y Diaz, *Petits discriminants des corps de nombres totalement imaginaires de degré* 8, J. Number Theory **25** (1987), 34–52.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SAITAMA UNIVERSITY, SHIMO-OKUBO URAWA 338, JAPAN