A \(B_2\)-SEQUENCE WITH LARGER RECIPROCAL SUM

ZHENXIANG ZHANG

Abstract. A sequence of positive integers is called a \(B_2\)-sequence if the pairwise differences are all distinct. The Mian-Chowla sequence is the \(B_2\)-sequence obtained by the greedy algorithm. Its reciprocal sum \(S^*\) has been conjectured to be the maximum over all \(B_2\)-sequences. In this paper we give a \(B_2\)-sequence which disproves this conjecture. Our sequence is obtained as follows: the first 14 terms are obtained by the greedy algorithm, the 15th term is 229, from the 16th term on, the greedy algorithm continues. The reciprocal sum of the first 300 terms of our sequence is larger than \(S^*\).

1. Introduction

A sequence of positive integers \(a_1 < a_2 < \cdots\) is called a \(B_2\)-sequence or a Sidon sequence if the pairwise differences are all distinct, or in other words, if all the sums \(a_i + a_j\) (\(i = j\) is permitted) are different. The Mian-Chowla sequence is the \(B_2\)-sequence obtained by the greedy algorithm; i.e., each term is the least integer greater than earlier terms which does not violate the distinctness of differences (or sums) condition.

If \(M\) is the maximum of reciprocal sums over all \(B_2\)-sequences and \(S^*\) is the sum of the reciprocals of the Mian-Chowla sequence, then \(M \geq S^* > 2.156\). But Levine observes that

\[
M \leq \sum_{n \geq 0} \frac{1}{1 + \frac{n(n+1)}{2}} < 2.374
\]

and would like to see a proof or disproof of \(M = S^*\) [2, pp. 127–128].

The main task of this paper is to construct a \(B_2\)-sequence which disproves this conjecture. Most work is done on a personal computer IBM PC/XT. We state our results in the following two theorems:

Theorem 1. We have \(S^* < 2.1596\).

Theorem 2. We have \(M > 2.1597\).

In §2 we prove both theorems. The first 445 terms of the Mian-Chowla sequence \(\{a_i : 1 \leq i \leq 445\}\) and the first 300 terms of our \(B_2\)-sequence \(\{b_i : \ldots\} Which theorem states the upper limit for S*?
2. PROOFS OF THE THEOREMS

Theorem 1. We have \( S^* < 2.1596 \).

Proof. Let

\[
S_1 = \sum_{i=1}^{445} \frac{1}{a_i}, \quad S_2 = \sum_{i=446}^{11153} \frac{1}{a_i}, \quad S_3 = \sum_{i=11154}^{\infty} \frac{1}{a_i}.
\]

Then by computer calculations we have

\[
s_1 = 2.15828 \ldots < 2.15829
\]

and

\[
s_2 \leq \frac{1}{a_{445} + d_1} + \frac{1}{a_{445} + d_1 + d_2} + \ldots + \frac{1}{a_{445} + d_1 + d_2 + \ldots + d_{10708}} = 0.00110 \ldots < 0.00111,
\]

where \( d_1 = 33, d_2 = 88, \ldots, d_{10708} = 16960 \) are the first 10708 positive integers not belonging to the set \( \{a_i - a_j : 1 \leq i < j < 445\} \). Put

\[
h = a_{445} + d_1 + d_2 + \ldots + d_{10708} = 100005214 \quad \text{and} \quad d = d_{10708} = 16960.
\]

Then

\[
s_3 \leq \frac{1}{h + d + 1} + \frac{1}{h + d + 1 + d + 2} + \ldots
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{h + kd + k(k+1)/2}
\]

\[
= \sum_{k=1}^{\infty} \frac{2}{k^2 + k(2d + 1) + 2h} < \sum_{k=1}^{\infty} \frac{2}{(k + 14000)(k + 14001)}
\]

\[
= 2 \sum_{k=1}^{\infty} \left( \frac{1}{k + 14000} - \frac{1}{k + 14001} \right)
\]

\[
= \frac{2}{14001} < 0.00015.
\]

Thus \( S^* = s_1 + s_2 + s_3 < 2.15829 + 0.00111 + 0.00015 = 2.15955 < 2.1596 \).

This completes the proof. \( \Box \)

Theorem 2. We have \( M > 2.1591 \).

Proof. We construct our sequence \( \{b_i\} \) as follows: the first 14 terms are obtained by the greedy algorithm. Thus, they are the same as the first 14 terms of the Mian-Chowla sequence. Let \( b_{15} = b_{14} + 47 = 229 \). Note that \( a_{15} = a_{14} + 22 = 202 \). From the 16th term on, the greedy algorithm continues. Since

\[
\{b_{i_5} - b_i : 1 \leq i \leq 14\} \cap \{b_j - b_i : 1 \leq i < j \leq 14\} = \emptyset,
\]

our sequence is a \( B_2 \)-sequence. The reciprocal sum of the first 300 terms is greater than 2.1597. This completes the proof. \( \Box \)
3. BEGINNING TERMS OF THE TWO SEQUENCES

The first 445 terms of the Mian-Chowla sequence are as follows:

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<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>13</th>
<th>21</th>
<th>31</th>
<th>45</th>
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<td>70</td>
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</table>

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The beginning terms of the two sequences:
Remark. We see that 22 does occur as a difference, in fact $22 = a_{15} - a_{14}$. This answers a question of Erdös and Graham [1]. But we do not know if 33 occurs as a difference.

The first 300 terms of our new sequence are:

A table showing the first 300 terms of the sequence with values ranging from 97 to 47823.

4. Search procedure

We first use a search procedure to find a finite $B_2$-sequence $1 = b_1 < b_2 < \cdots < b_{120}$ with reciprocal sum as large as possible. The procedure contains six main steps:

1. For a given integer $u$ ($2 \leq u \leq 20$), use the greedy algorithm to get the first $u$ terms $1 = b_1 < b_2 < \cdots < b_u$.

2. Let $d_1 < d_2 < \cdots$ be the consecutive integers which can be a candidate for $b_{u+1} - b_u$ so that the set $\{b_1, b_2, \ldots, b_u, b_{u+1}\}$ does not violate the distinctness of differences condition.

3. For a given integer $v$ ($2 \leq v \leq 5$), let $b_{u+1} = b_u + d_v$.

4. Continue the greedy algorithm to get the last $119 - u$ terms: $b_{u+2} < b_{u+3} < \cdots < b_{120}$.
(5) Calculate $F_{u,v} = \sum_{i=1}^{120} 1/b_i$, then go to (3) for another choice of $v$ until $v > 5$.

(6) Go to (1) for another choice of $u$ until $u > 20$.

We find that $F_{14,4} = 2.15848\ldots \geq F_{u,v}$ for $2 \leq u \leq 20$, $2 \leq v \leq 5$.

Note that the reciprocal sum of the first 120 terms of the Mian-Chowla sequence is 2.15686\ldots. Thus, it is reasonable to conjecture that if we extend the sequence with $u = 14$, $v = 4$ ($b_{14} = 182$, $d_4 = 47$, and $b_{15} = 229$) to an infinite sequence, we will get a $B_2$-sequence with larger reciprocal sum than that of the Mian-Chowla sequence. To prove this we must have enough terms of both sequences as indicated in §§2 and 3. The Pascal programs for proving Theorems 1 and 2 ran about 5.5 hours in total on an IBM PC/XT.

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BIBLIOGRAPHY


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