MATHEMATICS OF COMPUTATION
VOLUME 60, NUMBER 202
APRIL 1993, PAGES 827-834

ON A PROBLEM OF ERDŐS CONCERNING PRIMITIVE SEQUENCES

ZHENXIANG ZHANG

Dedicated to Paul Erdős on the occasion of his 80th birthday

ABSTRACT. A sequence $A = \{a_i\}$ of positive integers $a_1 < a_2 < \cdots$ is said to be primitive if no term of A divides any other. Let $\Omega(a)$ denote the number of prime factors of a counted with multiplicity. Let $p(a)$ denote the least prime factor of a and $A(p)$ denote the set of $a \in A$ with $p(a) = p$. The set $A(p)$ is called *homogeneous* if there is some integer s_p such that either $A(p) = \varnothing$ or $\Omega(a) = s_p$ for all $a \in A(p)$. Clearly, if $A(p)$ is homogeneous, then $A(p)$ is primitive. The main result of this paper is that if A is a positive integer sequence such that $1 \notin A$ and each $A(p)$ is homogeneous, then

$$\sum_{a \le n,\, a \in A} \frac{1}{a \log a} \le \sum_{p \le n,\, p \text{ prime}} \frac{1}{p \log p} \quad \text{for } n > 1.$$

This would then partially settle a question of Erdős who asked if this inequality holds for any primitive sequence A.

1. INTRODUCTION

A sequence $A = \{a_i\}$ of positive integers $a_1 < a_2 < \cdots$ is said to be primitive if no term of A divides any other (cf. [3] or [5]). We denote by p_m the mth prime, by p a variable prime and by $p(a)$ the least prime factor of a. We define the degree of an integer a, denoted by $\Omega(a)$, to be the number of prime factors of a counted with multiplicity. The degree of an integer sequence A, denoted by $d^\circ(A)$, is defined as the maximum degree of its terms. We take $d^\circ(A) = 0$ if $A = \{1\}$ or $\varnothing$.

For a primitive sequence A with $d^\circ(A) > 0$ we define

$$f(A) = \sum_{a \in A} 1/(a \log a).$$

We take $f(A) = 0$ if $d^\circ(A) = 0$. Erdős [1] proved that there exists an absolute constant C such that $f(A) \le C$ for any primitive sequence A. Recently he [2] has asked if the inequality

$$\sum_{a \le n,\, a \in A} \frac{1}{a \log a} \le \sum_{p \le n,\, p \text{ prime}} \frac{1}{p \log p} \quad \text{for } n > 1 \tag{1}$$

Received by the editor January 19, 1990 and, in revised form, November 5, 1990 and September 16, 1991.

1991 *Mathematics Subject Classification*. Primary 11N64, 11B83.

Key words and phrases. Primitive sequences.

is always true for any primitive sequence A. Zhang [8] proved that if A is primitive with $d^\circ(A) \leq 4$, then the inequality is true. Erdős and Zhang [4] proved that $f(A) < 1.84$ for any primitive sequence A, and gave a necessary and sufficient condition for the inequality (1), namely $\sum_{b\in B} 1/(b\log b) \leq \sum 1/(p\log p)$ for any primitive sequence B. Clearly, if (1) is true then $C = \sum 1/(p\log p) < 1.64$.

In this paper we partially settle this question of Erdős in another direction. To give our result, we need some more notation and concepts. Let $A(p)$ denote the set of $a \in A$ with $p(a) = p$. A sequence B is called *homogeneous* if either $B = \varnothing$ or $\Omega(b) = d^\circ(B)$ for all $b \in B$. Clearly, if B is homogeneous, then B is primitive. Now we state our main result as the following

Theorem. *If A is a positive integer sequence such that $1 \notin A$ and each $A(p)$ is homogeneous, then the inequality* (1) *is true.*

The basic idea for proving the theorem is the same as that used in [8]; i.e., we consider the least prime factors of the terms of A. The key point of this paper is to prove that, for a given prime p, if $B = B(p)$ is homogeneous and nonempty, then

$$\sum_{b\in B} \frac{1}{b\log b} \leq \frac{1}{p\log p}. \tag{2}$$

It is clear that (2) immediately implies the theorem. In fact we have the stronger result where "$a \leq n$" is replaced in (1) with "$(a, n!) > 1$".

2. Proof of the theorem

We first define two functions:

$$w(s, m) = \sum_{\Omega(a)=s-1,\, p(a)\geq p_{m+1}} \frac{1}{a\log(p_{m+1}a)}$$

for integers $s \geq 2$, $m \geq 0$, and

$$h(m) = \sum_{i>m} \frac{1}{p_i \log(i-1)}$$

for integers $m \geq 2$.

We need nine lemmas.

Lemma 1. *We have $p_n > n\log n$ for $n \geq 1$ and $p_n < n(\log n + \log\log n)$ for $n \geq 6$.*

These results may be found in [6] and [7].

Lemma 2. *We have $h(m) < 1/\log m$ for $m \geq 2$.*

Proof. Note that for each $i \geq 3$, we have

$$\frac{1}{i\log i\log(i-1)} < \frac{\log(i/(i-1))}{\log i\log(i-1)} = \frac{1}{\log(i-1)} - \frac{1}{\log i}.$$

Thus, from Lemma 1,

$$h(m) < \sum_{i>m} \frac{1}{i\log i\log(i-1)} < \sum_{i>m}\left(\frac{1}{\log(i-1)} - \frac{1}{\log i}\right) = \frac{1}{\log m}. \quad \square$$

In the following we define $i(a) = i$ if the largest prime factor of a is p_i.

Lemma 3. *For* $m \geq 2$, $s \geq 1$, *we have*

$$\sum_{p(a)>p_m,\,\Omega(a)=s} \frac{1}{a\log(i(a)-1)} \leq h(m) < \frac{1}{\log m}.$$

Proof. We proceed by induction on s. If $s = 1$, then this is just Lemma 2. Assume the lemma for s. For the $s+1$ case, we have, by Lemma 2,

$$\begin{aligned}
&\sum_{p(a)>p_m,\ \Omega(a)=s+1} \frac{1}{a\log(i(a)-1)} \\
&\quad = \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{1}{b} \sum_{j\geq i(b)} \frac{1}{p_j \log(j-1)} \\
&\quad < \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{1}{b\log(i(b)-1)} \leq h(m) < \frac{1}{\log m}. \quad \square
\end{aligned}$$

Lemma 4. *For* $i \geq 2$, $B \geq 2$, *we have*

$$\begin{aligned}
\sum_{j>i} \frac{1}{p_j \log(Bp_j)} &< \frac{\log(1+\log B/\log i)}{\log B} \\
&\leq \min\left\{\frac{1}{\log i}, \frac{1}{e\log i} + \frac{1}{e\log B}\right\},
\end{aligned}$$

where $e = 2.718\cdots$ *is the base of the natural logarithms.*

Proof. We have, by Lemma 1,

$$\begin{aligned}
\sum_{j>i} \frac{1}{p_j \log(Bp_j)} &< \int_i^\infty \frac{dx}{x\log x \log(Bx)} \\
&= \frac{\log(1+\log B/\log i)}{\log B} \leq \min\left\{\frac{1}{\log i}, \frac{1}{e\log i} + \frac{1}{e\log B}\right\},
\end{aligned}$$

observing that the last inequality follows from

$$\log(1+x) < x \quad \text{and} \quad \log x = 1 + \log(1+(x-e)/e) \leq x/e$$

for all $x > 0$. $\square$

Lemma 5. *For* $m \geq 2$, $B \geq 2$, $s \geq 2$, *we have*

$$\begin{aligned}
&\sum_{p(u)>p_m,\ \Omega(u)=s} \frac{1}{u\log(Bu)} \\
&\quad < (e^{-1} + \cdots + e^{1-s})h(m) + e^{1-s} \sum_{j>m} \frac{1}{p_j \log(Bp_j)}.
\end{aligned}$$

Proof. We proceed by induction on s. If $s = 2$, then we have, by Lemma 4,

$$\begin{aligned}
\sum_{p(u)>p_m,\ \Omega(u)=2} \frac{1}{u\log(Bu)} &= \sum_{j>m} \frac{1}{p_j} \sum_{k\geq j} \frac{1}{p_k \log(Bp_jp_k)} \\
&< e^{-1}h(m) + e^{-1} \sum_{j>m} \frac{1}{p_j \log(Bp_j)}.
\end{aligned}$$

For the $s+1$ case, we have, by Lemmas 3 and 4 and the s case,

$$\sum_{p(u)>p_m,\ \Omega(u)=s+1} \frac{1}{u\log(Bu)} = \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{1}{b} \sum_{j\geq i(b)} \frac{1}{p_j \log(Bbp_j)}$$
$$< \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{e^{-1}}{b}\left(\frac{1}{\log(i(b)-1)} + \frac{1}{\log(Bb)}\right)$$
$$< (e^{-1}+\cdots+e^{-s})h(m) + e^{-s}\sum_{j>m}\frac{1}{p_j\log(Bp_j)}. \quad \square$$

Lemma 6. *For* $m \geq 5$, $s \geq 2$, *we have* $w(s, m) < 1/\log p_{m+1}$.

Proof. We have, by Lemmas 2, 4, and 5,

$$w(s, m) < W(s, m),$$

where

$$W(s, m) = \frac{e^{-1}+\cdots+e^{1-s}}{\log m} + \frac{e^{1-s}}{\log p_{m+1}}.$$

By Lemma 1 we have

$$\frac{\log p_{m+1}}{\log m} < \frac{\log(m+1)+\log(\log(m+1)+\log\log(m+1))}{\log m}$$
$$\leq \frac{\log 6+\log(\log 6+\log\log 6)}{\log 5} = 1.65\cdots < e-1.$$

Thus,

$$W(s, m) - W(s+1, m) = e^{-s}\left(\frac{e-1}{\log p_{m+1}} - \frac{1}{\log m}\right) > 0$$

for $m \geq 5$, $s \geq 2$. Therefore,

$$w(s, m) < W(s, m) \leq W(2, m) = \frac{1}{e\log m} + \frac{1}{e\log p_{m+1}}$$
$$< \frac{e-1}{e\log p_{m+1}} + \frac{1}{e\log p_{m+1}} = \frac{1}{\log p_{m+1}}. \quad \square$$

Lemma 7. *For* $0 \leq m \leq 4$, *we have* $w(2, m) < 1/\log p_{m+1}$.

Proof. We have, by Lemma 4,

$$w(2, m) < w(m) \quad \text{for } 0 \leq m \leq 4,$$

where

$$w(m) = \frac{1}{p_{m+1}\log(p_{m+1}^2)} + \frac{1}{p_{m+2}\log(p_{m+1}p_{m+2})}$$
$$+ \frac{1}{\log p_{m+1}}\log\left(1+\frac{\log p_{m+1}}{\log(m+2)}\right) \quad \text{for } 1 \leq m \leq 4$$

and

$$w(0) = \frac{1}{2\log 4} + \frac{1}{3\log 6} + \frac{1}{5\log 10} + \frac{1}{\log 2}\log\left(1+\frac{\log 2}{\log 3}\right).$$

By calculation we have Table 1.

TABLE 1

m	$w(m)$	p_{m+1}	$1/\log p_{m+1}$
4	0.388...	11	0.417...
3	0.464...	7	0.513...
2	0.581...	5	0.621...
1	0.856...	3	0.910...
0	1.339...	2	1.442...

Thus, $w(2, m) < w(m) < 1/\log p_{m+1}$ for $0 \le m \le 4$. □

Lemma 8.1. *For $s \ge 3$, $2 \le m \le 4$, we have $w(s, m) < 1/\log p_{m+1}$.*

Proof. For a fixed m, put

$$\gamma_s = (e^{-1} + \cdots + e^{2-s})h(m) + e^{2-s}w(m),$$

where $w(m)$ is the upper bound of $w(2, m)$, defined in the proof of Lemma 7. Then by Lemma 5 we have for $s \ge 3$ that

$$w(s, m) < (e^{-1} + \cdots + e^{2-s})h(m) + e^{2-s}w(2, m) < \gamma_s.$$

If $h(m)/w(m) < e - 1$ and $m \le 4$, then we have, from Table 1,

$$\gamma_s < ((e^{-1} + \cdots + e^{2-s})(e-1) + e^{2-s})w(m) = w(m) < 1/\log p_{m+1}.$$

For $m = 4$, we have, by Lemma 2,

$$h(4) = \sum_{i=5}^{10} \frac{1}{p_i \log(i-1)} + h(10) < 0.6442,$$

using $h(10) < 1/\log 10$. Thus, $h(4)/w(4) < 1.7 < e - 1$, so that the case $m = 4$ is done.

For $m = 3$ we have

$$h(3) = 1/(7\log 3) + h(4) < 0.7743, \quad \text{and} \quad h(3)/w(3) < 1.7 < e - 1.$$

Thus the $m = 3$ case is done.

For $m = 2$, since

$$h(2) = 1/(5\log 2) + h(3) < 1.063,$$

we use the upper bound $H = 1.063$ for $h(2)$ and we see that

$$H/w(2) > e - 1.$$

However, we then have

$$\gamma_s < (e^{-1} + \cdots + e^{2-s})H + e^{2-s}\frac{H}{e-1} = \frac{H}{e-1} < 0.62 < 1/\log 5,$$

so that the $m = 2$ case is done. □

Lemma 8.2. *We have $w(s, 1) < 1/\log p_2$ for $s \ge 3$.*

Proof. We have $w(s, 1) = u(s) + v(s)$, where

$$u(s) = \frac{1}{3} \sum_{\substack{\Omega(b)=s-2 \\ p(b)\ge p_2}} \frac{1}{b\log(9b)} \quad \text{and} \quad v(s) = \sum_{\substack{\Omega(b)=s-1 \\ p(b)\ge p_3}} \frac{1}{b\log(3b)}.$$

Taking

$$h(2) < \sum_{i=3}^{25} \frac{1}{p_i \log(i-1)} + \frac{1}{\log 25} < 1.0396$$

and

$$\sum_{i>2} \frac{1}{p_i \log(3p_i)} < \sum_{i=3}^{25} \frac{1}{p_i \log(3p_i)} + \frac{1}{\log 25} < 0.5779 < \frac{1.0396}{e-1},$$

we have, by Lemma 5,

$$v(s) < 1.0396(e^{-1} + \cdots + e^{2-s}) + 0.5779e^{2-s} < \frac{1.0396}{e-1} < 0.6051 < \frac{2/3}{\log 3}.$$

Since $w(2, 1) < 1/\log 3$ by Lemma 7 and $u(s) < w(s-1, 1)/3$, we have, for $s \geq 3$,

$$w(s, 1) < w(s-1, 1)/3 + v(s) < (1/3)/\log 3 + (2/3)/\log 3 = 1/\log 3. \quad \square$$

Lemma 8.3. *We have* $w(s, 0) < 1/\log 2$ *for* $s \geq 3$.

Proof. Put

$$u_i(s) = \frac{1}{p_i} \sum_{\Omega(b)=s-2,\, p(b)\geq p_i} \frac{1}{b \log(2p_i b)} \quad \text{for } 1 \leq i \leq 9$$

and

$$v_i(s) = \sum_{\Omega(b)=s-1,\, p(b)\geq p_i} \frac{1}{b \log(2b)} \quad \text{for } 1 \leq i \leq 10.$$

Then for $1 \leq i \leq 9$, we have

$$v_i(s) = u_i(s) + v_{i+1}(s) \tag{3}$$

and

$$u_i(s) < \frac{v_i(s-1)}{p_i}. \tag{4}$$

Let $N = 800$. Put

$$h = \sum_{i=10}^{N} \frac{1}{p_i \log(i-1)} + \frac{1}{\log N} < 0.403693$$

and

$$g = \sum_{i=10}^{N} \frac{1}{p_i \log(2p_i)} + \frac{1}{\log N} < 0.306441.$$

Then

$$h(9) < h \quad \text{and} \quad \sum_{i>9} \frac{1}{p_i \log(2p_i)} < g.$$

We have, by Lemma 5, $v_{10}(s) < V_{10}(s)$, where

$$V_{10}(s) = (e^{-1} + \cdots + e^{2-s})h + e^{2-s}g.$$

By calculation we get the upper bounds of $V_{10}(s)$, for $3 \leq s \leq 9$, listed in Table 2, which serve as upper bounds of $v_{10}(s)$ for $3 \leq s \leq 9$.

By Lemma 4 we have

$$u_i(3) < \frac{1}{p_i}\left(\sum_{j=i}^{N} \frac{1}{p_j \log(2p_ip_j)} + \frac{1}{\log N}\right).$$

By calculation we get the upper bounds of $u_i(3)$, for $1 \le i \le 9$, listed in Table 2.

Since we now have upper bounds for $v_{10}(3)$ and $u_i(3)$, we can, by equation (3), get upper bounds of $v_i(3)$ for $i = 9, 8, \dots, 2, 1$. Then, by equation (3), inequality (4) and the upper bounds of $v_{10}(s)$, we can get upper bounds of $v_i(s)$ for $i = 9, 8, \dots, 2, 1$; $s = 4, 5, \dots, 9$.

In this way we get upper bounds (listed in Table 2) of

$$w(s, 0) = v_1(s) < 1/\log 2 \quad \text{for } 3 \le s \le 9.$$

In the above calculations we also get the upper bounds of $v_i(9)$, for $1 \le i \le 10$, listed in Table 2.

Let $k_1 = 1/\log 2$ and

$$k_i = \frac{\prod_{j=1}^{i-1}(1 - 1/p_j)}{\log 2} \quad \text{for } 2 \le i \le 10.$$

We list the values of k_i, for $1 \le i \le 10$, in Table 2.

TABLE 2. Upper bounds of $V_{10}(s)$, $u_i(3)$, $w(s, 0) = v_1(s)$ and $v_i(9)$; and values of k_i

s or i	$V_{10}(s)$	$u_i(3)$	$w(s, 0) = v_1(s)$	$v_i(9)$	k_i
1		0.4264		1.4412	1.4426...
2		0.1885		0.7204	0.7213...
3	0.2613	0.0843	1.1049	0.4795	0.4808...
4	0.2447	0.0512	1.2814	0.3835	0.3847...
5	0.2385	0.0287	1.3787	0.3286	0.3297...
6	0.2363	0.0228	1.4224	0.2987	0.2997...
7	0.2355	0.0164	1.4380	0.2757	0.2767...
8	0.2352	0.0141	1.4417	0.2595	0.2604...
9	0.2351	0.0112	1.4412	0.2458	0.2467...
10				0.2351	0.2360...

We see that

$$v_i(9) < k_i \quad \text{for } 1 \le i \le 10. \tag{5}$$

Since $V_{10}(9) < k_{10}$ and $V_{10}(s+1) - V_{10}(s) = e^{1-s}(h - (e-1)g) < 0$, we have

$$v_{10}(s) < V_{10}(s) < k_{10} \quad \text{for } s \ge 9. \tag{6}$$

For $i = 9$ down to 1, for $s = 9, 10, \dots,$ we have, by (3), (4), (5), and (6),

$$v_i(s+1) < \frac{v_i(s)}{p_i} + v_{i+1}(s+1) < \frac{k_i}{p_i} + k_{i+1} = k_i.$$

Thus, $w(s, 0) = v_1(s) < k_1 = 1/\log 2$ for $s \ge 9$. □

Combining Lemmas 8.1, 8.2, and 8.3, we have the following

Lemma 8. *We have* $w(s, m) < 1/\log p_{m+1}$ *for* $s \geq 3$, $0 \leq m \leq 4$.

Lemma 9. *For a given prime* p, *if* $B = B(p)$ *is homogeneous and nonempty, then*

$$\sum_{b \in B} \frac{1}{b \log b} \leq \frac{1}{p \log p}.$$

Proof. This follows from Lemmas 6, 7, and 8. □

As we have seen above, Lemma 9 immediately implies the theorem.

Acknowledgments

I would like to thank Professor Paul Erdős for sending me the paper [3], and Professors G. Robin and J. P. Massias for their help when the original manuscript was written in France. Special thanks go to the referee for kind, friendly and helpful suggestions, for improvements and simpler proofs of almost all of the lemmas, especially for pointing out an error in the original proof of Lemma 8, and for advice on the organization and presentation of this paper.

Bibliography

1. P. Erdős, *Note on sequences of integers no one of which is divisible by any other*, J. London Math. Soc. **10** (1935), 126–128.
2. ——, Seminar at the University of Limoges, 1988.
3. P. Erdős, A. Sárközy, and E. Szemerédi, *On divisibility properties of sequences of integers*, Number Theory, Debrecen, Colloq. Math. Soc. János Bolyai, vol. 2, North-Holland, Amsterdam, and New York, 1968.
4. P. Erdős and Zhenxiang Zhang, *Upper bound of* $\sum 1/(a_i \log a_i)$ *for primitive sequences*, Proc. Amer. Math. Soc. (to appear).
5. H. Halberstam and K. F. Roth, *Sequences*, Chapter V, Oxford Univ. Press, London, 1966.
6. J. Barkley Rosser, *The n-th prime is greater than* $n \log n$, Proc. London Math. Soc. (2) **45** (1939), 21–44.
7. J. Barkley Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.
8. Zhenxiang Zhang, *On a conjecture of Erdős on the sum* $\sum_{p \leq n} 1/(p \log p)$, J. Number Theory **39** (1991), 14–17.

Department of Mathematics, Anhui Normal University, 241000 Wuhu, Anhui, P. R. China

State Key Laboratory of Information Security, Graduate School USTC, 100039 Beijing, P. R. China