CYCLOTOMY AND DELTA UNITS

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To the memory of Derrick Henry Lehmer

ABSTRACT. In this paper we examine cyclic cubic, quartic, and quintic number fields of prime conductor $p$ containing units that bear a special relationship to the classical Gaussian periods: $\eta_j - \eta_{j+1} + c$ is a unit for periods $\eta_j$ and $c \in \mathbb{Z}$.

1. INTRODUCTION

In [10], Emma Lehmer discovered that certain well-known families of cubic and quartic fields contained translation units, where a translation unit $\delta$ differs from a Gaussian period $\eta$ by a rational integer. She then presented a family of quintic fields with the same property. Schoof and Washington [11] proved the converse of Lehmer's results for cubic fields and those quartic fields in which all units have norm +1.

Later D. H. and Emma Lehmer became interested in a cyclotomy where the Gaussian period $\eta$ was replaced by the difference $\delta_j$ of two periods $\eta_j - \eta_{j+1}$. We will show that the fields with analogously-defined delta units are, in the cubic and quartic cases, the same as those already known. In Lehmer's quintic case the situation is more complicated because the ordering of the $\eta$'s is not unique. The Lehmers observed without proof in [9] that only half of the primitive roots mod $p$ induce an ordering of the $\eta$'s which give a delta unit in the quintic field of conductor $p$. We investigate this phenomenon.

2. DEFINITIONS

The cyclotomic classes of degree $e$ and prime conductor $p = ef + 1$ are

$\mathcal{C}_j = \{g^{\nu+j} \mod p : \nu = 0, \ldots, f-1\}, \quad j = 0, \ldots, e-1,$

where $g$ is any primitive root mod $p$. Here, $\mathcal{C}_0$ contains the $e$th-power residues, but the ordering of the other classes depends upon the choice of $g$.

The Gaussian periods $\eta$ are defined by

$$\eta_j = \sum_{\nu \in \mathcal{C}_j} r_{\nu}^j, \quad j = 0, \ldots, e-1,$$

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where $\zeta_p = \exp(2\pi i/p)$. The Lagrange resolvent $\tau$, sometimes called a Gauss sum, of a character $\chi$ of order $e$ (e.g., $\chi$ is a complex-valued $e$th-power residue symbol) is

$$\tau(\chi) = \sum_{j=0}^{p-1} \chi(j)\zeta_p^j.$$ 

When $\chi$ is taken to be the character defined by $\chi(g) = \zeta_e$, the well-known fundamental relations between Gaussian periods and Lagrange resolvents are given by

$$\tau(\chi^i) = \sum_{k=0}^{e-1} \zeta_e^{jk} \eta_k, \quad \eta_k = e^{-i} \sum_{j=0}^{e-1} \zeta_e^{-jk} \tau(\chi^j).$$

The delta cyclotomy is defined by

$$\delta_j = \eta_j - \eta_{j+1}.$$ 

Here and throughout, indices of $\eta$ and $\delta$ should be understood mod $e$; when omitted, we mean to refer to any $\eta$ or $\delta$'s. The different orderings of the $\eta$'s induce different values of the $\delta$'s.

A unit $\theta$ such that $\theta = \eta + c$ for some $c \in \mathbb{Z}$ is called a translation unit. If $\theta = \delta + c$ for some $\delta$ defined by (2.3), then $\theta$ is a generalized delta unit; if $\theta = \delta \pm 1$, then $\theta$ is a delta unit.

### 3. Cubic fields

Since the conductor $p \equiv 1 \mod 6$, we have the well-known decomposition

$$4p = L^2 + 27M^2, \quad L \equiv 1 \mod 3, \quad M > 0.$$ 

We may assume that $g$ is chosen such that [5, Proposition 1]

$$g^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \mod p.$$ 

**Theorem 1.** If $K$ is a cyclic cubic field of prime conductor $p$, the following are equivalent:

(i) $M = 1$, so $K$ is a simplest cubic as defined by Shanks [12].

(ii) $K$ has a translation unit.

(iii) $K$ has a delta unit.

(iv) $K$ has a generalized delta unit.

**Proof.** (i) $\Rightarrow$ (ii) & (iii): Shanks showed that the polynomials

$$Y^3 - \frac{L - 3}{2} Y^2 - \frac{L + 3}{2} Y - 1 = \prod_{j=0}^{2} (Y - \theta_j)$$

generate the cubic fields with $M = 1$. Emma Lehmer showed that $\eta + (L - 1)/6$ is one of the units $\theta$ [10]. The Lehmers showed in [9] that if $M = 1$, then $\delta - 1$ is a unit.

(iii) $\Rightarrow$ (iv): Trivial.

(ii) $\Rightarrow$ (i): This is shown in [11].
(iv) ⇒ (i): We can find the minimal polynomial \( \text{Irr}_{Q \delta} \) from the definition (2.3) and the cyclotomic numbers of order 3. These are defined (for fixed \( g \)) by

\[(h, k) = \#\{\nu \in (\mathbb{Z}/p\mathbb{Z})^* : \nu \in \mathbb{C}^{(g)}_h, \nu + 1 \in \mathbb{C}^{(g)}_k\}.
\]

There are a number of well-known general formulas satisfied by the cyclotomic numbers (see, e.g., [1, 13]), including

\[
\eta_a\eta_{a+k} = \epsilon^{(k)f} + \sum_{h=0}^{e-1} (h, k)\eta_{a+h},
\]

\[(3.3) \quad \epsilon^{(k)} = \begin{cases} 1, & k = 0, f \text{ even}, \text{ or } k = e/2, f \text{ odd}, \\ 0, & \text{ otherwise}. \end{cases}
\]

The cyclotomic numbers for \( e = 3 \) were determined in principle by Gauss. For \( g \) normalized by (3.1), we have [5, Proposition 1, misprint corrected]

\[
(00) = (p - 8 + L)/9,
(11) = (20) = (02) = (2p - 4 - L - 9M)/18,
(01) = (10) = (22) = (2p - 4 - L + 9M)/18,
(12) = (21) = (p + 1 + L)/9.
\]

It is now a routine computation to find that

\[\text{Irr}_{Q \delta} = X^3 - pX + Mp.\]

We are therefore looking to solve

\[(3.4) \quad N^K_Q(\delta + c) = c^3 - p(c + M) = \pm 1.\]

If \( c = -1 \), it is immediate that the only solution is \( M = 1 \) and a norm of \(-1\). If \( c = 1 \), there are no units. First, \( p = 7 \) (where \( M = 1 \)) can be checked as a special case. For \( p > 7 \), we have \( 1 - p + M < 1 + 2\sqrt{p} - p < -1 \). This shows (iii) ⇒ (i).

Generalized delta units of norm +1 would be, from (3.4), solutions to

\[(c - 1)(c^2 + c + 1) = p(c + M).\]

Since \( p \) is prime, it divides one of the factors on the left. If

\[(3.5) \quad dp = c^2 + c + 1,\]

then

\[(3.6) \quad d(c - 1) = c + M.\]

Isolating \( M \), gives

\[(3.7) \quad M = cd - c - d = (c - 1)(d - 1) - 1.\]

From (3.5) and \( p > 0 \) we have \( d > 0 \). Combining this with (3.7) and \( M > 0 \) forces \( d \geq 2 \) and \( c \geq 2 \). When \( c = 2 \), hence \( p = 7 \) and \( M = 1 \), (3.6) is not satisfied. When \( c = 3 \), then \( d = 1 \), a contradiction. When \( c = 4 \), then \( p = 7 \) and \( d = 3 \), which gives \( M = -5 \), also a contradiction. Therefore, we
may assume \( c \geq 5 \). Starting from (3.5), we have
\[
d p < 2c^2 \Rightarrow L^2 + 27M^2 < \frac{8c^2}{d} \Rightarrow M < \frac{2\sqrt{2c}}{3\sqrt{3d}} < \frac{5c}{9}.
\]
Plugging this back into (3.6), we have
\[
d(c - 1) < \frac{14c}{5} \Rightarrow d < \frac{14c}{5(c - 1)} < 2
\]
(since \( c \geq 5 \), a contradiction.

Now suppose
\[
(3.8) \quad d p = c - 1,
\]
so
\[
(3.9) \quad M = d(c^2 + c + 1) - c.
\]
If \( c = 1 \), we would have from (3.8) that \( d = 0 \) and then from (3.9), \( M = -1 \), impossible. Moreover, \( \text{sgn } d = \text{sgn } c \) by (3.8). When both are negative,
\[
M < d(c^2 + c + 1) + dc = d(c + 1)^2 \leq 0,
\]
a contradiction. For \( c > 1 \), we must have that \( c > 8 \), since \( p \geq 7 \). Now
\[
p \leq d p < c \Rightarrow M^2 < \frac{4c}{27} \Rightarrow M < \sqrt{c}.
\]
Combining this with (3.9) gives the inequality \( c^2 + 1 < \sqrt{c} \), which never holds. Hence, there are no generalized delta units of norm \( +1 \).

For the norm \(-1\) case we are looking for solutions to
\[
(c + 1)(c^2 - c + 1) = p(c + M).
\]
Proceeding similarly to the positive-norm case, we first consider the possibility that \( d p = c^2 - c + 1 \) and \( M = cd - c + d = (c + 1)(d - 1) + 1 \). As before, \( d > 0 \). If \( d = 1 \), we see that \( M = 1 \) is a solution to (3.4), regardless of \( c \). From now on, assume \( d > 1 \). If \( c \leq 2 \), then either \( p < 7 \) or \( M < 0 \), which are impossible. Assume \( c \geq 3 \). Then
\[
d p < 2c^2 \Rightarrow M < \frac{2\sqrt{2c}}{3\sqrt{3d}} \Rightarrow d(c + 1) < \frac{14c}{5} \Rightarrow d < \frac{14c}{9(c + 1)} < 2,
\]
contradicting the assumption \( d \geq 2 \).

The remaining case is \( d p = c + 1 \). We have \( M = d(c^2 - c + 1) - c \). If \( c = -1 \), then \( d = 0 \) and \( M = 1 \), a solution to (3.4). If \( c < -1 \), then \( d < 0 \). Now
\[
M = d(c^2 - c + 1) - c < d(c^2 - c + 1) + dc < d(c^2 + 1) < 0,
\]
a contradiction. It remains to check only \( c \geq 0 \). Immediately we get \( d > 0 \). But then, as with \( d p = c - 1 \), we quickly get a contradiction:
\[
p < d p < 2c \Rightarrow M < \sqrt{c} \Rightarrow c^2 - c + 1 < c + \sqrt{c},
\]
and since \( c \geq 6 \), this, too, is impossible. \( \Box \)
We found all solutions to (3.4) during the proof of the theorem and summarize this result.

**Corollary 3.1.** All generalized delta units have norm $-1$. If $M \neq 1$, there are no generalized delta units. If $M = 1$, then $\delta - 1$ is a unit. If, in addition, there exists $c \in \mathbb{Z}$ such that $p = c^2 - c + 1$, then $\delta + c$ and $\delta - (c - 1)$ are also units.

Shanks [12] showed that when $M = 1$, the group generated by $-1$ and any two of the units $\theta_j$ in (3.2) is the full unit group, and that Galois action on the units $\theta$ is given by the map $\theta \mapsto -(\theta + 1)^{-1}$. Since $\eta_0$ is invariant under choice of $g$, we fix $\theta_0$.

**Proposition 3.2.** The ordering of the $\eta$ induced by $\theta_0 = \eta_0 - (L + 1)/6$ and Shanks's map $\theta_{j+1} = -(\theta_j + 1)^{-1}$ coincides with the ordering obtained by (2.1) and (3.1).

**Proof.** We find that
\[
(\eta_1 + (L - 1)/6)(\eta_0 + (L + 5)/6)
= \frac{1}{36}(36 \eta_0 \eta_1 + 6 \eta_1 L + 30 \eta_1 + 6 L \eta_0 + L^2 + 4 L - 6 \eta_0 - 5)
= \frac{1}{36}(4 \eta_0 p + 10 \eta_0 - 2 \eta_0 L + 4 \eta_1 p - 26 \eta_1 - 2 \eta_1 L + 4 \eta_2 p + 4 \eta_2 + 4 \eta_2 L)
= -1,
\]
expanding $\eta_0 \eta_1$ by (3.3) and substituting in $\eta_2 = -1 - \eta_0 - \eta_1$ and $p = (L^2 + 27)/4$. Therefore, $\theta_1 = -(\theta_0 + 1)^{-1}$. Applying Galois action to both sides proves the general case. □

Hasse [4] wrote elements of cyclic cubic fields as $[x, y]$, where
\[
[x, y] = x - y \tau(\chi) - y \tau(\chi) \in K,
\]
$x \in \mathbb{Z}$, $y \in \mathbb{Q}[\zeta_3]$, $\chi(\cdot) = \left(\frac{L + 3\sqrt{-3M}}{2}\right)_3$.

He normalized Galois action so that $[x, y] \rightarrow [x, \zeta_3 y]$. (Warning: Hasse used $L \equiv -1 \mod 3$.)

**Proposition 3.3.** Shanks's map is the inverse of Galois action as normalized by Hasse.

**Proof.** It is evident from the relations (2.2) that Hasse’s map takes
\[
\eta_0 = (1 + \tau(\chi) + \tau(\chi))/3 \rightarrow (1 + \zeta_3 \tau(\chi) + \zeta_3^2 \tau(\chi))/3 = \eta_2,
\]
whereas the previous proposition shows that Shanks’s map increments the index of $\eta$. □

**Delta units and the choice of $g$.** Fix, for the moment, the choice of $g$. In general, redefining the periods using a generator $g' \in \mathfrak{C}(g)$ yields $\eta_{\nu'} = \eta_{\nu^j}$. If $g' \in \mathfrak{C}_{-1}(g)$, then $\delta_{\nu'} = -\delta_{-\nu}$. Therefore, in looking for delta units, $\mathfrak{C}_j(g)$ and $\mathfrak{C}_{-j}(g)$ can be paired, so $\phi(\epsilon)/2$ essentially distinct delta polynomials must be considered. Therefore, when $e < 5$, the existence of delta units does not depend on the choice of $g$. For cubic fields, choosing a primitive root from the
other class of cubic nonresidues $C_2$ changes the signs of $\delta$, $c$, and the norm of the delta units.

4. QUARTIC FIELDS

Because we are interested in both cyclotomy and units, we will consider only the real fields, where $p \equiv 1 \pmod{8}$. (The unit groups of the imaginary quartic fields are generated, up to torsion, by quadratic units.) Here we will use the normalization

$$p = a^2 + b^2, \quad b \equiv 0 \pmod{4}, \quad b > 0, \quad a \equiv 1 \pmod{4},$$

and a primitive root $g$ is chosen (per [7]) with

$$g^{(p-1)/4} \equiv a/b \pmod{p}.$$

Theorem 2. If $K$ is a real cyclic quartic field of prime conductor $p$, the following are equivalent:

(i) $b = 4$, so $K$ is a simplest quartic field as defined by Gras [3].

(ii) $K$ has a translation unit of norm +1.

(iii) $K$ has a delta unit.

(iv) $K$ has a generalized delta unit of norm +1.

Proof. (i) $\Rightarrow$ (ii) & (iii): Emma Lehmer showed that if $b = 4$, then $-\eta + (a - 1)/4$ is a root of the Gras quartic polynomial [3]

$$Y^4 - aY^3 - 6Y^2 + aY + 1,$$

so it is a unit of norm +1 [10, equation (4.5), corrected]. The Lehmers later showed that if $b = 4$, then either $\delta + 1$ or $\delta - 1$ is a unit [9], without determining which sign held for a particular $g$.

(iii) $\Rightarrow$ (iv) & (i): Since Hasse’s [4] normalization for quartic fields agrees with ours, we will use it to obtain $\text{Irr}_Q \delta$. The symbol $[x_0, x_1, y_0, y_1]$ will represent the element of $K$ given by

$$[x_0, x_1, y_0, y_1] = \frac{1}{4}((x_0 - x_1)\sqrt{p} + (y_0 + iy_1)\tau(x) + (y_0 - iy_1)\overline{\tau(x)}),$$

where $\chi$ is the quartic character belonging to $K$, viz., the quartic residue symbol $\left(\frac{a+b}{4}\right)$. (Condition (4.1) is equivalent to $\chi(g) = i$ [7].) A general formula for the minimal polynomial of any element written in this way appears in [8] (or see Gras [3]). From (2.2),

$$\delta_0 = \eta_0 - \eta_1 = [-1, -1, 1, 0] - [-1, 1, 0, -1] = [0, -2, 1, 1].$$

The minimal polynomial formula now gives

$$\text{Irr}_Q \delta = Y^4 - p(Y + b')^2, \quad b' = b/4,$$

whence

$$N^K_Q(\delta + c) = c^4 - p(b' - c)^2.$$

Immediately we have $c = 1 \Rightarrow b = 4$ and norm +1; $c = -1$ is impossible.

(ii) $\Rightarrow$ (i): Proven in [11].
(iv) $\Rightarrow$ (i): From (4.3), units of norm $+1$ will be solutions to

$$
(4.4) \quad c^4 - 1 = (c + 1)(c - 1)(c^2 + 1) = p(b' - c)^2.
$$

There are no primes $\equiv 1 \mod 8$ dividing the left side for $c = \pm 2, \pm 3$, and when $c = \pm 4$, the prime $p = 17$ divides the left side, but $p = 17$ implies $b' = 1$ and (4.4) is not satisfied. The cases $c = \pm 1$ have been handled above, so we may assume $|c| \geq 5$.

Supposing, first, that $dp = c + 1$, we have $b' = c \pm \sqrt{d(c - 1)(c^2 + 1)}$. The minus root gives $b' < 0$, impossible. The plus root gives $b' > |c|^{3/2} + c > |c|^{3/2}/4$. Then $b > |c|^{3/2}$, so $p > |c|^3$. Since $(b' - c)^2 > \frac{124}{25}|c|^3$, we are reduced to the inequality $c^4 > \frac{124}{25}c^6$, which is never true for $|c| \geq 5$. The case $dp = c - 1$ is virtually identical. The case $dp = c^2 + 1$ is similar. Here, $b' = c \pm \sqrt{d(c^2 - 1)}$. Since $b' \in \mathbb{Z}$ and $c \neq \pm 1$, we cannot have $d = 1$, so the minus root is impossible. Then

$$
b' > \frac{\sqrt{24}(\sqrt{2} - 1)}{5}|c| > \frac{2|c|}{5} \Rightarrow p > \frac{64}{25}c^2 \Rightarrow c^4 - 1 = p(b' - c)^2 > 3c^4,
$$

which again has no solution. $\Box$

We have also proved en passant:

**Corollary 4.1.** A generalized delta unit of norm $+1$ is a delta unit with $c = 1$. If $\theta = \delta \pm 1$ is a delta unit, then $b = 4$, the plus sign holds, and $N^K \theta = 1$.

Gras showed that Galois action on the roots $\theta$ of (4.2) is given by $\theta_{j+1} = (\theta_j - 1)/(\theta_j + 1)$.

**Proposition 4.2.** The ordering of the $\eta$ induced by $\theta_0 = -\eta_0 + (a - 1)/4$ and Gras's map $\theta_{j+1} = (\theta_j - 1)/(\theta_j + 1)$ coincides with the ordering obtained by (2.1) and (4.1). Gras's map is the inverse of Galois action as normalized by Hasse.

**Proof.** The identity $\theta_1(\theta_0 + 1) = \theta_0 - 1$, which suffices to prove the first statement, was verified using the rule for multiplication in Hasse's basis [4, §8(1)]. Hasse normalized Galois action so that $[x_0, x_1, y_0, y_1] \rightarrow [x_0, -x_1, -y_1, y_0]$, and the proof of the second statement is analogous to Proposition 3.3. $\square$

**Remarks.** (1) Choosing a generator from the other class of nonresidues $\in_3$ changes the sign of all $\delta$, hence $c$.

(2) The only known example of a translation unit of norm $-1$ is $\eta - 2$ in the field of conductor 401 [11]. This field does not contain a generalized delta unit. The only generalized delta unit of norm $-1$ which we have found is $\delta + 2$ in the field of conductor 17, which also contains delta units; no others can exist for $c^4 + 1$ squarefree.

### 5. Quintic Fields

Dickson showed [2] that the conductor $p \equiv 1 \mod 5$ may be decomposed as

$$
16p = x^2 + 50u^2 + 50v^2 + 125w^2,
$$

subject to

$$
xw = v^2 - 4uv - u^2, \quad x \equiv 1 \mod 5.
$$
If \((x, u, v, w)\) is one solution to this system, the others are \((x, -v, u, -w)\), \((x, v, -u, -w)\), and \((x, -u, -v, w)\). If \(g\) is a primitive root mod \(p\), Katre and Rajwade proved in [6] that \((x, u, v, w)\) can be defined unambiguously, given \(g\), by the additional condition

\[
g^{(p-1)/5} \equiv (a - 10b)/(a + 10b) \mod p, \quad a = x^2 - 125w^2, \\
b = 2xu - xv - 25vw.
\]

Conversely, if a choice of \((x, u, v, w)\) is fixed, primitive roots \(g\) in only one of the four classes of quintic nonresidues in \(\mathbb{Z}/p\mathbb{Z}\) will satisfy (5.1). The cyclotomic numbers for such \(g\) are given by

\[
\begin{align*}
(00) &= (p - 14 + 3x)/25, \\
(01) &= (10) = (44) = (4p - 16 - 3x + 50v + 25w)/100, \\
(02) &= (20) = (33) = (4p - 16 - 3x + 50u - 25w)/100, \\
(03) &= (30) = (22) = (4p - 16 - 3x - 50u - 25w)/100, \\
(04) &= (40) = (11) = (4p - 16 - 3x - 50v + 25w)/100, \\
(12) &= (21) = (34) = (43) = (41) = (2p + 2 + x - 25w)/50, \\
\end{align*}
\]

If we set \(\delta_j = \eta_j - \eta_{j+1}\), we have, by direct computation,

\[
\text{Irr}_Q \delta = \Delta(Y) = Y^5 - Y^3p + Y^2vp \\
+ p \frac{((3u + v)(u - v) + 5w^2)}{4} Y \\
+ p(4v + 4u)w^2
\]

In the quintic case, defining the periods \(\eta'\) with \(g' \in \mathbb{Z}_p^{(g)}\) effects the substitution \((x, u, v, w) \rightarrow (x, -v, u, -w)\). Hence, the minimal polynomial of \(\delta'_j = \eta'_j - \eta'_{j+1} = \eta_{2j} - \eta_{2(j+1)}\) is given by

\[
\Delta'(Y) = Y^5 - Y^3p + Y^2up \\
+ p \frac{((3v - u)(v + u) + 5w^2)}{4} Y \\
- p(4v + 4u)w^2
\]

The quintic analogue to a simplest field was given by Emma Lehmer in [10]. For \(n \in \mathbb{Z}\) set

\[
u = n + 1, \quad v = n + 2, \quad w = \left(\frac{n}{5}\right)_2,
\]

from which it follows that \(x = -\left(\frac{g}{5}\right)_2(4n^2 + 10n + 5)\) and

\[
p = n^4 + 5n^3 + 15n^2 + 25n + 25.
\]
Lehmer showed that

\[ \theta = w \eta - (w - n^2)/5 \]

is a translation unit up to sign.

The normalization (5.1) of \( g \) reduces to

\[ g^{(p-1)/5} \equiv (a - 10b)/(a + 10b) \mod p, \]

(5.7)

\[ a = 4(4n^4 + 30n^2 + 25), \quad b = -2 \left( \frac{n}{5} \right) \left( 2n^3 + 20n + 25 \right). \]

**Theorem 3.** Suppose \( p \) is of type (5.5) and \( g \) is chosen such that (5.7) holds. Then \( \delta - 1 \) is a unit. If \( p \neq 11 \),

(i) \( \delta - 1 \) is the only generalized delta unit, and

(ii) \( \delta' + c \) is never a unit.

**Proof.** For such \( p \), \( \Delta(Y) \) reduces to

\[ Y^5 - pY^3 + p(n+2)Y^2 - pnY - p \]

\[ = 1 + (Y - 1)(Y^4 + Y^3 - (p - 1)Y^2 + [p(n + 1) + 1]Y + p + 1). \]

Clearly, \( \delta - 1 \) is a unit of norm \(-1\). The equations \( N_k^\Sigma(\delta - c) \pm 1 = \Delta(c) \pm 1 = 0 \) may be considered as quintic polynomials in \( c \). The lack of integer solutions to the unit equations may be proved by locating their irrational solutions between consecutive integers. If \( n \geq 1 \), then \( \Delta(c) + 1 \) has a root in each open interval \((\hat{c}, \hat{c} + 1)\) for

\[ \hat{c} \in \{-n^2 - 3n - 6, -1, 0, n + 1, n^2 + 2n + 3\}. \]

In each case, \( \text{sgn}(\Delta(\hat{c}) + 1) \neq \text{sgn}(\Delta(\hat{c} + 1) + 1) \). This accounts for all five roots, so there are no generalized delta units when \( n \geq 1 \). The polynomial \( \Delta(c) - 1 \) has an exact root at \( c = 1 \) instead of an irrational root in \((0, 1)\); otherwise, its four irrational roots are located in the same intervals. Similar results hold for \( n < -3 \). The case \( n = -3 \) yields no solutions for \( c \), which leaves only \( p = 11 \). Hence (i). For the proof of (ii), replace \( \Delta \) by \( \Delta' \) and proceed in the same way. □

**Corollary 5.1.** Take \( x, u, v, w, p, a, \) and \( b \) as above and define the periods with an arbitrary primitive root \( g \). If \( p \neq 11 \), all \( g \) define an ordering such that \( \Delta(Y) \) has delta units. Otherwise, \( \Delta(Y) \) has delta units if and only if \( g \) satisfies

\[ g^{(p-1)/5} \equiv \left( \frac{a - 10b}{a + 10b} \right)^{\pm 1} \mod p. \]

These are the \( g \) in two (i.e., half) of the four nonresidue classes.

**Proof.** This is immediate from the theorem and (5.1). □

The field of conductor 11 is a special case. It is of type (5.5) with either \( n = -2 \) or \( n = -1 \). (One can show that 11 is the only integer represented
nonuniquely by the polynomial (5.5).) The period polynomial for $p = 11$ is
\[ Y^5 + Y^4 - 4 Y^3 - 3 Y^2 + 3 Y + 1, \]
so the periods $\eta$ are themselves units. Also $\eta \pm 1$ and $\eta + 2$ are Galois-conjugate units (but not conjugate to $\eta$). Choosing to use $n = -2$, we have from (5.3) and (5.4) that $\delta - 1$, $\delta + 2$, $\delta - 3$, $\delta' \pm 1$, and $\delta' + 2$ are all units, no two conjugate.

The converse of Theorem 3 is false. In the field of conductor 211 using $(x, u, v, w) = (1, 1, 2, -5)$, $\delta - 1$ is a unit of norm $-1$. There is a generalized delta unit $\delta - 3$ for $p = 61$ and $(x, u, v, w) = (1, 1, 4, -1)$.

Schoof and Washington showed that Galois action on the quintic translation units (5.6) can be given by
\begin{equation}
(5.8) \quad \theta \to \frac{(n + 2) + n\theta - \theta^2}{1 + (n + 2)\theta}.
\end{equation}

When $g$ satisfies (5.7), then (5.6) induces an ordering of the $\theta_j$. The method of Proposition 3.2 can be used to show that with this ordering the image of $\theta_0$ under (5.8) is $\theta_2$ when $w = 1$, and $\theta_3$ when $w = -1$. In [11], the map (5.8) was derived from (5.6) and the canonical ordering of the $\eta_j$, but we have changed the normalization of $(x, u, v, w)$ from [10] and [11]. The normalizations (3.1), (4.1), and (5.1) all follow naturally from Jacobi sums; they insure that the character defined by $\chi(g) = \zeta_p$ coincides with the particular $\chi$th-power residue symbol modulo $p$ belonging to the field $K$ [5]. Using Lehmer's $u$ and $v$ with normalized $g$ makes the units translates of $\delta'$ instead of $\delta$. Changing $u$ and $v$ seemed the lesser evil.

**Remark.** We were unable to find any infinite family of quintic fields with generalized delta units containing either $p = 61$ or $p = 211$. Furthermore, we were unable to make any progress on the conjecture of Schoof and Washington in [11] that all quintic fields with translation units are of Emma Lehmer's form (5.5).

**Bibliography**


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