THE CUSP FORMS OF WEIGHT 3 ON $\Gamma_2(2, 4, 8)$

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Abstract. The congruence subgroup $\Gamma_2(2, 4, 8)$ of the group $\Gamma_2$ of $4 \times 4$ integral symplectic matrices is contained in $\Gamma_2(4)$ and contains $\Gamma_2(8)$, with $\Gamma_2(n)$ the principal congruence subgroup of level $n$. The Satake compactification of the quotient of the three-dimensional Siegel upper half space by $\Gamma_2(2, 4, 8)$ is shown to be a complete intersection of ten quadrics in $\mathbb{P}^{13}$. We determine the space of global holomorphic three forms on this space, which coincides with the space of cusp forms of weight 3 on $\Gamma_2(2, 4, 8)$; it has dimension 2283. Finally, we study the action of the Hecke operators on this space and consider the Andrianov $L$-functions of some eigenforms.

1. Introduction

1.1. In this paper we study the cusp forms of weight 3 on the congruence subgroup $\Gamma_g(2, 4, 8)$ of $\Gamma_g := Sp_{2g}(\mathbb{Z})$ in case $g = 2$.

Recall that $\Gamma_g(n)$ consists of the matrices which are $= I \mod n$, that

$$\Gamma_g(4, 8) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(4) : \text{diag}(B) \equiv \text{diag}(C) \equiv 0 \mod 8 \right\},$$

and in [6] the following (normal) subgroup of $\Gamma_g$ was defined:

$$\Gamma_g(2, 4, 8) := \left\{ \begin{pmatrix} I + 4A' \\ C \\ & I + 4D' \end{pmatrix} \in \Gamma_g(4, 8) : \text{trace}(A') \equiv 0 \mod 2 \right\}.$$ 

In particular,

$$\Gamma(8) \hookrightarrow \Gamma(2, 4, 8) \hookrightarrow \Gamma(4, 8) \hookrightarrow \Gamma(4).$$

The Siegel upper half plane $\mathbb{H}_g$ is the analytic variety consisting of $g \times g$ complex symmetric matrices with positive definite imaginary part. For a function $f : \mathbb{H} \to \mathbb{C}$, $M \in \Gamma_g$ and $k \in \mathbb{N}$ one defines

$$f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix} (\tau) = \text{det}(C\tau + D)^{-k} f((A\tau + B)(C\tau + D)^{-1}).$$

Let $\Gamma'$ be a congruence subgroup of $\Gamma_g$, that is, $\Gamma_g(n) \subset \Gamma'$ for some $n$. A modular form of weight $k$ for $\Gamma'$ is a holomorphic function $f$ on $\mathbb{H}_g$. 

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satisfying $f|_k M = f$ for all $M \in \Gamma'$. The $\mathbb{C}$-vector space of such functions is denoted by $M_k(\Gamma')$.

One defines the Siegel operator $\Psi$, mapping $f \in M_k(\Gamma')$ to a function on $\mathbb{H}_{g-1}$, by

$$\Psi(f)(\tau) := \lim_{t \to \infty} f(((5,0)j, t)), \quad \tau \in \mathbb{H}_{g-1}. $$

The subspace $S_k(\Gamma')$ of $M_k(\Gamma')$, called the space of cusp forms, is defined by

$$S_k(\Gamma') = \{ f \in M_k(\Gamma') : \Psi(f|_k M)(\tau) = 0 \ \forall \tau \in \mathbb{H}_{g-1}, \forall M \in \Gamma_g \}. $$

1.2. In case the group $\Gamma'$ acts without fixed points on $\mathbb{H}_g$ (for example, if $\Gamma' \subset \Gamma_g(n)$ and $n \geq 3$), the space $M_{g+1}(\Gamma')$ corresponds to the space of holomorphic $\frac{1}{2}g(g+1)$-forms on the quasi-projective variety $X^0 = \mathbb{H}_g/\Gamma'$. This correspondence is given by $\omega \mapsto f$ when

$$\pi : \mathbb{H}_g \rightarrow X^0, \quad \pi^* \omega = f \left( \bigwedge d \tau_{ij} \right).$$

The subspace of those forms which extend to (any) smooth compactification $\bar{X}$ of $X^0$ is exactly $S_{g+1}(\Gamma')$. In particular,

$$S_{g+1}(\Gamma') \cong H^0(\bar{X}, \Omega^\frac{1}{2}g(g+1)).$$

A remarkable aspect of this result is that the 'cusp form condition' need only be checked at points in the boundary of the Satake compactification which are in quotients of $\mathbb{H}_g$, rather than at all points which are in quotients of $\mathbb{H}_k$ with $0 \leq k \leq g-1$ (this can be generalized to other symmetric domains, see [1, Chapter IV]). We will happily exploit this fact.

1.3. In the case $g = 2$ (where we will omit the subscript $g$) and $\Gamma' = \Gamma(2,4,8)$, the variety $X^0$ can be described explicitly as a Zariski open subset of a projective variety $X \subset \mathbb{P}^{13}$. The embedding of $X^0$ into $\mathbb{P}^{13}$ is given by certain theta constants. The variety $X$ is the complete intersection of ten quadrics, which can easily be written down explicitly. Using this, and combinatorics of theta constants, we can determine the space $H^0(\tilde{X}, \Omega^\frac{1}{2}g(g+1))$, and thus also the space $S_3(\Gamma(2, 4, 8))$. (We use the computer program 'Macaulay' for the computations with ideals of polynomials.)

On the space $S_3(\Gamma(2, 4, 8))$ there acts the finite group $\Gamma/\Gamma(2,4,8)$, and we determine the decomposition into irreducible subrepresentations.

In the last sections we study the action of the Hecke algebra on $S_3(\Gamma(2, 4, 8))$. The action of this algebra is induced by correspondences. In this case these are codimension 3 cycles on $X^0 \times X^0$ and by 'pullback-push forward' they give linear maps on $S_3(\Gamma(2, 4, 8))$. The definition of these cycles is in terms of isogenies of abelian varieties. Similar to the elliptic modular case, one has a congruence relation which relates the action of the Hecke operators on $S_3(\Gamma(2, 4, 8))$ to the action of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^3(\tilde{X}, \mathbb{Q}_l)$. It is therefore of some interest to determine the eigenspaces and eigenvalues of these operators. We determined the Hecke polynomials, which describe the Hecke action, for several cusp forms and for some small primes $p$. 

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Most of the forms we consider appear to be obtained via liftings from modular forms on subgroups of $SL_2(\mathbb{Z})$. In one case the Hecke polynomials suggest that the modular form is related to a Hecke character of the field of eight roots of unity (the form $g_1$). There is one case in which the Hecke polynomials of the cusp form do not allow one of these interpretations (the form $g_2$). In this paper we do not actually try to prove that most of the forms are indeed liftings.

2. Combinatorics of theta characteristics

2.1. The modular forms we consider are linear combinations of products of theta constants. For $m = \frac{1}{2}(m', m'') \in \mathbb{R}^2 \times \mathbb{R}^2$ with $m'_j, m''_j \in \{0, 1\}$ we define the theta constant $\theta_m : \mathbb{H}_2 \to \mathbb{C}$ with (half-integral) (theta) characteristic $m$ by

$$\theta_m(\tau) := \sum_{k \in \mathbb{Z}^2} \exp(2\pi i \frac{1}{2}(k + m'_j)(k + \frac{1}{2}) + (k + \frac{1}{2})'(m''_j)).$$

The theta constant is not identically zero if and only if the theta characteristic $m$ is even, i.e., $m'_j m''_j \in 2\mathbb{Z}$. There are ten even theta characteristics. If $m = \frac{1}{2}(a, b, c, d)$ we will also write

$$\theta_m(\tau) = \theta[\frac{a}{c} \frac{b}{d}](\tau).$$

Under the action of $\Gamma$ on $\mathbb{H}_2$ these ten theta null’s are permuted (up to a root of unity times a common factor, cf. 5.2). Therefore, $\Gamma$ acts on the set $C_1$ of the ten even characteristics. The action of $M \in \Gamma$ is given by (cf. [8, V.6]):

$$M : C_1 \to C_1, \quad M \circ m := n, \quad n = mM^{-1} + \frac{1}{2}(C'D)_0, (A'B)_0) \mod 1.$$  

Here, $M \in \Gamma$ is the matrix with blocks

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and $(C'D)_0, (A'B)_0$ are the diagonals of the matrices $C'D$ and $A'B$, respectively, viewed as row vectors.

This action of $\Gamma$ factors over $\Gamma/\Gamma(2) \cong S_6$, the group of permutations of the set $S = \{1, 2, \ldots, 6\}$. The ten even theta characteristics then correspond to the $\frac{1}{2}(\frac{5}{3}) = 10$ partitions of $S$ into two subsets with three elements each (cf. [9]); such a partition is called a triadic syntheme. The action of $S_6$ on $C_1$ is then easy to follow.

2.2. Associated with each $m \in C_1$ there is a quadratic form $Q_m$ in the variables $X_0, X_1, X_2, X_3$ and a quadric $V_m = V(Q_m)$ in $\mathbb{P}^3 = \mathbb{P}(X_0, X_1, X_2, X_3)$. The $Q_m$’s are defined by the relation (cf. [8, IV.1]):

$$\theta^2_m(\tau) = Q_m(\theta[0 0]1(2\tau), \theta[1 0]0(2\tau), \theta[1 1]0(2\tau), \theta[0 1]0(2\tau)).$$
The ten quadrics determine an interesting configuration of 30 lines (15 pairs of skew lines) and 60 points (vertices of 15 tetrahedrons). By a tetrahedron we mean the algebraic variety consisting of the union of six lines, the edges, meeting in four points, the vertices, as in the figure.

We let $C_i$ be the set of subsets of cardinality $i$ of $C_1$ and we put $C = \bigcup_i C_i$. We describe the orbit structure of $S_6$ on $C$ and that part of the geometry of the quadrics which is relevant for our purposes.

2.4. Proposition. The orbits of the $S_6 = \Gamma/\Gamma(2)$ action on the sets $C_n$ are as follows:

1. The group $S_6$ acts transitively on $C_1$; $\#C_1 = 10$.
2. The group $S_6$ acts transitively on $C_2$; $\#C_2 = 45$.

Two quadrics $V_m$ and $V_n$ intersect in a 4-gon of lines, thus determining a tetrahedron.

3. There are two orbits on $C_3$, denoted by $C_3^+$ and $C_3^-$.

$$C_3 = C_3^+ \cup C_3^-,$$
$$\#C_3 = \binom{10}{3} = 120,$$  \quad $\#C_3^+ = \#C_3^- = 60.$$

A triple $\{m_1, m_2, m_3\}$ is in $C_3^+$ if and only if $m_1 + m_2 + m_3$ is an even theta characteristic (such triples are called syzygeous).

A triple $\{m_1, m_2, m_3\}$ is in $C_3^-$ if and only if $m_1 + m_2 + m_3$ is an odd theta characteristic (such triples are called asyzygeous).

The quadrics of a syzygeous triple intersect in eight points, vertices of two tetrahedrons.
The quadrics of an asyzygeous triple intersect in a pair of skew lines.

4. There are three orbits on $C_4$, denoted by $C_4^+$, $C_4^-$, and $C_4^*$.

$$C_4 = C_4^+ \cup C_4^- \cup C_4^*, \quad \#C_4 = \binom{10}{4} = 210, \quad \#C_4^+ = \#C_4^- = 15, \quad \#C_4^* = 180.$$  

A 4-tuple $\{m_1, m_2, m_3, m_4\}$ is in $C_4^+$ if and only if any subtriple is in $C_3^+$.

4-tuple is in $C_4^+$ if and only if any subtriple is in $C_3^-$. We call such 4-tuples syzygeous.

4-tuple is in $C_4^*$ if and only if the sum of two subtriples is even and the sum of the other two subtriples is odd.

The sets $C_4^+$ and $C_4^-$ are in natural 1-1 correspondence with the set of the 15 tetrahedra and the set of 15 line pairs respectively as follows:

For $S \in C_4^+$, the complementary set $\overline{S} \in C_5$ consists of six characteristics whose corresponding quadrics pass through the vertices of a unique tetrahedron $T_S$. The union of the four quadrics $V_m$, $m \in S$, contains 24 of the 30 lines, but none of the six lines of the tetrahedron $T_S$.

For $S \in C_4^-$, the quadrics $V_m$, $m \in S$, all pass through a line pair $l_S$. Conversely, any of the 15 line pairs is cut out by four quadrics, thus setting up a 1-1 correspondence. The union of the four quadrics contains all the 15 line pairs.

5. There are three orbits on $C_5$, we denote them by $C_5^+$, $C_5^-$, and $C_5^*$.

$$C_5 = C_5^+ \cup C_5^- \cup C_5^*, \quad \#C_5 = \binom{10}{5} = 252, \quad \#C_5^+ = \#C_5^- = 90, \quad \#C_5^* = 72.$$  

A 5-tuple is in $C_5^+$ if and only if it contains a (unique) syzygeous 4-tuple.

A 5-tuple is in $C_5^-$ if and only if it contains a (unique) asyzygeous 4-tuple.

A 5-tuple is in $C_5^*$ if and only if the sum of the five characteristics is odd.

For any $S \in C_5^*$, the union $\bigcup_{m \in S} V_m$ also contains all the 15 line pairs.

6. For $n \geq 6$ the orbit structure of $C_n$ can be obtained by taking complements from the above. We use the notation $C_{10-i}^+ := \{\overline{S}: S \in C_i^\perp\}$,

$$C_{10-i}^- := \{\overline{S}: S \in C_i^+\}.$$  

Proof: This follows easily from [8, V.6, especially Prop. 2]. The transitivity of $S_6$ on $C_2$ is in fact the corollary of Prop. 2. Note also that the sum of an even number of theta characteristics transforms linearly, so orbits may be distinguished by such a sum being 0 or not, whereas the sum of an odd number of theta characteristics transforms like a theta characteristic, so such orbits may be distinguished by the sum being even or odd. □

2.5. The complete incidence structure between points, lines, and quadrics is easily obtained, and is left to the reader as amusing passtime. We only note:
A line pair $l_S$, $S \in C_4^-$, lies on a quadric $V_m$, $m \in C_1$, if and only if $m \in S$.

Furthermore, on each line there are six points, thus on each line pair there are 12 points, and these 12 points make up three tetrahedra. Conversely, through each point there are three lines and each tetrahedron is formed out of a triple of line pairs, etc.

2.6. Lemma. The $S_6$-orbit structure on the $C_i$, $i = 2, \ldots, 8$, is given by:

$$
\begin{array}{cccc}
  C_4^+(15) & \downarrow & C_5^+(90) & \varnothing & C_6^+(15) \\
  C_2(45) & \overleftarrow{3} & C_4^-(60) & \overleftarrow{2} & C_5^-(60) \\
  C_3(60) & \overleftarrow{4} & C_5^+(72) & \overleftarrow{2} & C_6^-(180) \\
  C_7^-(60) & \overleftarrow{4} & C_5^-(90) & \varnothing & C_6^-(15) \\
\end{array}
$$

where $A \xrightarrow{n} B$ means: each $S \in B$ contains exactly $n$ $S' \in A$. There is also a dual interpretation: $\overleftarrow{A \xrightarrow{n} B}$ means: an element $S \in B$ can be extended in $n$ ways to get an element $S' \in A$.

3. The space $X \subset P^{13}$ and its singular locus $\Sigma$

3.1. In [6], the map

$$
\Theta : A_2(2, 4, 8) := H_2/\Gamma_2(2, 4, 8) \rightarrow P^{13},
$$

$$
\tau \mapsto \left(\ldots : \theta \begin{bmatrix}
a & b \\
0 & 0
\end{bmatrix} (2\tau) : \ldots : \theta_m(\tau) : \ldots \right),
$$

where $m$ runs over the ten even characteristics and $a, b$ run over $\{0, 1\}$, is shown to be an embedding. We denote the image by $X^0$ and the closure of $X^0$ in $P^{13}$ will be denoted by $X$.

We define two morphisms:

$$
p : X \rightarrow P^3, \quad q : X \rightarrow P^9
$$

obtained by projection on the first four and the last ten coordinates.

The map $p$ corresponds to the natural map $A(2, 4, 8) \rightarrow A(2, 4)$; in fact, $P^3$ can be identified with the Satake compactification $\bar{A}(2, 4)$ of $A(2, 4)$. The boundary components of $\bar{A}(2, 4)$ correspond to the 30 lines in the $P^3$.

The map $q$ corresponds to the natural map $A(2, 4, 8) \rightarrow A(4, 8)$.

The equations of $X$ are very simple. To describe them, we choose for each $m \in C_1$ a variable $Z_m$ and consider

$$
F_m := Z_m^2 - Q_m \in \mathbb{C}[X, Z].
$$

3.2. Lemma. The projective variety $X$ has the following properties:

1. $X$ is the complete intersection of the ten quadrics $F_m$, $m \in C_1$.

2. The singular locus $\Sigma$ of $X$ is exactly the inverse image of the union of the 30 lines in $P^3$ under the map $p$. The locus $\Sigma$ consists of $30 \cdot 2^3 = 240$ irreducible components, each one isomorphic to a degree-8, genus-5 curve.
3. \( X \) is (projectively) normal and is in fact isomorphic to the Satake compactification of \( A_2(2, 4, 8) \):

\[ X \cong A_2^+(2, 4, 8). \]

**Proof.** Let \( X' = \{(X, Z) \in \mathbb{P}^{13} : F_m(X, Z) = 0 \ \forall m \in C_1 \} \). Then by equation (2.3), we have that \( X' \subset X \). Furthermore, the projection \( p : X' \to \mathbb{P}^3, (X, Z) \mapsto X \) represents \( X' \) as an iterated branched cover of \( \mathbb{P}^3 \), branching along the quadrics \( V_m, m \in C_1 \). It follows that \( X' \) is purely 10-codimensional and hence is a complete intersection.

An easy local computation shows that \( X' \) is singular exactly above the points where at least two of the quadrics \( V_m \) intersect. When we restrict to the line \( X_2 = X_3 = 0 \) the equations \( F_m \) reduce to

\[
\begin{align*}
(A) \quad & Z_1^2 = X_0^2 + X_1^2, \\
& Z_2^2 = X_0^2 - X_1^2, \\
& Z_5^2 = 2X_0X_1, \\
(B) \quad & Z_0^2 = Z_7^2 = Z_9^2 = Z_{10}^2 = 0, \\
& Z_1^2 = Z_3^2, \\
& Z_2^2 = Z_4^2, \\
& Z_5^2 = Z_8^2.
\end{align*}
\]

The ideal generated by (A) defines a degree-8, genus-5 curve in \((Z_1 : Z_2 : Z_5 : X_0 : X_1)-\text{space} \) (in fact, this is the elliptic-modular curve \( X(8) \)). The equations (B) show that the solution set consists of \( 2^3 = 8 \) copies of this curve.

As \( \dim \Sigma = 1 \) and \( X' \) is a complete intersection, it follows that \( X' \) is irreducible and thus \( X' = X \).

Furthermore, as a complete intersection is arithmetically Cohen-Macaulay, it follows from Serre’s criterion for normality that \( X \) is (projectively) normal. Since the map \( \Theta \) is given by modular forms, there exists a morphism \( \psi : X \to A^+(2, 4, 8) \). Since \( \Theta : \mathbb{H}_2/\Gamma(2, 4, 8) \to X^0 \) is an isomorphism ([6, Theorem 2.2]), the map \( \psi \) is a birational isomorphism. Comparing the description of the Satake compactification in [10] with \( X \), we see that \( \psi \) is a bijection. By Zariski’s main theorem it follows that \( X \cong A^+(2, 4, 8) \). \( \square \)

Now let

\[ I_X = (F_m : m \in C_1) \subset \mathbb{C}[X, Z], \quad R_X := \mathbb{C}[X, Z]/I_X \]

(the affine coordinate ring of the cone over \( X \)). Furthermore, we let \( I_\Sigma \) be the ideal of (the affine cone over) the singular locus \( \Sigma \) with its reduced structure (i.e., \( I_\Sigma \) is radical).

3.3. **Lemma.** We have \( I_\Sigma = \bigcap_{S \in C^+_1} (Z_m, m \in S; I_X) \).

**Proof.** Clearly, we have

\[ I_\Sigma = \bigcap_{S \in C^+_1} I(l_S), \]

where \( I(l_S) \) is the ideal in \( \mathbb{C}[X, Z] \) of \( p^{-1}(l_S) \), the inverse image of the line pair \( l_S \subset \mathbb{P}^3 \) in \( \mathbb{P}^{13} \), with reduced structure. The ideal of a line pair \( l_S \) is

\[ J(l_S) = (Q_m, m \in S) \subset \mathbb{C}[X], \]

since every pair of skew lines in \( \mathbb{P}^3 \) is cut out by four quadrics, and the four \( Q_m, m \in S \), vanish on \( l_S \).
The ideal-theoretic inverse image of $J(I_S)$ is given by the ideal
\[ J(I_S) = (Q_m, m \in S, I_X) \subset \mathbb{C}[X, Z] \]
\[ = (Q_m, Z_m^2 - Q_m, m \in S, F_m, m \notin S) \]
\[ = (Z_m^2, m \in S, I_X). \]
So, $(Z_m, m \in S, I_X) \subset \sqrt{J(I_S)} = I(I_S)$.

But the ideal on the left is in fact radical: by transitivity of $S_6$ on $C_4^-$ we may assume $S = \{6, 7, 9, 10\}$ and then
\[ (Z_m, m \in S, I_X) \]
\[ = (F_m, m \notin S, Z_m, m \in S, X_0X_2, X_1X_3, X_0X_3, X_1X_2) \]
\[ = (F_m, m \notin S, Z_m, m \in S, X_0, X_1) \]
\[ \cap (F_m, m \notin S, Z_m, m \in S, X_2, X_3), \]
and both of the ideals are radical (see the proof of Lemma 3.2). Thus the inclusion is actually an equality and the lemma is proved. □

4. The cusp forms

4.1. Proposition. The space of cusp forms of weight three for $\Gamma(2, 4, 8)$ is canonically isomorphic to the degree-6 part of $I_{\Sigma, X} := I_{\Sigma}/I_X \subset \mathbb{C}[X, Z]/I_X$: $S_3(\Gamma(2, 4, 8)) \cong I_{\Sigma, X, 6}$.

Proof. There are in fact two ‘natural isomorphisms’. We describe them both.

1. The map $\Theta: \mathbb{H}_2 \rightarrow \mathbb{P}^{13}$ induces by pull-back an isomorphism
\[ \Theta^*: H^0(X^0, \mathcal{O}_{X^0}(6)) \xrightarrow{\sim} M_3(\Gamma(2, 4, 8)). \]
As $X$ is normal, we have $H^0(X^0, \mathcal{O}_{X^0}(6)) \cong H^0(X, \mathcal{O}_X(6))$ and because $X$ is projectively normal
\[ H^0(X, \mathcal{O}_X(6)) \cong \mathbb{R}_X.6. \]
A polynomial $P \in \mathbb{C}[X, Z]_6$ pulls back to a cusp form if and only if it vanishes on $\Sigma$ (the boundary components of $X$).

2. With any polynomial $P$ of degree 6 we can associate a (meromorphic) differential form on $X$ as follows. There is an isomorphism
\[ \text{Res}: \mathcal{O}_X(6) \rightarrow \omega_X, \quad P \mapsto \omega_P := \text{Res} \left( \frac{P\Omega}{F_1 \cdots F_{10}} \right), \]
where $\Omega := \sum_{i=0}^{13} (-1)^i Y_i dY_0 \wedge \cdots \wedge \hat{d}Y_i \wedge \cdots \wedge dY_{13}$ and where the $Y_i$ are the coordinates on $\mathbb{P}^{13}$.

The differential forms which extend holomorphically on a desingularization $\pi: \tilde{X} \rightarrow X$ correspond, via $\text{Res}$, to an ideal $\mathcal{I}_A \subset \mathcal{O}_X$, which is independent of the desingularization (see [11]). We will study the ‘adjunction ideal’ $I_A \subset \mathbb{C}[X, Z]$ corresponding to $\mathcal{I}_A$.

Now a simple local calculation shows that transverse to a general point of $\Sigma$ the variety $X$ has a singularity which is isomorphic to the cone over an elliptic curve (of degree 4). This singularity ‘imposes precisely one adjunction
condition \( p_g = 1 \)''. This means that \( P \) has to vanish on \( \Sigma \) if \( \omega_P \) is to extend holomorphically. Therefore, \( I_A \subset I_{\Sigma} \).

That in fact \( I_A = I_{\Sigma} \) follows from the principle that forms which extend to the general point of the inverse image of \( \Sigma \) in \( \tilde{X} \) extend to all of \( \tilde{X} \) (see [4, Satz 3], [5, 'Anmerkung' to Satz III, 2.6, p. 156]). That is, the 60 special points do not impose further adjunction conditions (!). This can also be checked directly by pulling back the differential forms to an explicit resolution of singularities of \( X \) above the \( 2^4 \cdot 60 \) special points. \( \Box \)

4.2. We now come to the crucial part of this paragraph: the explicit generators of the ideal \( I_{\Sigma} \subset R_X \) or \( I_{\Sigma} \subset \mathbb{C}[X, Z] \). To describe these, we need a little more notation. The tetrahedron \( T_S \), determined by an \( S \in C_4^+ \), gives rise to an ideal

\[ J_T \subset \mathbb{C}[X, Z] \]

of the functions vanishing on the six lines of \( T_S \).

4.3. **Lemma.** The ideal \( J_T \) is generated by four elements of degree 3.

**Proof.** For each tetrahedron, the four products of three of the four linear forms defining the faces of the tetrahedron vanish on the tetrahedral lines and in fact generate the ideal. \( \Box \)

The following theorem allows us to find all the cusp forms of weight 3 on \( \Gamma(2, 4, 8) \). We describe them in Theorem 6.4, where we also determine the \( \Gamma \)-action on the space \( S_3(\Gamma(2, 4, 8)) \).

4.4. **Theorem.** The ideal \( I_{\Sigma} \) is (minimally) generated by the following elements:

- \( F_m \quad \sharp = 10 \)
- \( Z_S, \quad S \in C_4^- \quad \sharp = 15 \)
- \( Z_S, \quad S \in C_5^+ \quad \sharp = 72 \)
- \( ZSF, \quad F \in J_{T_S, 3}, \quad S \in C_4^+, \quad S' \in C_3^+, \quad S' \subset S \quad \sharp = 240. \)

Here we use the notation \( Z_S := \prod_{m \in S} Z_m \) for any \( S \in C \).

**Proof.** We first show that the stated elements are in the ideal \( I_{\Sigma} \).

Because the union of the quadrics \( V_m, \ m \in S, \) contains all 30 lines in case \( S \in C_4^- \) and \( S \in C_5^+ \), it follows that in these cases \( Z_S \) vanishes on \( \Sigma \), and so \( Z_S \in I_{\Sigma} \).

The union of the quadrics \( V_m, \ m \in S, \ S \in C_4^+ \) only contains 24 lines, which are precisely the lines not in the tetrahedron \( T_S \) determined by \( S \). Furthermore, the union of any three of the four quadrics \( V_m, \ m \in S, \ S \in C_4^+ \), already contains the same 24 lines. Consequently, multiplying any \( Z_S', \ S' \in C_3^+, \ S' \subset S \) with any element of \( J_{T_S} \) will give a function vanishing on the whole of \( \Sigma \).

The difficult part of the theorem is to show that there is nothing more in \( I_{\Sigma} \). So far, this depends on an explicit computation of the intersection of the 15 ideals \( I(S), \ S \in C_4^- \). For this we used the computer program 'Macaulay'. The computer output consisted of 337 elements, generating this ideal, which were readily recognized as the elements above.
To give a computer independent proof, it seems necessary to understand the combinatorics much better. □

4.5. **Corollary.** We have

1. \( \dim(I_{\Sigma, x, 4}) = 15 \),
2. \( \dim(I_{\Sigma, x, 5}) = 282 \),
3. \( \dim(I_{\Sigma, x, 6}) = 2283 \).

**Proof.** It is convenient to use the following isomorphism:

\[
R_X = \bigoplus_{s \in C} \mathbb{C}[X]Z_s,
\]

stating that, modulo the \( F_m \), every polynomial can be reduced in a *unique* way to a sum of squarefree monomials \( Z_s \), with coefficients in \( \mathbb{C}[X] \). (A Cohen-Macaulay ring is a free module over a parameter system.) □

From Theorem 4.4 we have

\[
I_{\Sigma, x, 4} = \bigoplus_{s \in C_4^+} \mathbb{C}Z_s,
\]

so \( \dim(I_{\Sigma, x, 4}) = 15 \).

In degree 5 we thus find, apart from the 72 new generators \( Z_{s}, S \in C_5^+ \), the elements of \( I_{\Sigma, x, 4} \) multiplied by a linear factor. The following cases occur:

1. \( Z_mZ_s, \quad m \notin S \),
2. \( Z_mZ_s, \quad m \in S \),
3. \( X_iZ_s, \quad i = 0, 1, 2, 3 \).

From diagram 2.6 we see that

- in case (1) \( Z_mZ_s = Z_{s'}, \quad S' \in C_5^- \), \( \# = 90 \)
- in case (2) \( Z_mZ_s = Q_mZ_{s'}, \quad S' \in C_3^- \), \( \{ m, S' \} \in C_4^- \), \( \# = 60 \)
- in case (3) \( X_iZ_s \), \( i = 0, 1, 2, 3 \).

 Altogether, in degree 5 we find \( 72 + 90 + 60 + 60 = 282 \) monomials.

To get elements of degree 6, we proceed in the same way: apart from the 240 generators \( FZ_{S'}, F \in JT_X \), \( S' \subset S \subset C_4^+ \), we get all the other factors by multiplying something of degree 5 with a linear factor. Starting from the 90 elements \( Z_{s}, S \in C_5^- \), we get:

- in case (1) we have \( Z_mZ_s = Z_{s'}, \quad S' \in C_5^- \), \( \# = 15 \).
- In case (2) there are two subcases: (2a) \( S' := S - \{ m \} \in C_4^- \), \( Z_mZ_s = Q_mZ_{s'}, \quad \# = 360 \) and (2b) \( S' \in C_4^+ \), \( \# = 360 \).

Proceeding in this way with the other elements of degree 5 in the ideal, we get the following table (the last column relates them to the representation studied
in §6):

<table>
<thead>
<tr>
<th>elements</th>
<th>dim</th>
<th>representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_S$</td>
<td>15</td>
<td>$R_5^-$</td>
</tr>
<tr>
<td>$Q_m Z_S$</td>
<td>90</td>
<td>$R_4^- (0; 2)$</td>
</tr>
<tr>
<td>$Q_m Z_S$</td>
<td>360</td>
<td>$R_4^- (1; 1)$</td>
</tr>
<tr>
<td>$X_1 Z_S$</td>
<td>360</td>
<td>$R_4^- (0; 1)$</td>
</tr>
<tr>
<td>$Q_m Q_m Z_S$</td>
<td>90</td>
<td>$R_4^- (1; 1; 0)$</td>
</tr>
<tr>
<td>$X_1 X_1 Z_S$</td>
<td>240</td>
<td>$R_4^- (0; 1)$</td>
</tr>
<tr>
<td>$Z_S$</td>
<td>150</td>
<td>$R_4^- (0; 2) \oplus R_4^- (2; 0)$</td>
</tr>
<tr>
<td>$Q_m Z_S$</td>
<td>180</td>
<td>$R_5^*$</td>
</tr>
<tr>
<td>$Q_m Z_S$</td>
<td>360</td>
<td>$R_5^* (1; 0)$</td>
</tr>
<tr>
<td>$X_1 Z_S$</td>
<td>288</td>
<td>$R_5^*$</td>
</tr>
<tr>
<td>$F Z_S$</td>
<td>240</td>
<td>$R(3, 3)$</td>
</tr>
</tbody>
</table>

In particular, we find $\dim I_{x, x, 6} = \dim S_3(\Gamma(2, 4, 8)) = 2283$.

5. **The theta transformation formula**

5.1. Let $\Gamma'$ be a normal subgroup of $\Gamma$ and let $M_3(\Gamma')$ be the space of modular forms of weight 3 on $\Gamma'$. The group $\Gamma$ (in fact $\Gamma/\Gamma'$) acts on $M_3(\Gamma')$ by

$$ f \mapsto f|M, \quad \text{with} \quad (f|M)(\tau) = \det(C\tau + D)^{-3} f(M \cdot \tau). $$

To decompose the spaces of cusp forms with respect to this representation, we introduce the following symplectic matrices:

$$ e_1(n) = \begin{pmatrix} 1 & 1 \\ 2n & -2n \\ 1 & \end{pmatrix}, \quad e_3(n) = \begin{pmatrix} 1 & 2n \\ 1 & \end{pmatrix}, $$

$$ e_2(n) = e_1(n), \quad e_4(n) = e_3(n), $$

$$ e_5(n) = \begin{pmatrix} 1 & 1 \\ 2n & 1 \end{pmatrix}, \quad e_6(n) = \begin{pmatrix} a & 1 \\ c & \end{pmatrix}, $$

$$ e_7(n) = \begin{pmatrix} 1 & 2n \\ 1 & \end{pmatrix}, \quad e_8(n) = \begin{pmatrix} 1 & 2n \\ 1 & \end{pmatrix}, $$

$$ e_9(n) = e_7(n), \quad e_{10}(n) = e_8(n). $$

Here, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is some matrix in $\text{SL}_2(\mathbb{Z})$ which is congruent to $\begin{pmatrix} 1 + 2n & 0 \\ 0 & 1 + 2n \end{pmatrix}$ modulo $4n$. 
To find the action of $\Gamma$ on the modular forms, we use the transformation formula for theta functions ([8, V, §1, Corollary, p. 176 and V, §2, Theorem 3, p. 182]):

5.2. Lemma (Igusa's transformation formula). For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{Z})$ and $m \in \mathbb{R}^{2g}$ a theta characteristic, we define

$$M \cdot m := mM^{-1} + \frac{1}{2}((C'D)_0(A'B)_0),$$

with $(C'D)_0$ the diagonal of $C'D$. Define

$$\phi_m(M) = -\frac{1}{2} \left( m'^{t}DB'm' - 2m'^{t}BC'm'' + m'^{t}CA'm'' - (m'^{t}D - m''^{t}C)'(A'B)_0 \right).$$

Then

$$\theta_{M \cdot m}((A\tau + B)(C\tau + D)^{-1}) = \kappa(M) \exp(2\pi i \phi_m(M)) \sqrt{\det(C\tau + D)} \theta_m(\tau),$$

in which $\kappa(M)$ is a complex number of absolute value 1 which depends only on $M$ and the choice of the square root. In particular, it does not depend on the characteristic $m$. Thus, $\kappa(M)^2$ is well defined; for $M \in \Gamma_g(2)$ one has

$$\kappa(M)^2 = (-1)^{\text{trace}(D - 1)/2}.$$  

5.3. In the remainder of this paragraph we derive two lemmas from this formula. The first lemma gives an explicit form of the transformation formula for certain matrices. The second lemma studies the transformation behavior of the functions $\theta[^a_b^c_d](2\tau)$ which are an ingredient of some of the cusp forms.

5.4. Lemma. For every half-integral characteristic

$$m = \frac{1}{2}(a, b, c, d) \quad \text{with} \quad a, b, c, d \in \{0, 1\}$$

and every $M \in \Gamma(2))$ as below we have

$$\theta_m(M \cdot \tau) = \chi_m(M)\theta_m(\tau)$$

for all $\tau \in \mathbb{H}$ and with $\chi_m(M)$ as in the table.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$e_1(1)$</th>
<th>$e_2(1)$</th>
<th>$e_3(1)$</th>
<th>$e_4(1)$</th>
<th>$e_5(1)$</th>
<th>$e_6(1)$</th>
<th>$e_7(1)$</th>
<th>$e_8(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_m(M)$</td>
<td>$(-1)^{bc}$</td>
<td>$(-1)^{ad}$</td>
<td>$(-1)^{ab}$</td>
<td>$1$</td>
<td>$(-1)^{ac}$</td>
<td>$ia$</td>
<td>$ib$</td>
<td></td>
</tr>
</tbody>
</table>

Proof. We will write $m_1 := (a, b)$, $m_2 := (c, d)$. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(2)$ with $A = D = I$, $B = 2B'$, $C = 0$; note that $B'$ is then a symmetric matrix with integral coefficients. We find

$$\theta_m(M \cdot \tau) = \theta_m(\tau + B)$$

$$= \sum_k \exp(2\pi i \left( \frac{1}{2}(k + \frac{m_1}{2})(\tau + B)'(k + \frac{m_1}{2}) + (k + \frac{m_1}{2})'(\frac{m_2}{2}) \right))$$

$$= \sum_k \exp(2\pi i \left( \frac{1}{2}(k + \frac{m_1}{2})\tau' + k B'^{t}k + kB'm_1 + \frac{m_1}{2}B'\frac{m_1}{2} \right)$$

$$= \exp(\frac{2\pi i}{4}m_1B'm_1) \cdot \theta_m(\tau).$$

From this, $\chi_m(M)$ for $M = e_3, e_7, e_8$ is easily computed.
Let now $M = \begin{pmatrix} A & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$; this implies that $D = \tau A^{-1}$. Then

$$\theta_m(M \cdot \tau) = \theta_m(\tau A^{-1})$$

$$= \sum_k \exp(2\pi i \frac{1}{2} (k + \frac{m_1}{2}) \tau ((k + \frac{m_1}{2}) A + ((k + \frac{m_1}{2}) A)((\frac{m_2}{2})' A^{-1})))$$

$$= \theta_n(\tau),$$

the characteristic $n$ being given by

$$n = (n_1, n_2), \quad n_1 = m_1 A, \quad n_2 = n_2 A^{-1}.$$ 

In case $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -1 \end{pmatrix}$, so $M = e_1(1)$, one obtains:

$$n = (a, b, c, d) + (2b, 0, 0, -2c), \quad \text{thus} \quad \theta_n(\tau) = (-1)^{bc} \theta_m(\tau),$$

where we use $(\theta_2)$ from [8, p. 49]. The formula for $\chi_m(M)$, with $M = e_2(1), e_5(1), e_6(1)$, is derived analogously; note one may take $A = -I$ in $e_5$ and $e_6$.

Note that by comparing this result with Lemma 5.2, we find

$$\kappa(M) \sqrt{\det(C\tau + D)} = 1$$

for these matrices. □

5.5. Lemma. Let $M \in \Gamma_g(2)$ with

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I + 2A' & 2B' \\ 2C' & I + 2D' \end{pmatrix};$$

let $m = (m_1, \ldots, m_g)$ with $m_i \in \{0, 1\}$ and let $\tau \in \mathbb{H}_g$.

Then we have

$$\theta[\begin{pmatrix} m \\ 0 \end{pmatrix}](2M \cdot \tau) = \lambda(M, \tau) \cdot (-1)^{(m+y)y} x \theta[\begin{pmatrix} m+y \\ 0 \end{pmatrix}](2\tau),$$

with $\lambda(M, \tau)$ independent of the characteristic $m$ and

$$\lambda(M, \tau)^2 = \det(C\tau + D), \quad x := \text{diag}(B'), \quad y := \text{diag}(C'),$$

where we view the diagonals as row vectors.

In case $C = 0$, we have $\lambda(M, \tau) = 1$.

Proof. This is actually a special case of Igusa’s transformation formula 5.2. Indeed,

$$\theta[\begin{pmatrix} m \\ 0 \end{pmatrix}](2(M \cdot \tau)) := \theta[\begin{pmatrix} m \\ 0 \end{pmatrix}](2(\tau A' + B)(\tau C' + D)^{-1})$$

$$= \theta[\begin{pmatrix} m \\ 0 \end{pmatrix}](A(2\tau) + 2B)(2\tau C' + D)^{-1})$$

$$= \theta[\begin{pmatrix} m \\ 0 \end{pmatrix}](M' \cdot (2\tau)).$$

One easily verifies that the matrix $M'$ is also a symplectic matrix with integral coefficients, thus $M'^{-1}$ is easy to compute:

$$M' := \begin{pmatrix} A & 2B' \\ C' & D \end{pmatrix}, \quad M'^{-1} := \begin{pmatrix} iD & -2B' \\ -iC' & iA \end{pmatrix}.$$
The action of $M'$ on the characteristic $(\frac{m}{2}, 0)$ is then given by
\[
M' \cdot (\frac{m}{2}, 0) := (\frac{m}{2}, 0)M'^{-1} + \frac{1}{2}(C''D)_{0}, 2(A'B)_{0}) \]
\[
= (\frac{m}{2}, 0) \left( \begin{array}{cc}
 iD & -2iB \\
 -iC' & iA
\end{array} \right) + \frac{1}{2}(C''D)_{0}, 2(A'B)_{0}) \]
\[
= (\frac{m}{2}D, -m'B) + \frac{1}{2}(C''D)_{0}, 2(A'B)_{0}) \]
\[
= (\frac{m}{2}D, 0) + (m'D + y', 2(-m'B' + (A'B'))_{0}) \]
\[
= k + l. \]

Using (\theta .2) from [8, p. 49], we find $\theta_{k+l} = \theta_k$. It is then easy to verify that
\[
\theta[\frac{m}{2}](2M \cdot \tau) = \theta_{M'^{-1}}(M' \cdot (2\tau)). \]

Applying 5.2 to the right-hand side, we find
\[
\theta[\frac{m}{2}](2M \cdot \tau) = \lambda(M, \tau) \exp(2\pi i\phi_k(M')) \theta_k(2\tau). \]

Since $k = (\frac{m}{2} + y, 0)$, the only nonzero terms in $\phi_k(M')$ are
\[
\phi_k(M') = -\frac{1}{2}(m + y)(iDB'y(m + y) + \frac{1}{2}(m + y)iD(A'B)_{0}
\]
\[
eq \frac{1}{2}(m + y)'x + \mathbb{Z}, \]
where we use that $B = 2B'$, $D = I + 2D'$ and $x := \text{diag}(B')$.

In case $C = 0$ it follows from (the proof of) Lemma 5.4 that $\lambda(M, \tau) = 1$.

This completes the proof of Lemma 5.5. $\square$

6. THE REPRESENTATION OF $\Gamma/\Gamma(2, 4, 8)$ ON $S_3(\Gamma(2, 4, 8))$

6.1. Recall that the subgroup $\Gamma(2)$ fixes the characteristics. For $f = \prod_{i=1}^{2k} \theta_{m_i}$, a modular form of weight $k$ on $\Gamma(4, 8)$, we can then define the homomorphism:
\[
\chi_f : \Gamma(2)/\Gamma(2, 4, 8) \to \mathbb{C}^*, \quad \text{by} \quad f \mid M = \chi_f(M)f. \]

As Lemma 5.5 shows, $\Gamma(2)$ does not fix the $\theta[\frac{g}{0}](2\tau)$'s. We define a subgroup $\Gamma'(2)$ by
\[
\Gamma'(2) := \left\{ \left( \begin{array}{cc}
 A & B \\
 2C' & D
\end{array} \right) \in \Gamma(2) : \text{diag}(C') \equiv 0 \text{ mod } 2 \right\}. \]

For $g = \theta[\frac{a}{0} | b][\frac{c}{0} | d] \prod_{i=1}^{2k-1} \theta_{m_i}(\tau)$, a modular form of weight $k$ on $\Gamma(2, 4, 8)$, we define a character
\[
\chi_g : \Gamma'(2)/\Gamma(2, 4, 8) \to \mathbb{C}^*, \quad \text{by} \quad g \mid M = \chi_g(M)g. \]

The following proposition lists some character values.

6.2. Proposition. Let $m_i$ be a half-integral characteristic with $m_i := \frac{1}{2}(a_i, b_i, c_i, d_i)$.

(a) Let
\[
f(\tau) = \theta_{m_1}(\tau)\theta_{m_2}(\tau) \cdots \theta_{m_k}(\tau). \]

Then $f$ is a modular form of weight $k$ on $\Gamma(4, 8)$ and values of $\chi_f$ are listed below. One also has $\chi_f(e_3(1)) = 1$. 

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(b) Let \( e_1, e_2 \in \{0, 1\} \) and let
\[
g(\tau) = \theta[e_1 e_2] (2\tau) \theta_{m_1}(\tau) \theta_{m_2}(\tau) \cdots \theta_{m_{2k-1}}(\tau).
\]
Then \( g \) is a modular form of weight \( k \) on \( \Gamma(2, 4, 8) \), and some values of \( \chi_g \) are listed below. One also has \( \chi_g(e_5(1)) = 1 \).

<table>
<thead>
<tr>
<th>( M )</th>
<th>( e_1(1) )</th>
<th>( e_2(1) )</th>
<th>( e_3(1) )</th>
<th>( e_4(1) )</th>
<th>( e_5(1) )</th>
<th>( e_6(1) )</th>
<th>( e_7(1) )</th>
<th>( e_8(1) )</th>
<th>( e_9(1) )</th>
<th>( e_{10}(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)^{2b} c^j</td>
<td>(-1)^{2a} d^j</td>
<td>(-1)^{2c} b^j</td>
<td>(-1)^{2d} d^j</td>
<td>(-1)^{2e} c^j</td>
<td>(-1)^{2f} b^j</td>
<td>(-1)^{2g} c^j</td>
<td>(-1)^{2h} b^j</td>
<td>(-1)^{2i} c^j</td>
<td>(-1)^{2j} d^j</td>
<td>(-1)^{2k} c^j</td>
</tr>
</tbody>
</table>

Proof. That \( f \in M_3(\Gamma(4, 8)) \) follows from the corollary in [8, V.7]. Note that for the matrix \( A \) in \( e_5 \) and \( e_6 \) we can take \( A = -I \). Then \( e_5(1) = -I \), which acts trivially on \( \mathbb{H}_2 \). The lemma then follows easily from 5.2. Note that all matrices \( M \) except \( e_6(1) \) have trace \( D - I = 0 \), so \( \kappa(M)^2 = 1 \) by [8, V.3, Theorem 3], and that \( \kappa(e_6(1))^2 = -1 \).

For the second part, we observe that Lemmas 5.5 and 5.2 imply that \( \Gamma(2, 4) \) acts by a character on the modular form \( g \). In [6] it is shown that this character is trivial on \( \Gamma(2, 4, 8) \) (but it is not trivial on \( \Gamma(4, 8) \)). Therefore, \( g \) is a modular form on \( \Gamma(2, 4, 8) \).

The matrices \( e_1, e_2, e_3, e_5, e_7, e_8 \) have \( C = 0 \), so \( \chi_g \) can be computed directly from 5.2 and Lemmas 5.4 and 5.5. For the matrix \( A \) in \( e_5 \) and \( e_6 \) we can take \( A = -I \). Therefore, also \( e_6 \) has \( C = 0 \).

The remaining matrices \( e_4, e_9, e_{10} \) are of the form \( M = (I \ C \ 0) \), so \( D = I \), and \( \kappa(M)^2 = 1 \). In the formula for \( f|M \) there appears however the constant \( \kappa(M')\kappa(M)^{2k-1} \), with \( M' = (I \ 0 \ C) \), cf. the proof of Lemma 5.5.

To find \( \kappa(M')\kappa(M) \), we compute \( \theta_0(2M\tau)\theta_0(M\tau) \). Note that
\[
M = \begin{pmatrix}
I & 0 \\
C & I
\end{pmatrix} = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\begin{pmatrix}
I & -C \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}.
\]
First of all, we find
\[
\theta_0(-2\tau^{-1})\theta_0(-\tau^{-1}) = \frac{1}{2} \det(\tau)\theta_0(\tau/2)\theta_0(\tau).
\]
Up to a 4th root of unity this follows directly from 5.2. By specializing \( \tau = (\tau_{kl}) \) to a matrix with \( \tau_{kl} = 0 \) if \( k \neq l \), we get \( \theta[e_{f_1} e_{f_2}](\tau) = \theta[e_{f_1}](\tau_{11})\theta[e_{f_2}](\tau_{22}) \). The formula then follows from the identity
\[
\theta[0\ 0](-\tau^{-1}) = \sqrt{-i\tau_1} \cdot \theta[0\ 0](\tau_1), \quad \text{with} \quad \text{Re}(\sqrt{-i\tau_1}) > 0, \quad \tau_1 \in \mathbb{H}_1.
\]
Next we apply \( (I \ -C) = (0 \ -2C) \):
\[
\frac{1}{2} \det(\tau - C)\theta_0(\frac{1}{2} - C')\theta_0(\tau - 2C') = \frac{1}{2} \det(\tau - C)\theta_0(\frac{1}{2})\theta_0(\tau),
\]
where we use Lemma 5.5. Applying \( (1 \ 0\ 0) \), we obtain
\[
\frac{1}{2} \det(-\tau^{-1} - C)\theta_0(-2\tau^{-1})\theta_0(-\tau^{-1}) = \det(C\tau + I)\theta_0(2\tau)\theta_0(\tau).
\]
Comparison with 5.2 shows that \( \kappa(M')\kappa(M) = 1 \). The values of \( \chi_g(M) \) then follow from 5.2. (Note that in [6] it is proved that \( \kappa(M')\kappa(M) = -1 \) with \( M = e_6(2) \), the generator of \( \Gamma(4, 8)/\Gamma(2, 4, 8) \), thus some care is needed.) □
6.3. We will now determine the splitting of \( S_3(\Gamma(2, 4, 8)) \) into irreducible \( \Gamma \)-representations and we show that the characters \( \chi_f, \chi_g \) determine the cusp forms (within the space of cusp forms). This will be important when we study the Hecke action in the next section.

6.4. **Theorem.** The space \( S_3(\Gamma(2, 4, 8)) \) is the direct sum of 11 irreducible \( \Gamma \)-representations. The representation on \( S_3(\Gamma(4)) \) is irreducible, and \( S_3(\Gamma(4, 8)) \) is the direct sum of \( 7 = 1 + 6 \) irreducible representations.

Below we label the representations, their dimensions and a cusp form in each representation space.

<table>
<thead>
<tr>
<th>space</th>
<th>label</th>
<th>dim</th>
<th>cusp form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_3(\Gamma(4)) )</td>
<td>( R_5^- )</td>
<td>15</td>
<td>( f_1 = \theta(1,0,0,0,0,0) )</td>
</tr>
<tr>
<td>( R_4^- (0; 2) )</td>
<td></td>
<td>90</td>
<td>( f_2 = \theta(0,1,0,0,0,0) )</td>
</tr>
<tr>
<td>( R_4^- (1, 1; 0) )</td>
<td></td>
<td>90</td>
<td>( f_3 = \theta(0,0,1,0,0,0) )</td>
</tr>
<tr>
<td>( R_4^- (1; 1) )</td>
<td></td>
<td>360</td>
<td>( f_4 = \theta(0,0,0,1,0,0) )</td>
</tr>
<tr>
<td>( R_6^- )</td>
<td></td>
<td>180</td>
<td>( f_5 = \theta(0,0,0,0,1,0) )</td>
</tr>
<tr>
<td>( R_4^- (2; 0) )</td>
<td></td>
<td>60</td>
<td>( f_6 = \theta(0,0,0,0,0,1) )</td>
</tr>
<tr>
<td>( R_2^- (1; 0) )</td>
<td></td>
<td>360</td>
<td>( f_7 = \theta(0,0,0,0,0,0) )</td>
</tr>
<tr>
<td>( S_3(\Gamma(2, 4, 8)) )</td>
<td></td>
<td>( R(3, 3) )</td>
<td>240 ( g_1 = \theta(0,0,0,0,0,1)(2\tau) \theta(0,0,0,0,0,0)(\tau) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R(1, 0) )</td>
<td>240 ( g_2 = \theta(0,0,0,0,0,1)(2\tau) \theta(0,0,0,0,0,0)(\tau) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R(0, 1) )</td>
<td>240 ( g_3 = \theta(0,0,0,0,0,1)(2\tau) \theta(0,0,0,0,0,0)(\tau) )</td>
</tr>
</tbody>
</table>

So the first representation is equal to \( S_3(\Gamma(4)) \) and the sum of the first seven representations is \( S_3(\Gamma(4, 8)) \).

The space \( S_3(\Gamma(4, 8)) \) is a direct sum of one-dimensional spaces \( \mathbb{C} f \), where \( f \) is a monomial, i.e., a product of six theta constants, and for monomials \( f, f' \in S_3(\Gamma(4, 8)) \) we have

\[
\chi_f = \chi_{f'} \iff f = f'.
\]

The space \( S_3(\Gamma(2, 4, 8)) = S_3(\Gamma(4, 8)) \oplus W' \), where \( W' \) is spanned by linear combinations of monomials, which are products of one \( \theta_0(0,0,0,0,0,0) \) and five \( \theta_m(\tau) \)'s. Under the action of \( \Gamma'(2) \), the space \( W' \) is a direct sum of mutually distinct one-dimensional subrepresentations:

\[
W' = \bigoplus_{g} \mathbb{C} g, \quad \text{and} \quad \chi_g = \chi_{g'} \iff g = g'.
\]

**Proof.** The meaning of the names of the representation spaces is as follows: \( R_4^- (1, 1; 0) \) is the space obtained by taking (linear combinations of) all products of one of the 15 monomials \( \theta_S, S \in C_4^- \), and squaring two different terms occurring. Thus we get \( 15 \cdot \binom{4}{2} = 90 \) different monomials. Note that Theorem 4.4 implies that \( \theta_S \) is a cusp form. Similarly, \( R_4^- (1; 1) \) is spanned by multiplying a monomial \( \theta_S, S \in C_4^- \), by \( \theta_m \) with \( m \in S \) and \( \theta_n \) with \( n \not\in S \). The dimension of this space is then \( 15 \cdot 4 \cdot 6 = 360 \). The meaning of the other terms is similar.
It follows from Theorem 4.4 that the 11 spaces are contained in $S_3(\Gamma(2, 4, 8))$ and that $S_3(\Gamma(2, 4, 8))$ is in fact a direct sum of these spaces.

Using Proposition 2.4 (and also [8, V.6] if the six characteristics are not distinct), it is not hard to verify that the first seven spaces are stable under the $\Gamma$-action and that $\Gamma$ permutes the monomials in each space transitively. Since only the monomials without a $\theta [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}]$ are on $\Gamma(4, 8)$, we see that $S_3(\Gamma(4, 8))$ is spanned by the monomials in the orbits of $f_1, \ldots, f_7$. In particular, $\dim S_3(\Gamma(4, 8)) = 1155$.

To show the irreducibility of the seven subrepresentations in $S_3(\Gamma(4, 8))$, we actually need that $\chi_f: \Gamma(2) \rightarrow \mathbb{C}^*$ determines $f$ in $S_3(\Gamma(4, 8))$. To prove it, we use the $\Gamma$-action, so we may assume that $f = f_i$, with $f_i$ one of the seven forms listed. Then one determines all 6-tuples of characteristics which give the same character and one observes, for each $i$, that there is only one set of characteristics which gives a cusp form (to wit, the set of characteristics of $f_i$ itself).

Suppose now that a linear combination of monomials from $S_3(4, 8)$ lies in a subrepresentation. Then, using the action of $\Gamma(2)$ and taking linear combinations, we see that each monomial in the combination lies in that subrepresentation. As each monomial in $S_3(4, 8)$ lies in the $\Gamma$-span of one of the seven cusp forms listed, the subrepresentation is a direct sum of some of the seven listed. Therefore, $S_3(\Gamma(4, 8))$ is a direct sum of seven irreducible $\Gamma$-representations.

Since $f_i$ is invariant under $\Gamma(4)$, we have $f_i \in S_3(\Gamma(4))$. Since the orbit defining the 6-tuple of $f_i$ is $C_6^-$, which has 15 elements, and $\dim S_3(\Gamma(4)) = 15$ [12], it follows that $S_3(\Gamma(4))$ is spanned by $f_i$.

We now consider all of $S_3(\Gamma(2, 4, 8))$. Let $W$ be the subspace of $M_3(\Gamma(2, 4, 8))$ spanned by products of one $\theta [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ and five $\theta_m(\tau)$'s. Using $\theta^2_m(\tau) = Q_m(\theta [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau))$, we also have that the cusp form $g_4$ is in $W$, in fact

$$\theta [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}](2\tau) = \frac{1}{4}(\theta^2 [\begin{smallmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}](\tau) - \theta^2 [\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{smallmatrix}](\tau)).$$

Similarly, all of $R(3, 3)$ is contained in $W$. Therefore,

$$S_3(\Gamma(2, 4, 8)) = S_3(\Gamma(4, 8)) \oplus W', \quad \text{with} \quad W' := W \cap S_3(\Gamma(2, 4, 8)).$$

(Indeed, a product of one $\theta_m(2\tau)$ and five $\theta_n(\tau)$'s is never in $S_3(\Gamma(4, 8))$.)

Under the action of $\Gamma$, the four $\theta [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ are mapped to linear combinations of these four theta nulls (cf. [8, II.5, Theorem 6]), whereas the $\theta_m(\tau)$'s are permuted. The space $W$ is thus stable under the action of $\Gamma$. Since the 5-tuples of characteristics in the cusp forms $g_1, \ldots, g_4$ are in different orbits for the $\Gamma$-action, we already find four distinct subrepresentations in $W' = S_3(\Gamma(2, 4, 8)) \cap W$, each spanned by the $\Gamma$-transforms of a $g_i$.

To see that these four subrepresentations span $W'$, let $f \in W'$ be the product of one $\theta [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ and five $\theta_m$'s. Then there is a transformation in $\Gamma$ which maps the five $\theta_m$'s to the five $\theta_m$'s of one of the first three cusp forms. Therefore, in the subrepresentation generated by $f$ there is a linear combination of the $\theta [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$'s multiplied by the product of the five $\theta_m$ from such a cusp form. Using the action of $e_7(1)$ and $e_8(1)$, we find that, for some $a, b$, $\theta [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ times the product of the same five $\theta_m$ lies in the subrepresentation generated by $f$ (cf. Lemma 5.5). Applying $e_7(1)^ae_{10}(1)^b$, which is in $\Gamma(2)$
and thus fixes the five characteristics but acts on the other (see Lemma 5.5), we get \( \theta[0 \, 0 \, 1 \, 1](2\tau) \) times the same product of the five \( \theta_m \), i.e., one of the \( g_i \) \( (i = 1, 2, 3) \). The monomials from \( R(3, 3) \) are in fact permuted transitively (up to a scalar multiple), as can be seen from the geometry of the tetrahedron or by a similar argument of above.

To prove the irreducibility of these four representations, we need that \( \chi_g \) determines the cusp form \( g \in W' \). This is done as in the \( S_3(\Gamma(4, 8)) \) case by explicit verification. In fact, for a monomial \( g \) obtained from a \( g_i \), \( i = 1, 2, 3 \), the restriction of \( \chi_g \) to \( \Gamma(2, 4) \) determines the five \( \theta_m \) among the possible 5-tuples obtained in this way. Since \( \Gamma \) has three orbits (corresponding to the \( i \)) on these 5-tuples and \( \Gamma(2, 4) \) is a normal subgroup, one need only verify that the 5-tuples of the \( g_i \) are uniquely determined by their character. The action of \( e_7 \) and \( e_8 \) allows one to recover the \( \theta[0 \, 0 \, 1 \, 1](2\tau) \) from \( \chi_g \). Similarly, using the action of \( \Gamma \) on the \( \Gamma \)-orbit of \( g_4 \), one need only check that \( g_4 \) is determined by its character.

The irreducibility of the four representations is then proven as in the \( S_3(\Gamma(4, 8)) \) case. □

7. Hecke eigenforms

7.1. The Hecke algebra, generated by the Hecke operators \( T_p \) and \( T_{p^2} \) for primes \( p > 2 \), acts on the space \( S_3(\Gamma(2, 4, 8)) \). We want to determine a basis of eigenvectors. For an eigenform \( f \) and a prime \( p > 2 \) such that

\[
T_p f = \lambda_p f, \quad T_{p^2} f = \lambda_{p^2} f
\]

one defines the Hecke polynomial

\[
H_p(X) := X^4 - a_p X^3 + a_p^2 X^2 - a_p p^3 X + p^6, \quad \text{with} \quad \begin{cases} 
\lambda_p = a_p, \\
\lambda_{p^2} = \lambda_p^2 - a_p - p^2.
\end{cases}
\]

7.2. For modular forms on \( \Gamma(8) \) there appears a character \( \chi_2: (\mathbb{Z}/8\mathbb{Z})^* \to \{ \pm 1 \} \) in the Hecke polynomial. This character is defined by \( f|\!| M_p = \chi_2(p)f \) with \( M_p \in \Gamma \) a matrix with \( M_p \equiv \text{diag}(p^{-1}, p^{-1}, p, p) \) mod 8. We will show that \( \chi_2 \) is trivial for modular forms on \( \Gamma(2, 4, 8) \).

If \( p \equiv 1 \) mod 8, then one may take \( M_p = -I \), and thus \( \chi_2(p) = +1 \) since \(-I \) acts trivially on \( f \). If \( p \equiv 5 \) mod 8, then put \( A = (\frac{5}{8} \quad 0) \in SL_2(\mathbb{Z}) \) and take \( M_p := (A \quad 0 \quad 0 \quad 0) \). Since \( M_p \in \Gamma(2, 4, 8) \), it also acts trivially on \( f \). Therefore the character \( \chi_2 \) is trivial (for any modular form on \( \Gamma(2, 4, 8) \)).

7.3. The action of the Hecke operators is given by the formulas in [2]. In fact, if

\[
f(\tau) = \sum_N a_N \exp \left( \frac{2\pi i}{8} \text{trace}(N\tau) \right), \quad (T_p f)(\tau) = \sum_N b_N \exp \left( \frac{2\pi i}{8} \text{trace}(N\tau) \right),
\]

are the Fourier-Jacobi series of \( f \) and \( T_p f \), then explicit formulas expressing \( b_N \) in terms of \( a_N \) and \( p^i \) are given in [2]. We will write

\[
a_N = a(n, r, m), \quad \text{with} \quad N = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}
\]
a positive definite half-integral matrix.
In case the quadratic form $nx^2 + rxy + my^2$ has no nontrivial zeros mod $p$, then one simply has $b(n, r, m) = a(p'n, p'r, p'm)$. We used this fact often in our computations.

To find the eigenspaces for the Hecke action, we will use the following proposition. It is an easy generalization of Satz 32 from [7].

7.4. **Proposition.** Let $\Gamma$ be a subgroup of $Sp_{2g}(\mathbb{Z})$, with $\Gamma_g(q) \subset \Gamma \subset Sp_{2g}(\mathbb{Z})$. Define

$$\overline{\Gamma} := \pi(\Gamma), \quad \text{with} \quad \pi: Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/q\mathbb{Z})$$

the reduction map. Assume that for every $n \in (\mathbb{Z}/q\mathbb{Z})^*$ and every $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \overline{\Gamma}$ one has that $\left( \begin{array}{cc} A & -B \\ C & D \end{array} \right) \in \overline{\Gamma}$.

Then the Hecke operator $T_n$ maps the space

$$M_k(\Gamma, \chi) := \{ f \in M_k(\Gamma_g(q)) : f|M = \chi(M)f, \quad \forall M \in \Gamma \}$$

to the space $M_k(\Gamma, \chi')$, where $\chi' : \overline{\Gamma} \to \mathbb{C}^*$ is given by

$$\chi'(M) := \chi(M'), \quad \text{with} \quad M' \in \Gamma, \quad M' \equiv \left( \begin{array}{cc} A & -B \\ nC & D \end{array} \right) \mod q.$$

**Proof.** The Hecke operator $T_n$ is defined as a sum:

$$T_n f := \sum_k f|H_k, \quad H_k \equiv D_n := \left( \begin{array}{cc} I & 0 \\ 0 & nI \end{array} \right) \mod q$$

and where $\Gamma(q)D_n\Gamma(q) = \bigsqcup_{k \in J} \Gamma(q)H_k$, a disjoint union. By [3, Lemma 1.1 (2)], one then also has

$$\Gamma D_n \Gamma = \bigsqcup_{k \in J} \Gamma H_k.$$

For any $M \in \Gamma$, the matrices $H_kM$, $k \in J$, are then also a set of coset representatives. Therefore, there is a permutation $\sigma = \sigma_M : J \to J$ and there are $M_k \in \Gamma$ such that

$$H_kM = M_kH_{\sigma(k)}, \quad \text{and thus} \quad M_k \equiv D_nMD_n^{-1} \mod q.$$

Given $M$, the matrices $M_k$ are thus all congruent mod $q$ to a matrix $M'$. By the assumption on $\Gamma$, we can choose $M' \in \Gamma$. Therefore, for $f \in M_k(\Gamma, \chi)$ we obtain

$$(T_n f)|M = \sum_k f|(H_kM) = \sum_k f|M_kH_{\sigma(k)}$$

$$= \sum_k \chi(M')f|H_{\sigma(k)} = \chi(M')T_n f.$$ 

The form $T_n f$ thus has the character $M \mapsto \chi(M')$. $\square$
7.5. **Proposition.** The following cusp forms are Hecke eigenforms:

\[ F_i := f_i, \quad i = 1, 5, 6, 7, \]
\[ F_2 := f_2 - 4f_2' = \theta[0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0], \]
\[ F_3 := f_3 + 16f_3' = \theta[0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0], \]
\[ F_4 := f_4 + 4f_4' = \theta[0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0], \]
\[ g_i, \quad i = 1, 2, 3, 4. \]

*Proof.* Recall that the \( f_i \) and the \( g_i \) are determined by their character (cf. Theorem 6.4). Since both \( \Gamma (2) \) and \( \Gamma (2)' \) satisfy the conditions of Proposition 7.4, we have that \( T_n f_i \) and \( T_n g_i \) are (up to a scalar multiple) determined by a character.

An explicit computation shows that \( T_p f \) has the same character if \( p \equiv 1 \mod 4 \), and it has character \( \overline{X_f} \), the complex conjugate of \( X_f \), if \( p \equiv 3 \mod 4 \), where in fact \( f \in S_3(\Gamma(2, 4, 8)) \) can be any cusp form determined with a character.

The space spanned by such a cusp form \( f \) and its translates by the Hecke action is thus at most 2-dimensional. In particular, if there is no cusp form with character \( \overline{X_f} \) or if \( X_f \) is real-valued, then \( f \) is an eigenform. Using a computer, one then finds the eigenforms listed. \( \square \)

7.6. In the table below we list the coefficients \( a_p, a_{p^2} \) of the Hecke polynomials corresponding to these eigenforms.

<table>
<thead>
<tr>
<th>coef.</th>
<th>( F_1 )</th>
<th>( F_2 )</th>
<th>( F_3 )</th>
<th>( F_4 )</th>
<th>( F_5 )</th>
<th>( F_6 )</th>
<th>( g_1 )</th>
<th>( g_2 )</th>
<th>( g_3 )</th>
<th>( g_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_3 )</td>
<td>8</td>
<td>8</td>
<td>16</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>( a_{3^2} )</td>
<td>6</td>
<td>6</td>
<td>102</td>
<td>54</td>
<td>54</td>
<td>6</td>
<td>-18</td>
<td>6</td>
<td>30</td>
<td>6</td>
</tr>
<tr>
<td>( a_5 )</td>
<td>28</td>
<td>28</td>
<td>28</td>
<td>20</td>
<td>12</td>
<td>-12</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>( a_{5^2} )</td>
<td>190</td>
<td>190</td>
<td>190</td>
<td>350</td>
<td>30</td>
<td>30</td>
<td>-130</td>
<td>70</td>
<td>-10</td>
<td>230</td>
</tr>
<tr>
<td>( a_7 )</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>32</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a_{7^2} )</td>
<td>2030</td>
<td>2030</td>
<td>-658</td>
<td>686</td>
<td>686</td>
<td>686</td>
<td>238</td>
<td>686</td>
<td>-18</td>
<td>-210</td>
</tr>
<tr>
<td>( a_{11} )</td>
<td>88</td>
<td>88</td>
<td>176</td>
<td>-40</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>88</td>
</tr>
<tr>
<td>( a_{11^2} )</td>
<td>-3146</td>
<td>-3146</td>
<td>8470</td>
<td>2662</td>
<td>2662</td>
<td>2662</td>
<td>6</td>
<td>1694</td>
<td>-330</td>
<td>462</td>
</tr>
<tr>
<td>( a_{13} )</td>
<td>204</td>
<td>204</td>
<td>204</td>
<td>-28</td>
<td>60</td>
<td>-60</td>
<td>84</td>
<td>0</td>
<td>0</td>
<td>-80</td>
</tr>
<tr>
<td>( a_{13^2} )</td>
<td>8398</td>
<td>8398</td>
<td>494</td>
<td>2290</td>
<td>4238</td>
<td>3094</td>
<td>-442</td>
<td>3510</td>
<td>390</td>
<td></td>
</tr>
<tr>
<td>( a_{17} )</td>
<td>356</td>
<td>356</td>
<td>356</td>
<td>4</td>
<td>-60</td>
<td>-60</td>
<td>36</td>
<td>-180</td>
<td>-92</td>
<td>20</td>
</tr>
<tr>
<td>( a_{17^2} )</td>
<td>25126</td>
<td>25126</td>
<td>8806</td>
<td>6630</td>
<td>-3162</td>
<td>15878</td>
<td>9894</td>
<td>-6970</td>
<td>25126</td>
<td></td>
</tr>
<tr>
<td>( a_{19} )</td>
<td>424</td>
<td>424</td>
<td>336</td>
<td>40</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>424</td>
</tr>
<tr>
<td>( a_{19^2} )</td>
<td>30438</td>
<td>30438</td>
<td>-3002</td>
<td>13718</td>
<td>13718</td>
<td>13718</td>
<td>10982</td>
<td>-12274</td>
<td>-8474</td>
<td>9918</td>
</tr>
</tbody>
</table>

8. The Andrianov L-functions

In the table below we list the Fourier coefficients of some elliptic modular new forms which appear to be related to the Siegel cusp forms listed above.
**THE CUSP FORMS OF WEIGHT 3 ON $\Gamma_2(2, 4, 8)$**

<table>
<thead>
<tr>
<th>form</th>
<th>space</th>
<th>$a_3$</th>
<th>$a_5$</th>
<th>$a_7$</th>
<th>$a_{11}$</th>
<th>$a_{13}$</th>
<th>$a_{17}$</th>
<th>$a_{19}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>$S_2(\Gamma_0(32))$</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>$S_3(\Gamma_0(32, (-1)))$</td>
<td>4i</td>
<td>2</td>
<td>-8i</td>
<td>-4i</td>
<td>-14</td>
<td>18</td>
<td>12i</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>$S_4(\Gamma_0(8))$</td>
<td>-4</td>
<td>-2</td>
<td>24</td>
<td>-44</td>
<td>22</td>
<td>50</td>
<td>44</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>$S_4(\Gamma_0(32))$</td>
<td>0</td>
<td>22</td>
<td>0</td>
<td>0</td>
<td>-18</td>
<td>-94</td>
<td>0</td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>$S_4(\Gamma_0(32))$</td>
<td>8</td>
<td>-10</td>
<td>16</td>
<td>-40</td>
<td>-50</td>
<td>-30</td>
<td>40</td>
</tr>
</tbody>
</table>

8.1. The cusp form $F_1$ was studied in [6], where it was proven that $F_1$ is the Saito-Kurokawa lift of the elliptic modular form $\rho_1 \in S_4(\Gamma_0(8))$. Therefore,

$$L(F_1, s) = \zeta_Q(s-1)\zeta_Q(s-2)L(\rho_1, s).$$

(One easily checks that indeed $H_p(X) = (X - p)(X - p^2)(X^2 - a_p X + p^3)$ with the $a_p$ from $\rho_1$.)

8.2. The first Hecke polynomials of $F_2$ suggest that its L-function is the same as that of $F_1$:

$$L(F_2, s) \overset{?}{=} \zeta_Q(s-1)\zeta_Q(s-2)L(\rho_1, s).$$

8.3. The L-function of $F_3$ also appears to be a twisted form of the L-function of $F_1$:

$$L(F_3, s) \overset{?}{=} \zeta_Q(s-1)\zeta_Q(s-2)L(\rho_1^{(3)}, s),$$

where the $(3)$ stands for twisting at the primes $3 \mod 4$ (the L-function $L(\rho_1^{(3)}, s)$ is the L-function of a cusp form of weight 3 on $\Gamma_0(16)$).

8.4. The L-function of $F_4$ appears to be the product of two elliptic modular L-functions:

$$L(F_4, s) \overset{?}{=} L(\phi_1^{(-2)}, s-1)L(\rho_3^{(-2)}, s),$$

where the $(2)$ stands for twisting at primes $\equiv 5, 7 \mod 8$.

8.5. The modular form $F_5$ was also studied in [6], in fact it defines the holomorphic 3-form on the threefold $Y$ studied there. Its L-series seems to be

$$L(F_5, s) \overset{?}{=} L(\phi_1, s-1)L(\rho_2, s).$$

8.6. The L-function of $F_6$ seems similar to the L-function of $F_5$:

$$L(F_6, s) \overset{?}{=} L(\phi_1^{(5)}, s-1)L(\rho_2^{(5)}, s),$$

where the $(5)$ stands for twisting at the primes $5 \mod 8$ (note that one can also twist at the primes $3 \mod 8$ (or $7 \mod 8$) without changing the L-functions).
8.7. The L-function of $F_7$ seems to be related to the L-function of a Galois representation $\pi$ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is the tensor product of the Galois representations corresponding to $\phi_1$ (a CM representation) and to $\psi_1$. At least for all primes $\leq 19$ we have that the roots of the Hecke polynomial of $F_7$ are of the form $\alpha_i\beta_j$ with $\alpha_i$ the roots of the Hecke polynomial of $\phi_1$ and $\beta_j$ the roots of the Hecke polynomial $\psi_1$.

8.8. The form $g_1$ appears to be related to a Hecke character $\chi$ of the field $K = \mathbb{Q}(\zeta_8)$.

We define a Hecke character

$$\chi : A_K^* \rightarrow \mathbb{C}^*, \quad \chi(x) = \prod_{p} \chi_p(x),$$

with $A_K^*$ the ideles of $K$ and the product is taken over all places of $K$. The character will be unramified outside the prime over 2, which we will denote by $\nu$. Since the class number of $K$ is one and $\chi$ is trivial on $K^*$ (embedded diagonally), it suffices to define only the infinite components and the component $\chi_\nu$ at the prime over 2. In fact, it suffices to define only the restriction of $\chi_\nu$ to $\mathcal{O}_K^*$. We give these data below.

As places at infinity we choose the complex embeddings $\sigma_i : K \hookrightarrow \mathbb{C}^*$,

$$\sigma_1 : \zeta_8 \mapsto e^{\pi iz}, \quad \sigma_2 : \zeta_8 \mapsto e^{\pi i2z},$$

The infinity components of $\chi$ we then define by

$$\chi_{\infty,i} : \mathbb{C}^* \rightarrow \mathbb{C}^*, \quad \chi_{\infty,1}(z) := z^{-3}, \quad \chi_{\infty,3}(z) := z^{-1}z^{-2}.$$ We define $\mathcal{O}_K^*/(1 + \pi_\nu^4 \mathcal{O}) = \mathcal{O}_K^*/(1 + 2\mathcal{O}) \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ (where the first factor is generated by the image of $\zeta_8$), the projection to the second factor will give a character, which is the restriction to $\mathcal{O}_K^*$ of the desired one:

$$\chi_\nu : \mathcal{O}_K^* \rightarrow \{\pm 1\} \subset \mathbb{C}^*.$$ (With $\pi_\nu = 1 - \zeta_8$ as local parameter at $\nu$, the subgroup generated by $\zeta_8$ mod $1 + 2\mathcal{O}_K$ is just: $1, 1 + \pi_\nu, 1 + \pi_\nu^2, 1 + \pi_\nu + \pi_\nu^3$, so $\chi_\nu$ is trivial on these, and not trivial on the other four). Since any unit in $\mathcal{O}_K^*$ can be written as $u = \zeta_8^j(1 - \sqrt{2})^{\pm j}$, one has that $\chi_{\infty,1}(u)\chi_{\infty,3}(u)\chi_\nu(u) = 1$, and thus these data define indeed a Hecke character.

The L-function of $\chi$ is given by

$$L(\chi, s) := (2\pi)^{-1-2s}\Gamma(s)\Gamma(s-1) \prod_p (1 - \chi_p(\pi_\nu))^N \varphi^{-s})^{-1},$$

where the product is now over all primes except the ones at infinity and $\nu$, which is ramified. To facilitate comparison with the Hecke polynomials, we define

$$H_{\chi, p}(X) := \prod_{p|\nu} (X^{e_p} - \chi_p(\pi_p)),$$

where $e_p := [\mathcal{O}_p/(\varphi) : \mathbb{F}_p]$ is the degree of the residue field extension. Then the equality $L(g_1, s) = L(\chi, s)$ is equivalent to $H_p(X) = H_{\chi, p}(X)$ for all $p$. 
8.9. To compute the $H_{\chi, p}$, we choose a generator $\pi_p$ in $\mathcal{O}_K$ for each of the primes $\varphi$ over $p$. Then

$$\chi_p(\pi_p) = \chi(1, 1, \ldots, 1, \pi_p, 1, \ldots)$$

$$= \chi(\pi_p^{-1}, \pi_p^{-1}, \ldots, \pi_p^{-1}, 1, \pi_p^{-1}, \ldots)$$

$$= \chi_{\infty, 1}(\pi_p^{-1})\chi_{\infty, 3}(\pi_p^{-1})\chi_\nu(\pi_p^{-1}),$$

where the last step follows because $\chi$ is unramified outside $\nu$.

In particular, if $p \equiv 7 \pmod{8}$, then there are two primes over $p$, and the generators $\pi_p$ and $\pi'_p$ of these prime ideals can be chosen to lie in $\mathbb{Z}[\sqrt{2}]$. Writing $\pi_p = a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$, we have $a^2 - 2b^2 = p$, and thus $a$ and $b$ are odd. Since $\sqrt{2} = \pi_p^2 + \pi'_p^2 \in \mathcal{O}_\nu/(2)$, we get $\pi_p = 1 + \pi_p^2 + \pi'_p^2 \in \mathcal{O}_\nu^*/(1 + 2\mathcal{O}_\nu)$, and thus $\chi_\nu(\pi_p^{-1}) = -1$. For the infinite places one finds (with $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$) that $\chi_{\infty, 1}(\pi_p^{-1}) = (a + b\sqrt{2})^3$ and $\chi_{\infty, 3}(\pi_p^{-1}) = (a - b\sqrt{2})^3$. Therefore,

$$\chi(\pi_p) = -p^3, \quad \text{and} \quad H_{\chi, p} = (X^2 + p^3)^2.$$

If $p \equiv 5 \pmod{8}$, then the generators for the two primes over $p$ can be chosen to lie in $\mathbb{Z}[i]$. Choosing a generator $\pi_p = a + bi$ with $a$ odd and $b$ even for such a prime, one finds that $\chi_\nu(\pi_p^{-1}) = +1$, $\chi_{\infty, 1}(\pi_p^{-1}) = (a + bi)^3$ and $\chi_{\infty, 3}(\pi_p^{-1}) = p(a + bi)$. Therefore,

$$\chi(\pi_p) = p(a + bi)^4, \quad \text{and} \quad H_{\chi, p} = (X^2 - p(a + bi)^4)(X^2 - p(a - bi)^4).$$

If $p \equiv 3 \pmod{8}$, then we choose the generators in $\mathbb{Z}[\sqrt{-2}]$ and let $\pi_p = a + b\sqrt{-2}$ be one of them. Since $a^2 + 2b^2 = p$, we must have $a$ and $b$ odd. Then $\pi_p = 1 + \pi_p^2 + \pi'_p^2 + \ldots \in \mathcal{O}_\nu^*$ and thus $\chi_\nu(\pi_p^{-1}) = -1$. Furthermore, $\chi_{\infty, 1}(\pi_p^{-1}) = (a + b\sqrt{-2})^3$ and $\chi_{\infty, 3}(\pi_p^{-1}) = p(a - b\sqrt{-2})$. Therefore,

$$\chi(\pi_p) = -p^2(a + b\sqrt{-2})^2$$

and

$$H_{\chi, p} = (X^2 + p^2(a + b\sqrt{-2})^2)(X^2 + p^2(a - b\sqrt{-2})^2).$$

For $p \equiv 1 \pmod{8}$, there are four primes over $p$ and the Hecke polynomial is not so easy to describe. However, one can check that indeed $H_{\chi, 17} = H_{17}$.

8.10. We were not able to write the L-function of $g_2$ as a (product of) 'known' L-functions.

8.11. The Hecke polynomials of $g_3$ have similar properties to those of $F_7$, with $\psi_1$ replaced by a form in $S_3(\Gamma_0(2^3), (-2))$.

8.12. The modular form $g_4$ also seems to be a (twisted) Saito-Kurokawa lift of the form $\rho_1$:

$$L(g_4, s) \equiv L(\chi^{(-2)}, s - 1)L(\chi^{(-2)}, s - 2)L(\rho_1, s),$$
with \( \chi^{(-2)} : (\mathbb{Z}/8\mathbb{Z})^* \to \{ \pm 1 \} \) the Dirichlet character with \( \chi^{(-2)}(5) = \chi^{(-2)}(7) = -1 \).

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