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A SEARCH FOR ALIQUOT CYCLES AND AMICABLE PAIRS

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ABSTRACT. A search for aliquot cycles below $3.6 \cdot 10^{10}$ and amicable pairs below 10^{11} is described. Three new cycles of length 4 and one new cycle of length 6 are exhibited. Four triples of amicable pairs with the same pair-sum are also exhibited.

The idea of an *aliquot cycle* is an extension of the concepts of *perfect numbers* and *amicable pairs*, which are aliquot cycles of lengths 1 and 2, respectively. Let $\sigma(n)$ be the sum of the divisors of n, where n is a natural number. Define $s(n)$ to be the sum of the divisors of n exclusive of n, that is, $s(n) = \sigma(n) - n$. An *aliquot cycle* of length k is then a finite sequence of distinct natural numbers $(a_1, \dots, a_k)$ such that $a_1 = s(a_k)$, and for each $i = 1, \dots, k-1$, $a_{i+1} = s(a_i)$. In an earlier paper [7] we described a search for aliquot cycles that covered some of the cycles with length 32 or less and smallest element below 10^{10}. A. Flammenkamp [5] recently reported a search for aliquot cycles that covered some of the cycles with length 50 or less and smallest element below $5 \cdot 10^9$. To date, including the results of these searches, 35 cycles with length at least 3 are known, 30 of length 4, two of length 8, and one each of lengths 5, 9, and 28 [1, 2, 4, 5, 7, 8, 13]. In this paper we describe a new search to find all aliquot cycles, of any length, such that the element preceding the largest element of the aliquot cycle does not exceed $3.6 \cdot 10^{10}$, and all amicable pairs with smaller element not exceeding 10^{11}.

In the aliquot cycle search, the method used was similar to that used in the third search in [7]. For all a_1 between 2 and $3.6 \cdot 10^{10}$, $a_2 = s(a_1)$ was constructed. The number a_2 was then the putative largest member of the aliquot cycle; hence if a_2 was less than a_1, the search stopped immediately. Otherwise, $a_i = s(a_{i-1})$ was constructed for $i = 3, 4$, and so on. If a_i ever became equal to a_1, iteration stopped and a new aliquot cycle was recorded. On the other hand, if a_i became equal to 1, a_i exceeded a_2, the sequence of a_i's fell into a cycle of length dividing 4 not involving a_1, a_i equalled 12496, the smallest member of the only known aliquot cycle of length 5, or a_i equalled 14316, the smallest member of the only known aliquot cycle of length 28, iteration stopped and a new cycle was not recorded. If i became equal to 5001, iteration stopped

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TABLE 1. New aliquot cycles with length at least 3

1	$(15837081520 = 2^4 \cdot 5 \cdot 13 \cdot 263 \cdot 57901$,	$23967995792 = 2^4 \cdot 13 \cdot 115230749$,
	$26042149708 = 2^2 \cdot 37 \cdot 137 \cdot 1284383$,	$21105018164 = 2^2 \cdot 4457 \cdot 1183813)$
2	$(17616303220 = 2^2 \cdot 5 \cdot 79 \cdot 227 \cdot 49117$,	$20012014220 = 2^2 \cdot 5 \cdot 11 \cdot 2357 \cdot 38593$,
	$25854330388 = 2^2 \cdot 17 \cdot 1123 \cdot 338567$,	$22095024044 = 2^2 \cdot 37 \cdot 149290703)$
3	$(21548919483 = 3^5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 431$,	$23625285957 = 3^5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 29 \cdot 277$,
	$24825443643 = 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 20719$,	$26762383557 = 3^4 \cdot 7^2 \cdot 13 \cdot 19 \cdot 27299$,
	$25958284443 = 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 167 \cdot 1427$,	$23816997477 = 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 218651)$
4	$(21669628904 = 2^3 \cdot 7 \cdot 11 \cdot 35177969$,	$28986647896 = 2^3 \cdot 15349 \cdot 236063$,
	$25367088104 = 2^3 \cdot 29 \cdot 179 \cdot 610843$,	$24111275896 = 2^3 \cdot 79 \cdot 38150753)$

and the sequence was examined by hand. In all these latter cases, 153 in all, it was found that the sequence had fallen into a known aliquot cycle of length 8 or 9.

As discussed in [7], the computation of $s(a_1)$ can be sped up using a sieving method, and the computation of subsequent $s(a_i)$'s, which must fall into a specified interval if the computation is to continue, can be sped up by noting that a partial factorization of a_i is sometimes enough to show that $s(a_i)$ falls outside an interval. Searching for aliquot cycles of arbitrary length is successful because the average length of the aliquot sequence which must be searched for each integer is small. Each new member of the aliquot sequence requires the factorization of the previous member. For integers near 10^9 the average number of factorizations required per integer was 2.40, while for integers near $2.5 \cdot 10^{10}$, the average number of factorizations required was 2.53.

The search was performed on an IBM ES 9000-580 and took about 2500 CPU hours. Besides already-known aliquot cycles, three cycles of length 4 and a cycle of length 6 were the only aliquot cycles found with length at least 3. These cycles are shown in Table 1. We believe this to be the first example of a cycle of length 6.

The search just described found all amicable pairs with smaller member less than $3.6 \cdot 10^{10}$. The search was continued to find all amicable pairs with smaller member less than 10^{11}, taking approximately 350 hours on the ES 9000-580. For all a_1 between $3.6 \cdot 10^{10}$ and 10^{11} not divisible by six, $a_2 = s(a_1)$ was computed. If a_2 was less than a_1, $s(a_2)$ was not computed, since a_1 was supposed to be the smaller member of the amicable pair. If a_1 and a_2 were both even and a_2 was twice a_1 or more, (a_1, a_2) could not have been an amicable pair [6], so $s(a_2)$ was not computed. Otherwise, $s(a_2)$ was checked for equality to a_1, possibly giving a new amicable pair. If a_1 was divisible by six, an amicable pair (a_1, a_2) would have had to have been of the form $(2^\mu M^2, N^2)$ for some μ and odd M and N [9]. For all a_1 less than 10^{11} divisible by six and equal to a square or twice a square, $s(a_1)$ was checked to see if it was a perfect square, which it never was. Hence no amicable pairs with smaller member divisible by six were found.

In all, a total of 3340 amicable pairs with smaller member less than 10^{11} were found. A list of all amicable pairs with smaller member less than 10^{10} [9] contains 1427 of these. Of the remaining 1913 pairs, 419, those with smaller member between 10^{10} and $2 \cdot 10^{10}$, were computed by te Riele in a 1990 search [10], 275 can be found in [12], three in [9], two in [3], and 40 in an unpublished list [11]. (Some can be found in more than one of these places.) We expect that many of the remaining 1263 are new. C. Pomerance [9] has asked whether there

TABLE 2. Triples of amicable pairs with the same pair-sum

$(29912035725 = 3^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 431 \cdot 719,$	$34883817075 = 3^3 \cdot 5^2 \cdot 127 \cdot 503 \cdot 809)$
$(31695652275 = 3 \cdot 5^2 \cdot 7 \cdot 19 \cdot 53 \cdot 167 \cdot 359,$	$33100200525 = 3 \cdot 5^2 \cdot 7 \cdot 29 \cdot 971 \cdot 2239)$
$(32129958525 = 3 \cdot 5^2 \cdot 7 \cdot 31 \cdot 41 \cdot 179 \cdot 269,$	$32665894275 = 3 \cdot 5^2 \cdot 7 \cdot 31 \cdot 59 \cdot 34019)$
Sum:	$64795852800 = 2^{14} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 31$
$(54666647145 = 3^3 \cdot 5 \cdot 11 \cdot 19 \cdot 1049 \cdot 1847,$	$57100392855 = 3^3 \cdot 5 \cdot 11 \cdot 251 \cdot 307 \cdot 499)$
$(54853467435 = 3^3 \cdot 5 \cdot 13 \cdot 17 \cdot 199 \cdot 9239,$	$56913572565 = 3^3 \cdot 5 \cdot 23 \cdot 29 \cdot 43 \cdot 14699)$
$(55171066784 = 2^5 \cdot 83 \cdot 109 \cdot 149 \cdot 1279,$	$56595973216 = 2^5 \cdot 499 \cdot 1583 \cdot 2239)$
Sum:	$111767040000 = 2^{12} \cdot 3^4 \cdot 5^4 \cdot 7^2 \cdot 11$
$(52025880375 = 3^2 \cdot 5^3 \cdot 7 \cdot 29 \cdot 37 \cdot 47 \cdot 131,$	$65160720585 = 3^2 \cdot 5 \cdot 43 \cdot 113 \cdot 233 \cdot 1279)$
$(53734975875 = 3^2 \cdot 5^3 \cdot 7 \cdot 19 \cdot 197 \cdot 1823,$	$63451625085 = 3^2 \cdot 5 \cdot 19 \cdot 103 \cdot 607 \cdot 1187)$
$(55477298835 = 3^2 \cdot 5 \cdot 7 \cdot 31 \cdot 37 \cdot 233 \cdot 659,$	$61709302125 = 3^2 \cdot 5^3 \cdot 47 \cdot 59 \cdot 131 \cdot 151)$
Sum:	$117186600960 = 2^{13} \cdot 3^4 \cdot 5 \cdot 11 \cdot 13^2 \cdot 19$
$(57826671370 = 2 \cdot 5 \cdot 13 \cdot 37 \cdot 47 \cdot 179 \cdot 1429,$	$60486723830 = 2 \cdot 5 \cdot 13 \cdot 151 \cdot 439 \cdot 7019)$
$(58557943665 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 47 \cdot 197 \cdot 569,$	$59755451535 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 131 \cdot 41039)$
$(58906037421 = 3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 43 \cdot 89 \cdot 167,$	$59407357779 = 3^2 \cdot 7 \cdot 11^2 \cdot 19 \cdot 53 \cdot 71 \cdot 109)$
Sum:	$118313395200 = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19$

exist pairs, triples, and so forth of amicable pairs having the same pair-sum. In [9], 37 pairs of amicable pairs having the same pair-sum are listed. We believe that we have located the first triple of amicable pairs with the same pair-sum. In fact, we found four triples in all, shown in Table 2.

There were, in all, 70 pairs of amicable pairs with the same pair-sum in our list of pairs. The pair with the smallest ratio of smaller member to larger member was

$$(35049418250 = 2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 23 \cdot 43 \cdot 263,$$
$$54192685558 = 2 \cdot 11 \cdot 191 \cdot 967 \cdot 13337),$$

with ratio 0.646756, and the pair with the largest ratio was

$$(85617896265 = 3^4 \cdot 5 \cdot 11 \cdot 53 \cdot 349 \cdot 1039,$$
$$85625175735 = 3^4 \cdot 5 \cdot 11 \cdot 59 \cdot 199 \cdot 1637),$$

with ratio 0.999915.

The authors will provide interested parties with a list of amicable pairs found upon request. In addition, we have deposited this list in the UMT file.

Added in proof. We have since extended our aliquot cycle search to cover all cycles with member preceding the largest not exceeding $6.5 \cdot 10^{10}$. Nothing new was found except for a 4-cycle with smallest member 44379752648.

BIBLIOGRAPHY

1. W. Borho, *Über die Fixpunkte der k-fach iterierten Teilersummenfunktion*, Mitt. Math. Gesellsch. Hamburg **9** (1969), 34–48.
2. H. Cohen, *On amicable and sociable numbers*, Math. Comp. **24** (1970), 423–429.
3. P. Costello, *Amicable pairs of the form* $(i, 1)$, Math. Comp. **56** (1991), 859–865.
4. J. S. Devitt, R. K. Guy, and J. L. Selfridge, *Third report on aliquot sequences*, Proc. 6th Manitoba Conf. Numerical Math. 1976, Congr. Numer. XVIII, Utilitas Math., Winnipeg, Manitoba, 1977, pp. 177–204.

5. A. Flammenkamp, *New sociable numbers*, Math. Comp. **56** (1991), 871–873.

6. E. J. Lee and J. S. Madachy, *The history and discovery of amicable numbers—Part* I, J. Recreational Math. **5** (1972), 77–93.

7. D. Moews and P. C. Moews, *A search for aliquot cycles below* 10^{10}, Math. Comp. **57** (1991), 849–855.

8. P. Poulet, #4865, L'intermédiare des math. **25** (1918), 100–101.

9. H. J. J. te Riele, *Computation of all the amicable pairs below* 10^{10}, Math. Comp. **47** (1986), 361–368.

10. ——, personal communication, Dec. 9, 1991.

11. ——, personal communication, Jan. 28, 1992.

12. H. J. J. te Riele, W. Borho, S. Battiato, H. Hoffmann, and E. J. Lee, *Table of amicable pairs between* 10^{10} *and* 10^{52}, Centrum voor Wiskunde en Informatica, Note NM-N8603, Stichting Math. Centrum, Amsterdam, 1986.

13. S. C. Root, Item 61, M.I.T. Artificial Intelligence Memo no. 239, Feb. 29, 1972.

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