THE DECOMPOSITION NUMBERS
OF THE HECKE ALGEBRA OF TYPE $E_6^*$

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Abstract. Let $E_6(q)$ be the Chevalley group of type $E_6$ over a finite field with $q$ elements, $l$ be a prime not dividing $q$, and $H_E(q)$ be the endomorphism ring of the permutation representation (over a valuation ring $R$ with residue class field of characteristic $l$) of $E_6(q)$ on the cosets of a standard Borel subgroup $B(q)$. Then the $l$-modular decomposition matrix $D_l$ of the algebra $H_E(q)$ is a submatrix of the $l$-modular decomposition matrix of the finite group $E_6(q)$. In this paper we determine the matrices $D_l$, for all $l, q$ as above. For this purpose, we consider the generic Hecke algebra $H$ associated with the finite Weyl group of type $E_6$ over the ring $A = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in an indeterminate $v$, and calculate the decomposition matrices of $H$ which are associated with specializations of $v$ to roots of unity over $\mathbb{Q}$ or values in a finite field. The computations were done by using the computer algebra systems MAPLE and GAP.

1. Introduction

Let $W$ be the finite Weyl group of type $E_6$, and let $S = \{s_1, \ldots, s_6\}$ be a corresponding set of simple reflections. The relations in $W$ are determined by the following diagram (the product of any two distinct generators has order 3 if they are joined by a line, and 2 otherwise):

\[
\begin{array}{cccccccc}
  & s_1 & s_2 & s_3 & s_5 & s_6 \\
 s_4 & & & & & & & \\
\end{array}
\]

Let $A = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in an indeterminate $v$. The generic Hecke algebra $H$ of $W$ over $A$ with parameter $u := v^2$ is a free $A$-module with basis $T_w$, $w \in W$, and multiplication defined by $T_s^2 = uT_s + (v - 1)T_s$, if $s \in S$, and $T_wT_{w'} = T_{ww'}$, if $l(ww') = l(w) + l(w')$, where $l(\cdot)$ denotes the usual length function on $W$. If $f: A \to K$ is a homomorphism into any field $K$ of characteristic 0, then we get a corresponding specialized algebra $K \otimes_A H$ over $K$ with parameter $f(u)$. The map $f$ induces a decomposition map $d_f$ between the Grothendieck groups of $\mathbb{Q}(v) \otimes_A H$ and...
Denoting by \( P(u) = \sum_{w \in W} u^{l(u)} \) the Poincaré polynomial of \( W \), we get that \( d_f \) is the "identity" if and only if \( P(f(u)) \neq 0 \) in \( K \). We have
\[
P(u) = \Phi_2(u)^4 \Phi_3(u)^3 \Phi_2(u)^2 \Phi_5(u) \Phi_6(u) \Phi_8(u) \Phi_9(u) \Phi_{12}(u),
\]
where \( \Phi_e(u) \) denotes the \( e \)th cyclotomic polynomial in \( \mathbb{Z}[u] \). So we only get nontrivial decomposition maps \( d_f \), if \( f(u) \) is a root of unity in \( K \).

Let \( 2e \) be a positive even integer, and \( \overline{v} \) be a primitive \((2e)\)th root of unity over \( \mathbb{Q} \). Then we have a specialization \( g_e : A \rightarrow \mathbb{Q}(\overline{v}) \), \( v \mapsto \overline{v} \), inducing the \( \Phi_e \)-modular decomposition map \( d_e \) of \( H \) (cf. [6]). We determine the associated decomposition matrices \( D_e \) for \( e = 2, 3, 4, 6 \), while the cases \( e = 5, 8, 9, 12 \) ("e-defect 1") have been settled in [6].

Let \( q \) be a prime power and \( l \) be a prime not dividing \( q \) such that \( l \) divides \( \Phi_e(q) \). It is shown in [6] that, in this situation and as a consequence of a theorem of R. Dipper (see [3, §2]), the matrix \( D_e \) is contained as a submatrix in the usual \( l \)-modular decomposition matrix of the finite Chevalley group \( E_6(q) \), provided that \( l \) is larger than a number \( N \) which only depends on \( W \) (and not on \( q, e \)). We determine this bound \( N \) and show what happens for "small" \( l \). To be more specific: There exists a valuation ring \( I \subseteq \mathbb{Q}(\overline{v}) \) with residue class field \( \mathbb{F}_l(\overline{q}^{1/2}) \) such that \( \overline{u} \) is mapped to \( \overline{q} \in \mathbb{F}_l \) under the canonical epimorphism. Thus we also get a decomposition map \( d'_l \) between the Grothendieck groups of \( \mathbb{Q}(\overline{v}) \otimes_A H \) and \( \mathbb{F}_l(\overline{q}^{1/2}) \otimes_A H \), and we have a factorization \( d_l = d'_l \circ d_e \) of the \( l \)-modular decomposition map of \( H \), hence a factorization of the corresponding decomposition matrices as \( D_l = D_e D'_l \).

Using the same techniques as in [7], we show that \( d'_l \) is the identity if \( l \) does not divide the order of \( W \) or, equivalently, that \( N = 5 \) (this is conjectured to be true for all finite Weyl groups in [6]). We also give explicit information for what happens if \( l = 2, 3, 5 \).

As an example of an application of the results of this paper, the first column (corresponding to the trivial module) of the \( l \)-modular decomposition matrix of the finite Chevalley group \( E_6(q) \) is explicitly known, for all \( l \) not dividing \( q \) (cf. [8, §2]).

2. The methods

The principal method for computing decomposition numbers is the same as in [7]: Construct a set of representations of \( H \) such that each irreducible appears at least once as a constituent, specialize the parameter \( v \), and decompose the specialized representations into irreducibles by using the Meat-Axe, see [9]. All computations were done by using the computer algebra systems GAP and MAPLE, see [10, 2].

One should note, however, that there is a principal difficulty arising from the fact that in characteristic 0 the Meat-Axe can only be applied if one has found elements in \( \mathbb{Q}(\overline{v}) \otimes_A H \) with null-space of dimension 1 in a given representation, and there is no simple or effective rule to produce such elements. And even if one has found such an element there are limitations on the degrees of the representations caused by the complexity of the necessary computations (spinning-up vectors, Gaussian elimination in characteristic 0, etc.).

Therefore, we had to keep computations in cyclotomic fields over \( \mathbb{Q} \) as limited as possible. This was achieved by combining results on \( l \)-modular decom-
position numbers with a result on the "generic degree" of a projective module over $\mathbb{Q}(v) \otimes_A H$, which we shall now explain.

For this result, we can assume that $W$ is an arbitrary finite Weyl group, and $H$ is the associated Hecke algebra over $A$ with parameter $u$.

Let $\phi_1, \ldots, \phi_n$ be the ordinary irreducible characters of $\mathbb{Q}(v) \otimes_A H$ and $\beta_1, \ldots, \beta_k$ be the $\Phi_e$-modular irreducible characters of $\mathbb{Q}(v) \otimes_A H$. Then the $\Phi_e$-modular decomposition numbers $d_{ij}$ are given by the equations

$$g_e(\phi_i) = \sum_{j=1}^k d_{ij} \beta_j, \quad i = 1, \ldots, n.$$ 

By Brauer reciprocity, the function $\Psi_j := \sum_{i=1}^n d_{ij} \phi_i$ is the character of a projective indecomposable $A_e \otimes_A H$-module corresponding to $\beta_j$, where $A_e$ is the localization of $\mathbb{Q}(v)$ at the irreducible polynomial $\Phi_{2e}(v)$. For $1 \leq i \leq n$, denote by $D_{\phi_i}(u) \in \mathbb{Q}[u]$ the generic degree associated with $\phi_i$. It has the property that $D_{\phi_i}(q)$ is the degree of the corresponding principal series representation of a finite Chevalley group $G(q)$ with Weyl group $W$. In [6] it is shown that $\Phi_e(u)$ divides the "generic degree"

$$D_{\Psi_j}(u) := \sum_{i=1}^n d_{ij} D_{\phi_i}(u) \in \mathbb{Q}[u]$$

of $\Psi_j$, for all $j$. This can be strengthened as follows.

**Proposition 2.1.** Let $\Phi_e(u)^d$ be the maximal power of $\Phi_e(u)$ dividing the Poincaré polynomial $P(u)$ of $W$. Then

$$\Phi_e(u)^d \text{ divides } D_{\Psi_j}(u) \text{ in } \mathbb{Q}[u].$$

(Note, however, that in general, $D_{\Psi_j}(u)$ is not a product of cyclotomic polynomials!)

**Proof.** Write $D_{\Psi_j}(u) = \Phi_e(u)^d f(u)$, where $d_j$ is a nonnegative integer and $f(u) \in \mathbb{Q}[u]$ is not divisible by $\Phi_e(u)$. Since $\Phi_e(u)$ is irreducible, we can find polynomials $a(u), b(u) \in \mathbb{Q}[u]$ such that

$$1 = a(u)\Phi_e(u)^d + b(u)f(u).$$

Now choose a prime $l$ which satisfies the following conditions: (i) $l > |W|$, (ii) $l$ does not divide the number $N$ of [6, Proposition 5.5], (iii) $l$ does not divide any denominator in the coefficients of the polynomials $a(u), b(u), f(u)$. Furthermore, choose a prime power $q$ such that $l$ divides $\Phi_e(q)$.

Let $G$ be a connected reductive group with Weyl group $W$, split over $\mathbb{F}_q$, with associated Frobenius map $F$. We may assume that $G$ is of adjoint type. Then the order of $G^F$ is given by $|G^F| = q^m(q-1)^r P(q)$, where $m$ is the number of positive roots of $W$, $r$ is the rank of $G$, and $P(u)$ is the Poincaré polynomial of $W$. Let $(K, R, \overline{R})$ be a splitting $l$-modular system for all subgroups of $G^F$. The specialized algebra $H_R(q)$ is isomorphic to the $RG^F$-endomorphism ring of the permutation module $P := 1^G_{BF}$, where $B$ is an $F$-stable Borel subgroup of $G$. The order of $B^F$ is given by $q^m(q-1)^r$. 
By condition (ii), the decomposition map \( d'_l \) is the identity, i.e., the \( l \)-
modular decomposition matrix of \( H \) coincides with the \( \Phi_e \)-modular decomposition matrix. By condition (i), the multiplicative order of \( q \) modulo \( l \) is \( e \), and \( \Phi_e(u) \) is the only cyclotomic factor in \( P(u) \) such that \( l \) divides \( \Phi_e(q) \) (see [5]). So the \( l \)-part of \(|G^F|\) is equal to the \( l \)-part of \( \Phi_e(q)^d \). This implies that the order of \( B^F \) is prime to \( l \), so \( P \) and every direct summand of \( P \) is a projective \( RG^F \)-module. Dipper's Theorem implies that the decomposition of \( P \) into indecomposable direct summands is determined by the decomposition of \( H \) into projective indecomposable modules (see [3, §2]). Putting things together, we see that \( D_{\Phi_j}(q) \) is the degree of a projective indecomposable \( RG^F \)-
module, for each \( j = 1, \ldots, k \), and it is therefore divisible by the \( l \)-part of \( \Phi_e(q)^d \).

By condition (iii) and equation (*), \( a(q), b(q), f(q) \) all lie in \( R \), and \( b(q), f(q) \) are units in \( R \). In particular, \( f(q) \) is not divisible by \( l \). So the \( l \)-part of \( D_{\Phi_j}(q) \) is equal to the \( l \)-part of \( \Phi_e(q)^d \). By what we just proved before, this \( l \)-part must be divisible by the \( l \)-part of \( \Phi_e(q)^d \), so we conclude that \( d \leq d_j \). □

From now on, \( W \) will denote the finite Weyl group of type \( E_6 \) with corre-

sponding generic Hecke algebra \( H \). The procedure for computing the \( \Phi_e \)-
modular and \( l \)-modular decomposition numbers of \( H \) can now be described

as follows.

(1) Construct a set of representations of \( H \) which have relatively “small” de-

gree and have a complete set of irreducibles for \( \mathbb{Q}(\nu) \otimes_A H \) as their constituents.

The following methods are available:

- Computing Kazhdan-Lusztig polynomials and left cell representations of \( H \). Here we used the GAP-programs of [7]. The decomposition of the various left cell representations is described in [1, p. 415]. In this way we constructed irreducible representations with characters \( \phi_1, 0, \phi_6, 1, \phi_20, 2, \phi_64, 4, \phi_60, 5, \phi_{81}, 6, \phi_{24}, 6, \phi_{15}, 4, \phi_{30}, 3, \phi_{15}, 5 \) (for the notations cf. [1, §13.2]; note that the irreducible characters of \( H \) are in bijective correspondence with the irreducible characters of \( W \)).

- Taking the dual of a representation. If \( \rho: H \to A^{n \times n} \) is a representation, then its dual \( \rho^* \) is defined by \( \rho^*(T_s) := \rho(-uT_s^{-1}) \), \( s \in S \) (cf. [6, (8.5)]). This construction is a substitute for tensoring with the sign representation in the Weyl group itself.

- Inducing representations from a parabolic subalgebra \( H' \) corresponding
to a subset \( S' \subseteq S \) (cf. [6, (8.6)]). In this way, we found representations containing the self-dual representations with characters \( \phi_{20}, 10, \phi_{60}, 8, \phi_{80}, 7, \phi_{90}, 8 \) as their constituents (using the parabolic subalgebra of type \( D_5 \) obtained by removing the node \( s_6 \) from the Dynkin diagram for which we could work out all irreducible representation as constituents of left cell representations).

- Extending a representation of a parabolic subalgebra. For example, an
irreducible representation \( \rho \) with character \( \phi_{10}, 9 \) remains irreducible when restricted to the parabolic subalgebra \( H' \) of type \( D_5 \) (remove node \( s_6 \)). If this restriction is known, then one has to find a matrix \( \rho_6 \) such that the defining relations for \( H \) are satisfied: \( \rho(T_{s_i})\rho_6 = \rho_6\rho(T_{s_i}) \), for \( i = 1, 2, 3, 4, \ldots \) (Of course, this can be carried out only for small degrees.)

(2) Fix \( e > 1 \) and choose a prime \( l > 5 \) (so \( l \) does not divide the order of
and a prime power $q$ such that $l$ and $q$ are coprime, $\bar{q} \in \mathbb{F}_l^\times$ has order $e$, and $\mathbb{F}_l$ contains a square root of $\bar{q}$ (example: $e = 3, l = 13, q = 3$). Then consider the specialization $v \mapsto \bar{q}^{1/2} \in \mathbb{F}_l$, and use the GAP implementation of R. Parker's Meat-Axe to decompose the specialized representations into irreducibles. Thus, we get the $l$-modular decomposition matrix $D_l$ of $H$. (For example, this is the matrix printed in Table B on p. 895.)

(3) Fix $e, q, l$ as in (2), and assume that we have found the $l$-modular decomposition matrix $D_l$ of $H$. By [6, Theorem 5.3], we have a factorization $D_l = D_eD_l'$, where $D_e$ is the $\Phi_e$-modular decomposition matrix of $H$. This means that the columns of $D_l$ are sums of the columns of $D_e$. Let $(a_1, \ldots, a_n)$ be a column of $D_e$. Then there is a column $(b_1, \ldots, b_n)$ of $D_l$ such that $a_i \leq b_i$ for all $i$. In particular, there are only finitely many possibilities for $(a_1, \ldots, a_n)$, and we have as an additional requirement that $\sum_i a_iD_{\phi_i(u)}$ must be divisible by the $\Phi_e(u)$-part of $P(u)$, by Proposition 2.1. Using MAPLE, we found that this condition forces $a_i = b_i$ for all $i$ in each case, with the only exception: $e = 3$ (with the choice $l = 13, q = 3$ as above) and columns 2 and 8 (in the matrix printed in Table B), where columns 5 (resp. 10) could possibly be subtracted. If this were the case, then the representations with characters $\phi_{30,3}$ and $\phi_{30,15}$ would remain irreducible under $\Phi_3$-modular reduction, which was shown to be false by directly using the Meat-Axe in characteristic 0 (use the element $T_S T_S T_S T_S T_S T_S T_S - T_1 \in H$: it has corank 1 in these representations and splits up proper subspaces). Thus, we get the $\Phi_e$-modular decomposition matrices of $H$.

(4) The decomposition maps $d_l'$ for $l \leq 5$ were determined again by using the Meat-Axe. In order to show that $d_l'$ is the identity for $l > 5$, we used the same techniques as explained in [7, Step IV in §3]. For this purpose, we had to construct explicitly matrices for a complete set of irreducible $\Phi_e$-modular representations in blocks of $e$-defect $> 1$. This was done by using the characteristic 0 Meat-Axe in GAP. As already mentioned, the most difficult problem is to find elements in $H$ which have corank 1 in a given representation. We systematically searched for such elements in the following way: (a) for $i \in \{1, \ldots, 5\}$, let $A := T_{s_i} \cdots T_{s_i}$ and $B := T_{s_{i+1}} \cdots T_{s_k}$; (b) compute the "Fingerprint" associated with $A$ and $B$ using the scheme described in [9, Table 4]; (c) replace $A$ by $AB$, or $B$ by $AB$, or $A$ by $B^{-1}AB$, and repeat Step (b); (d) for all elements computed so far, check whether 1 or $-1$ is an eigenvalue of multiplicity 1. (Of course, it is not guaranteed that we find elements with corank 1 in this way!)

In some cases it was also possible to show that an irreducible $\Phi_e$-modular representation remains irreducible under reduction modulo $l$ by restricting it to parabolic subalgebras of type $D_5$ (remove node $s_6$) and $A_5$ (remove node $s_4$), where computations were much easier. Consider the following example: There is an irreducible $\Phi_6$-modular representation $\rho$ of dimension 32. Its restriction to the parabolic subalgebra $D_5$ splits into irreducibles of dimensions 10, 11, 11; its restriction to the subalgebra $A_5$ splits into irreducibles of dimensions 16, 16. The cyclotomic polynomial $\Phi_6(u)$ divides the Poincaré polynomials of $A_5$ and $D_5$ just once. Hence we know that $d_l'$ for $A_5$ and $D_5$ is the identity if $l > 5$ by [6, Part II]. From the dimensions of the constituents of the restrictions of $\rho$ it is then clear that the reduction modulo $l$ of $\rho$ must remain irreducible, for $l > 5$. 

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3. The results

Now we collect the results obtained by the methods outlined in the previous section. Let \( e \in \{2, 3, 4, 6\} \), \( l \) be a prime, and \( q \) be a prime power such that \( l \) divides \( \Phi_e(q) \). As a result of the computations sketched in Step (4) of §2, we obtain:

(3.1) \textit{The decomposition map } \( d_l^e \textit{ is the identity if } l > 5 \).

This means that the decomposition matrix \( D_e \) is equal to the \( l \)-modular decomposition matrix \( D_l \) of \( H \), for \( l > 5 \). So, in this case, \( D_e \) is a submatrix of the \( l \)-modular decomposition matrix of the finite Chevalley group \( E_6(q) \). The ordering of the irreducible characters of \( H \) (and therefore of the corresponding principal series representations of \( E_6(q) \)) is chosen according to the general scheme described in [4, §3] (pulling back the partial order on unipotent conjugacy classes to unipotent characters via Lusztig's map). The notation for the irreducible characters of \( H \) is taken from [1, p. 415].

Table A. The \( \Phi_2 \)-blocks of \( H \) are: \( \{\phi_{1,0}, \phi_{6,1}, \phi_{20,2}, \phi_{15,5}, \phi_{30,3}, \phi_{15,4}, \phi_{60,5}, \phi_{24,6}, \phi_{81,6}, \phi_{20,10}, \phi_{90,8}, \phi_{60,8}, \phi_{10,9}, \phi_{81,10}, \phi_{60,11}, \phi_{24,12}, \phi_{15,17}, \phi_{30,15}, \phi_{15,16}, \phi_{20,20}, \phi_{6,25}, \phi_{1,36}\} \) with 2-defect 4, \( \{\phi_{64,4}, \phi_{64,13}\} \) with 2-defect 1, and \( \{\phi_{80,7}\} \) with 2-defect 0. The whole decomposition matrix is given as follows:

\[
\begin{array}{cccccccccc}
\phi_{1,0} & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{6,1} & \cdot & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{20,2} & \cdot & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{15,5} & \cdot & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{30,3} & \cdot & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{15,4} & \cdot & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\phi_{64,4} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdots & \cdots & \cdots \\
\phi_{60,5} & \cdot & \cdot & \cdot & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{24,6} & \cdot & \cdot & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{81,6} & \cdot & 1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots \\
\phi_{20,10} & \cdot & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\phi_{90,8} & \cdot & 1 & 2 & 1 & 1 & \cdots & \cdots & \cdots & \cdots \\
\phi_{60,8} & \cdot & \cdot & \cdot & 1 & 1 & \cdots & \cdots & \cdots & \cdots \\
\phi_{10,9} & \cdot & \cdot & \cdot & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{81,10} & \cdot & 1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots \\
\phi_{60,11} & \cdot & \cdot & \cdot & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{24,12} & \cdot & \cdot & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{64,13} & \cdot & \cdot & \cdot & \cdot & 1 & \cdots & \cdots & \cdots & \cdots \\
\phi_{15,17} & \cdot & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{30,15} & \cdot & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{15,16} & \cdot & \cdot & \cdot & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{20,20} & \cdot & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{6,25} & \cdot & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_{1,36} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdots & \cdots \\
\end{array}
\]

The elementary divisors of the Cartan matrix are 1, 1, 1, 2, 4, 4, 8, 128.
Table B. The $\Phi_3$-blocks of $H$ are: $\{\phi_{1,0}, \phi_{6,1}, \phi_{20,2}, \phi_{15,5}, 
\phi_{30,3}, \phi_{15,4}, \phi_{64,4}, \phi_{60,5}, \phi_{24,6}, \phi_{20,10}, \phi_{80,7}, \phi_{60,8}, \phi_{10,9}, \n\phi_{60,11}, \phi_{24,12}, \phi_{64,13}, \phi_{15,17}, \phi_{30,15}, \phi_{15,16}, \phi_{20,20}, \phi_{6,25}, \n\phi_{1,36}\}$ with 3-defect 3, and $\{\phi_{81,6}\}, \{\phi_{81,10}\}, \{\phi_{90,8}\}$ each with 3-defect 0. The whole decomposition matrix is given as follows:

|      | $\phi_{1,0}$ | $\phi_{6,1}$ | $\phi_{20,2}$ | $\phi_{15,5}$ | $\phi_{30,3}$ | $\phi_{15,4}$ | $\phi_{64,4}$ | $\phi_{60,5}$ | $\phi_{24,6}$ | $\phi_{20,10}$ | $\phi_{80,7}$ | $\phi_{60,8}$ | $\phi_{10,9}$ | $\phi_{81,6}$ | $\phi_{20,10}$ | $\phi_{80,7}$ | $\phi_{90,8}$ | $\phi_{60,8}$ | $\phi_{10,9}$ | $\phi_{81,10}$ | $\phi_{60,11}$ | $\phi_{24,12}$ | $\phi_{64,13}$ | $\phi_{15,17}$ | $\phi_{30,15}$ | $\phi_{15,16}$ | $\phi_{20,20}$ | $\phi_{6,25}$ | $\phi_{1,36}$ |
|------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
|      | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            | 1            |
|      |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |              |

The elementary divisors of the Cartan matrix are $1, 1, 1, 1, 1, 1, 3, 3, 3, 9, 9, 81$. 
Table C. The $Φ_4$-blocks of $H$ are: $\{ϕ_{1,0}, ϕ_{6,1}, ϕ_{15,5}, ϕ_{15,4}, ϕ_{81,6}, ϕ_{80,7}, ϕ_{90,8}, ϕ_{10,9}, ϕ_{81,10}, ϕ_{15,17}, ϕ_{15,16}, ϕ_{6,25}, ϕ_{1,36}\}$ with 4-defect 2, $\{ϕ_{20,2}, ϕ_{60,5}, ϕ_{60,11}, ϕ_{20,20}\}$ with 4-defect 1, and $\{ϕ_{30,3}, ϕ_{64,4}, ϕ_{24,6}, ϕ_{20,10}, ϕ_{60,8}, ϕ_{24,12}, ϕ_{64,13}, ϕ_{30,15}\}$ each with 4-defect 0. The whole decomposition matrix is given as follows:

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The elementary divisors of the Cartan matrix are $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$. 

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TABLE D. The $\Phi_6$-blocks of $H$ are: \{\phi_{1,0}, \phi_{6,1}, \phi_{20,2}, \phi_{30,3}, \\
\phi_{15,4}, \phi_{60,5}, \phi_{24,6}, \phi_{80,7}, \phi_{60,8}, \phi_{60,11}, \phi_{24,12}, \phi_{30,15}, \phi_{15,16}, \\
\phi_{20,20}, \phi_{6,25}, \phi_{1,36}\} with 6-defect 2, and \{\phi_{15,5}, \phi_{64,4}, \\
\phi_{81,6}, \phi_{20,10}, \phi_{90,8}, \phi_{10,9}, \phi_{81,10}, \phi_{64,13}, \\
\phi_{15,17}\} each with 6-defect 0. The whole decomposition matrix is given as follows:

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<td>$\phi_{81,6}$</td>
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The elementary divisors of the Cartan matrix are 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 10, 240.
Finally, let us discuss what happens for small \( l \), i.e., \( l = 2, 3, 5 \). The \( l \)-modular decomposition matrix \( D_l \) is equal to a product \( D_e D_\Phi \), where \( e \) is such that \( l \) divides \( \Phi_e(q) \). If \( l = 2 \), then \( q \) is odd, so we may take \( e = 2 \). Then we have

\[
\begin{array}{cccccc}
1 & . & . & . & . & . \\
. & 1 & . & . & . & . \\
. & . & 1 & . & . & . \\
. & . & . & 1 & . & . \\
. & . & . & . & 1 & . \\
2 & 1 & . & . & . & . \\
\end{array}
\]

\( D_2' \):

If \( l = 3 \), then either \( q \equiv 1 \mod 3 \), so that we may take \( e = 3 \), in which case we have

\[
\begin{array}{cccccccc}
1 & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . \\
. & . & . & 2 & 1 & 2 & 1 & . & . \\
. & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . \\
\end{array}
\]

\( D_3' \):

or \( q \equiv -1 \mod 3 \), so that we may take \( e = 2 \), in which case we have

\[
\begin{array}{cccccc}
1 & . & . & . & . & . \\
. & 1 & . & . & . & . \\
1 & . & 1 & . & . & . \\
. & . & 1 & . & . & . \\
. & . & . & 1 & . & . \\
2 & 1 & 2 & 1 & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
\end{array}
\]

\( D_3' \):

If \( l = 5 \), then either \( q \equiv 1 \mod 5 \), so that we may take \( e = 5 \), or \( q \equiv 2, 3 \mod 5 \), so that we may take \( e = 4 \), or \( q \equiv 4 \mod 5 \), so that we may take \( e = 2 \). In all these cases, the respective matrix \( D_5' \) is the identity matrix.

**Bibliography**


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