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THE DECOMPOSITION NUMBERS OF THE HECKE ALGEBRA OF TYPE E_6^*

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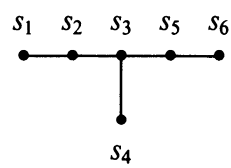
ABSTRACT. Let $E_6(q)$ be the Chevalley group of type E_6 over a finite field with q elements, l be a prime not dividing q, and $H_R(q)$ be the endomorphism ring of the permutation representation (over a valuation ring R with residue class field of characteristic l) of $E_6(q)$ on the cosets of a standard Borel subgroup $B(q)$. Then the l-modular decomposition matrix D_l of the algebra $H_R(q)$ is a submatrix of the l-modular decomposition matrix of the finite group $E_6(q)$. In this paper we determine the matrices D_l, for all l, q as above. For this purpose, we consider the generic Hecke algebra H associated with the finite Weyl group of type E_6 over the ring $A = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in an indeterminate v, and calculate the decomposition matrices of H which are associated with specializations of v to roots of unity over $\mathbb{Q}$ or values in a finite field. The computations were done by using the computer algebra systems MAPLE and GAP.

1. INTRODUCTION

Let W be the finite Weyl group of type E_6, and let $S = \{s_1, \dots, s_6\}$ be a corresponding set of simple reflections. The relations in W are determined by the following diagram (the product of any two distinct generators has order 3 if they are joined by a line, and 2 otherwise):



Let $A = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in an indeterminate v. The generic Hecke algebra H of W over A with parameter $u := v^2$ is a free A-module with basis T_w, $w \in W$, and multiplication defined by $T_s^2 = uT_1 + (u-1)T_s$, if $s \in S$, and $T_w T_{w'} = T_{ww'}$, if $l(ww') = l(w) + l(w')$, where $l(\ )$ denotes the usual length function on W. If $f: A \to K$ is a homomorphism into any field K of characteristic 0, then we get a corresponding specialized algebra $K \otimes_A H$ over K with parameter $f(u)$. The map f induces a decomposition map d_f between the Grothendieck groups of $\mathbb{Q}(v) \otimes_A H$ and

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$K \otimes_A H$ (see [6]). Denoting by $P(u) = \sum_{w \in W} u^{l(w)}$ the Poincaré polynomial of W, we get that d_f is the "identity" if and only if $P(f(u)) \neq 0$ in K. We have

$$P(u) = \Phi_2(u)^4 \Phi_3(u)^3 \Phi_2(u)^2 \Phi_5(u) \Phi_6(u)^2 \Phi_8(u) \Phi_9(u) \Phi_{12}(u),$$

where $\Phi_e(u)$ denotes the eth cyclotomic polynomial in $\mathbb{Z}[u]$. So we only get nontrivial decomposition maps d_f, if $f(u)$ is a root of unity in K.

Let $2e$ be a positive even integer, and $\overline{v}$ be a primitive $(2e)$th root of unity over $\mathbb{Q}$. Then we have a specialization $g_e\colon A \to \mathbb{Q}(\overline{v})$, $v \mapsto \overline{v}$, inducing the Φ_e-modular decomposition map d_e of H (cf. [6]). We determine the associated decomposition matrices D_e for $e = 2, 3, 4, 6$, while the cases $e = 5, 8, 9, 12$ ("e-defect 1") have been settled in [6].

Let q be a prime power and l be a prime not dividing q such that l divides $\Phi_e(q)$. It is shown in [6] that, in this situation and as a consequence of a theorem of R. Dipper (see [3, §2]), the matrix D_e is contained as a submatrix in the usual l-modular decomposition matrix of the finite Chevalley group $E_6(q)$, provided that l is larger than a number N which only depends on W (and not on q, e). We determine this bound N and show what happens for "small" l. To be more specific: There exists a valuation ring $I \subseteq \mathbb{Q}(\overline{v})$ with residue class field $\mathbb{F}_l(\overline{q}^{1/2})$ such that $\overline{u}$ is mapped to $\overline{q} \in \mathbb{F}_l$ under the canonical epimorphism. Thus we also get a decomposition map d'_l between the Grothendieck groups of $\mathbb{Q}(\overline{v}) \otimes_A H$ and $\mathbb{F}_l(\overline{q}^{1/2}) \otimes_A H$, and we have a factorization $d_l = d'_l \circ d_e$ of the l-modular decomposition map of H, hence a factorization of the corresponding decomposition matrices as $D_l = D_e D'_l$. Using the same techniques as in [7], we show that d'_l is the identity if l does not divide the order of W or, equivalently, that $N = 5$ (this is conjectured to be true for all finite Weyl groups in [6]). We also give explicit information for what happens if $l = 2, 3, 5$.

As an example of an application of the results of this paper, the first column (corresponding to the trivial module) of the l-modular decomposition matrix of the finite Chevalley group $E_6(q)$ is explicitly known, for all l not dividing q (cf. [8, §2]).

2. The methods

The principal method for computing decomposition numbers is the same as in [7]: Construct a set of representations of H such that each irreducible appears at least once as a constituent, specialize the parameter v, and decompose the specialized representations into irreducibles by using the Meat-Axe, see [9]. All computations were done by using the computer algebra systems GAP and MAPLE, see [10, 2].

One should note, however, that there is a principal difficulty arising from the fact that in characteristic 0 the Meat-Axe can only be applied if one has found elements in $\mathbb{Q}(\overline{v}) \otimes_A H$ with null-space of dimension 1 in a given representation, and there is no simple or effective rule to produce such elements. And even if one has found such an element there are limitations on the degrees of the representations caused by the complexity of the necessary computations (spinning-up vectors, Gaussian elimination in characteristic 0, etc.).

Therefore, we had to keep computations in cyclotomic fields over $\mathbb{Q}$ as limited as possible. This was achieved by combining results on l-modular decomposition numbers with a result on the "generic degree" of a projective module over $\mathbb{Q}(\overline{v}) \otimes_A H$, which we shall now explain.

For this result, we can assume that W is an arbitrary finite Weyl group, and H is the associated Hecke algebra over A with parameter u.

Let $\phi_1, \dots, \phi_n$ be the ordinary irreducible characters of $\mathbb{Q}(v) \otimes_A H$ and $\beta_1, \dots, \beta_k$ be the Φ_e-modular irreducible characters of $\mathbb{Q}(\overline{v}) \otimes_A H$. Then the Φ_e-modular decomposition numbers d_{ij} are given by the equations

$$g_e(\phi_i) = \sum_{j=1}^{k} d_{ij}\beta_j, \qquad i = 1, \dots, n.$$

By Brauer reciprocity, the function $\Psi_j := \sum_{i=1}^{n} d_{ij}\phi_i$ is the character of a projective indecomposable $\widehat{A}_e \otimes_A H$-module corresponding to β_j, where $\widehat{A}_e$ is the localization of $\mathbb{Q}(v)$ at the irreducible polynomial $\Phi_{2e}(v)$. For $1 \leq i \leq n$, denote by $D_{\phi_i}(u) \in \mathbb{Q}[u]$ the generic degree associated with ϕ_i. It has the property that $D_{\phi_i}(q)$ is the degree of the corresponding principal series representation of a finite Chevalley group $G(q)$ with Weyl group W. In [6] it is shown that $\Phi_e(u)$ divides the "generic degree"

$$D_{\Psi_j}(u) := \sum_{i=1}^{n} d_{ij} D_{\phi_i}(u) \in \mathbb{Q}[u]$$

of Ψ_j, for all j. This can be strengthened as follows.

Proposition 2.1. *Let $\Phi_e(u)^d$ be the maximal power of $\Phi_e(u)$ dividing the Poincaré polynomial $P(u)$ of W. Then*

$$\Phi_e(u)^d \textit{ divides } D_{\Psi_j}(u) \textit{ in } \mathbb{Q}[u].$$

(Note, however, that in general, $D_{\Psi_j}(u)$ is not a product of cyclotomic polynomials!)

Proof. Write $D_{\Psi_j}(u) = \Phi_e(u)^{d_j} f(u)$, where d_j is a nonnegative integer and $f(u) \in \mathbb{Q}[u]$ is not divisible by $\Phi_e(u)$. Since $\Phi_e(u)$ is irreducible, we can find polynomials $a(u)$, $b(u) \in \mathbb{Q}[u]$ such that

$$(*) \qquad 1 = a(u)\Phi_e(u)^{d_j} + b(u)f(u).$$

Now choose a prime l which satisfies the following conditions: (i) $l > |W|$, (ii) l does not divide the number N of [6, Proposition 5.5], (iii) l does not divide any denominator in the coefficients of the polynomials $a(u)$, $b(u)$, $f(u)$. Furthermore, choose a prime power q such that l divides $\Phi_e(q)$.

Let G be a connected reductive group with Weyl group W, split over $\mathbb{F}_q$, with associated Frobenius map F. We may assume that G is of adjoint type. Then the order of G^F is given by $|G^F| = q^m(q-1)^r P(q)$, where m is the number of positive roots of W, r is the rank of G, and $P(u)$ is the Poincaré polynomial of W. Let $(K, R, \overline{R})$ be a splitting l-modular system for all subgroups of G^F. The specialized algebra $H_R(q)$ is isomorphic to the RG^F-endomorphism ring of the permutation module $P := \mathbf{1}_{B^F}^{G^F}$, where B is an F-stable Borel subgroup of G. The order of B^F is given by $q^m(q-1)^r$.

By condition (ii), the decomposition map d_l' is the identity, i.e., the l-modular decomposition matrix of H coincides with the Φ_e-modular decomposition matrix. By condition (i), the multiplicative order of q modulo l is e, and $\Phi_e(u)$ is the only cyclotomic factor in $P(u)$ such that l divides $\Phi_e(q)$ (see [5]). So the l-part of $|G^F|$ is equal to the l-part of $\Phi_e(q)^d$. This implies that the order of B^F is prime to l, so P and every direct summand of P is a projective RG^F-module. Dipper's Theorem implies that the decomposition of P into indecomposable direct summands is determined by the decomposition of $H_R(q)$ into projective indecomposable modules (see [3, §2]). Putting things together, we see that $D_{\Psi_j}(q)$ is the degree of a projective indecomposable RG^F-module, for each $j = 1, \dots, k$, and it is therefore divisible by the l-part of $\Phi_e(q)^d$.

By condition (iii) and equation $(*)$, $a(q)$, $b(q)$, $f(q)$ all lie in R, and $b(q)$, $f(q)$ are units in R. In particular, $f(q)$ is not divisible by l. So the l-part of $D_{\Psi_j}(q)$ is equal to the l-part of $\Phi_e(q)^{d_j}$. By what we just proved before, this l-part must be divisible by the l-part of $\Phi_e(q)^d$, so we conclude that $d \leq d_j$. □

From now on, W will denote the finite Weyl group of type E_6 with corresponding generic Hecke algebra H. The procedure for computing the Φ_e-modular and l-modular decomposition numbers of H can now be described as follows.

(1) Construct a set of representations of H which have relatively "small" degree and have a complete set of irreducibles for $\mathbb{Q}(v)\otimes_A H$ as their constituents. The following methods are available:

- Computing Kazhdan-Lusztig polynomials and left cell representations of H. Here we used the GAP-programs of [7]. The decomposition of the various left cell representations is described in [1, p. 415]. In this way we constructed irreducible representations with characters $\phi_{1,0}$, $\phi_{6,1}$, $\phi_{20,2}$, $\phi_{64,4}$, $\phi_{60,5}$, $\phi_{81,6}$, $\phi_{24,6}$, $\phi_{15,4}$, $\phi_{30,3}$, $\phi_{15,5}$ (for the notations cf. [1, §13.2]; note that the irreducible characters of H are in bijective correspondence with the irreducible characters of W).
- Taking the dual of a representation. If $\rho\colon H \to A^{n\times n}$ is a representation, then its dual ρ^* is defined by $\rho^*(T_s) := \rho(-uT_s^{-1})$, $s \in S$ (cf. [6, (8.5)]). This construction is a substitute for tensoring with the sign representation in the Weyl group itself.
- Inducing representations from a parabolic subalgebra H' corresponding to a subset $S' \subseteq S$ (cf. [6, (8.6)]). In this way, we found representations containing the self-dual representations with characters $\phi_{20,10}$, $\phi_{60,8}$, $\phi_{80,7}$, $\phi_{90,8}$ as their constituents (using the parabolic subalgebra of type D_5 obtained by removing the node s_6 from the Dynkin diagram for which we could work out all irreducible representation as constituents of left cell representations).
- Extending a representation of a parabolic subalgebra. For example, an irreducible representation ρ with character $\phi_{10,9}$ remains irreducible when restricted to the parabolic subalgebra H' of type D_5 (remove node s_6). If this restriction is known, then one has to find a matrix ρ_6 such that the defining relations for H are satisfied: $\rho(T_{s_i})\rho_6 = \rho_6\rho(T_{s_i})$, for $i = 1, 2, 3, 4$, etc. (Of course, this can be carried out only for small degrees.)

(2) Fix $e > 1$ and choose a prime $l > 5$ (so l does not divide the order of

W) and a prime power q such that l and q are coprime, $\overline{q} \in \mathbb{F}_l^\times$ has order e, and $\mathbb{F}_l$ contains a square root of $\overline{q}$ (example: $e = 3$, $l = 13$, $q = 3$). Then consider the specialization $v \mapsto \overline{q}^{1/2} \in \mathbb{F}_l$, and use the GAP implementation of R. Parker's Meat-Axe to decompose the specialized representations into irreducibles. Thus, we get the l-modular decomposition matrix D_l of H. (For example, this is the matrix printed in Table B on p. 895.)

(3) Fix e, q, l as in (2), and assume that we have found the l-modular decomposition matrix D_l of H. By [6, Theorem 5.3], we have a factorization $D_l = D_e D_l'$, where D_e is the Φ_e-modular decomposition matrix of H. This means that the columns of D_l are sums of the columns of D_e. Let $(a_1, \dots, a_n)$ be a column of D_e. Then there is a column $(b_1, \dots, b_n)$ of D_l such that $a_i \leq b_i$ for all i. In particular, there are only finitely many possibilities for $(a_1, \dots, a_n)$, and we have as an additional requirement that $\sum_i a_i D_{\phi_i(u)}$ must be divisible by the $\Phi_e(u)$-part of $P(u)$, by Proposition 2.1. Using MAPLE, we found that this condition forces $a_i = b_i$ for all i in each case, with the only exception: $e = 3$ (with the choice $l = 13$, $q = 3$ as above) and columns 2 and 8 (in the matrix printed in Table B), where columns 5 (resp. 10) could possibly be subtracted. If this were the case, then the representations with characters $\phi_{30,3}$ and $\phi_{30,15}$ would remain irreducible under Φ_3-modular reduction, which was shown to be false by directly using the Meat-Axe in characteristic 0 (use the element $T_{s_2}T_{s_3}T_{s_4}T_{s_5}T_{s_6} - T_1 \in H$: it has corank 1 in these representations and splits up proper subspaces). Thus, we get the Φ_e-modular decomposition matrices of H.

(4) The decomposition maps d_l' for $l \leq 5$ were determined again by using the Meat-Axe. In order to show that d_l' is the identity for $l > 5$, we used the same techniques as explained in [7, Step IV in §3]. For this purpose, we had to construct explicitly matrices for a complete set of irreducible Φ_e-modular representations in blocks of e-defect > 1. This was done by using the characteristic 0 Meat-Axe in GAP. As already mentioned, the most difficult problem is to find elements in H which have corank 1 in a given representation. We systematically searched for such elements in the following way: (a) for $i \in \{1, \dots, 5\}$, let $A := T_{s_1} \cdot \cdots \cdot T_{s_i}$ and $B := T_{s_{i+1}} \cdot \cdots \cdot T_{s_6}$; (b) compute the "Fingerprint" associated with A and B using the scheme described in [9, Table 4]; (c) replace A by AB, or B by AB, or A by $B^{-1}AB$, and repeat Step (b); (d) for all elements computed so far, check whether 1 or -1 is an eigenvalue of multiplicity 1. (Of course, it is not guaranteed that we find elements with corank 1 in this way!)

In some cases it was also possible to show that an irreducible Φ_e-modular representation remains irreducible under reduction modulo l by restricting it to parabolic subalgebras of type D_5 (remove node s_6) and A_5 (remove node s_4), where computations were much easier. Consider the following example: There is an irreducible Φ_6-modular representation ρ of dimension 32. Its restriction to the parabolic subalgebra D_5 splits into irreducibles of dimensions 10, 11, 11; its restriction to the subalgebra A_5 splits into irreducibles of dimensions 16, 16. The cyclotomic polynomial $\Phi_6(u)$ divides the Poincaré polynomials of A_5 and D_5 just once. Hence we know that d_l' for A_5 and D_5 is the identity if $l > 5$ by [6, Part II]. From the dimensions of the constituents of the restrictions of ρ it is then clear that the reduction modulo l of ρ must remain irreducible, for $l > 5$.

3. The results

Now we collect the results obtained by the methods outlined in the previous section. Let $e \in \{2, 3, 4, 6\}$, l be a prime, and q be a prime power such that l divides $\Phi_e(q)$. As a result of the computations sketched in Step (4) of §2, we obtain:

(3.1) *The decomposition map d_l' is the identity if $l > 5$.*

This means that the decomposition matrix D_e is equal to the l-modular decomposition matrix D_l of H, for $l > 5$. So, in this case, D_e is a submatrix of the l-modular decomposition matrix of the finite Chevalley group $E_6(q)$. The ordering of the irreducible characters of H (and therefore of the corresponding principal series representations of $E_6(q)$) is chosen according to the general scheme described in [4, §3] (pulling back the partial order on unipotent conjugacy classes to unipotent characters via Lusztig's map). The notation for the irreducible characters of H is taken from [1, p. 415].

Table A. The Φ_2-blocks of H are: $\{\phi_{1,0}, \phi_{6,1}, \phi_{20,2}, \phi_{15,5}, \phi_{30,3}, \phi_{15,4}, \phi_{60,5}, \phi_{24,6}, \phi_{81,6}, \phi_{20,10}, \phi_{90,8}, \phi_{60,8}, \phi_{10,9}, \phi_{81,10}, \phi_{60,11}, \phi_{24,12}, \phi_{15,17}, \phi_{30,15}, \phi_{15,16}, \phi_{20,20}, \phi_{6,25}, \phi_{1,36}\}$ with 2-defect 4, $\{\phi_{64,4}, \phi_{64,13}\}$ with 2-defect 1, and $\{\phi_{80,7}\}$ with 2-defect 0. The whole decomposition matrix is given as follows:

$\phi_{1,0}$	1	·	·	·	·	·	·	·
$\phi_{6,1}$	·	1	·	·	·	·	·	·
$\phi_{20,2}$	·	1	1	·	·	·	·	·
$\phi_{15,5}$	1	·	1	·	·	·	·	·
$\phi_{30,3}$	·	1	1	1	·	·	·	·
$\phi_{15,4}$	1	·	·	·	1	·	·	·
$\phi_{64,4}$	·	·	·	·	·	1	·	·
$\phi_{60,5}$	·	·	1	·	·	·	1	·
$\phi_{24,6}$	·	·	1	1	·	·	·	·
$\phi_{81,6}$	1	1	1	·	1	·	1	·
$\phi_{20,10}$	·	1	·	·	1	·	·	·
$\phi_{80,7}$	·	·	·	·	·	·	·	1
$\phi_{90,8}$	·	1	2	1	·	·	1	·
$\phi_{60,8}$	·	·	·	·	1	·	1	·
$\phi_{10,9}$	·	·	·	1	·	·	·	·
$\phi_{81,10}$	1	1	1	·	1	·	1	·
$\phi_{60,11}$	·	·	1	·	·	·	1	·
$\phi_{24,12}$	·	·	1	1	·	·	·	·
$\phi_{64,13}$	·	·	·	·	·	1	·	·
$\phi_{15,17}$	1	·	1	·	·	·	·	·
$\phi_{30,15}$	·	1	1	1	·	·	·	·
$\phi_{15,16}$	1	·	·	·	1	·	·	·
$\phi_{20,20}$	·	1	1	·	·	·	·	·
$\phi_{6,25}$	·	1	·	·	·	·	·	·
$\phi_{1,36}$	1	·	·	·	·	·	·	·

The elementary divisors of the Cartan matrix are 1, 1, 1, 2, 4, 4, 8, 128.

Table B. The Φ_3-blocks of H are: $\{\phi_{1,0}, \phi_{6,1}, \phi_{20,2}, \phi_{15,5}, \phi_{30,3}, \phi_{15,4}, \phi_{64,4}, \phi_{60,5}, \phi_{24,6}, \phi_{20,10}, \phi_{80,7}, \phi_{60,8}, \phi_{10,9}, \phi_{60,11}, \phi_{24,12}, \phi_{64,13}, \phi_{15,17}, \phi_{30,15}, \phi_{15,16}, \phi_{20,20}, \phi_{6,25}, \phi_{1,36}\}$ with 3-defect 3, and $\{\phi_{81,6}\}$, $\{\phi_{81,10}\}$, $\{\phi_{90,8}\}$ each with 3-defect 0. The whole decomposition matrix is given as follows:

$\phi_{1,0}$	1	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{6,1}$	1	1	·	·	·	·	·	·	·	·	·	·	·
$\phi_{20,2}$	1	1	1	·	·	·	·	·	·	·	·	·	·
$\phi_{15,5}$	·	1	·	1	·	·	·	·	·	·	·	·	·
$\phi_{30,3}$	·	1	·	·	1	·	·	·	·	·	·	·	·
$\phi_{15,4}$	·	·	1	·	·	1	·	·	·	·	·	·	·
$\phi_{64,4}$	·	1	1	1	1	·	1	·	·	·	·	·	·
$\phi_{60,5}$	·	1	1	·	1	1	1	1	·	·	·	·	·
$\phi_{24,6}$	·	·	1	·	·	·	1	·	·	·	·	·	·
$\phi_{81,6}$	·	·	·	·	·	·	·	·	1	·	·	·	·
$\phi_{20,10}$	·	·	·	1	·	·	1	·	·	·	·	·	·
$\phi_{80,7}$	·	1	·	1	1	·	1	1	·	1	·	·	·
$\phi_{90,8}$	·	·	·	·	·	·	·	·	·	·	1	·	·
$\phi_{60,8}$	1	1	1	1	·	1	1	1	·	·	·	1	·
$\phi_{10,9}$	·	1	·	·	·	·	·	1	·	·	·	·	·
$\phi_{81,10}$	·	·	·	·	·	·	·	·	·	·	·	·	1
$\phi_{60,11}$	1	1	·	1	·	·	·	1	·	1	·	1	·
$\phi_{24,12}$	·	·	·	1	·	·	·	·	·	·	·	1	·
$\phi_{64,13}$	·	·	·	1	·	·	1	1	·	1	·	1	·
$\phi_{15,17}$	·	·	·	·	·	·	1	1	·	·	·	·	·
$\phi_{30,15}$	·	·	·	·	·	·	·	1	·	1	·	·	·
$\phi_{15,16}$	1	·	·	·	·	·	·	·	·	·	·	1	·
$\phi_{20,20}$	·	·	·	·	·	1	·	1	·	·	·	1	·
$\phi_{6,25}$	·	·	·	·	·	1	·	1	·	·	·	·	·
$\phi_{1,36}$	·	·	·	·	·	1	·	·	·	·	·	·	·

The elementary divisors of the Cartan matrix are $1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 9, 9, 81$.

TABLE C. The Φ_4-blocks of H are: $\{\phi_{1,0}, \phi_{6,1}, \phi_{15,5}, \phi_{15,4}, \phi_{81,6}, \phi_{80,7}, \phi_{90,8}, \phi_{10,9}, \phi_{81,10}, \phi_{15,17}, \phi_{15,16}, \phi_{6,25}, \phi_{1,36}\}$ with 4-defect 2, $\{\phi_{20,2}, \phi_{60,5}, \phi_{60,11}, \phi_{20,20}\}$ with 4-defect 1, and $\{\phi_{30,3}\}$, $\{\phi_{64,4}\}$, $\{\phi_{24,6}\}$, $\{\phi_{20,10}\}$, $\{\phi_{60,8}\}$, $\{\phi_{24,12}\}$, $\{\phi_{64,13}\}$, $\{\phi_{30,15}\}$ each with 4-defect 0. The whole decomposition matrix is given as follows:

$\phi_{1,0}$	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{6,1}$	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{20,2}$	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{15,5}$	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{30,3}$	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{15,4}$	1	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{64,4}$	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{60,5}$	·	·	1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
$\phi_{24,6}$	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
$\phi_{81,6}$	·	1	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
$\phi_{20,10}$	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
$\phi_{80,7}$	·	1	·	·	·	1	·	·	·	1	·	1	·	·	·	·	·	·	·
$\phi_{90,8}$	·	·	·	1	·	·	·	·	·	1	·	·	1	·	·	·	·	·	·
$\phi_{60,8}$	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
$\phi_{10,9}$	1	·	·	·	·	1	·	·	·	·	·	·	·	·	1	·	·	·	·
$\phi_{81,10}$	·	·	·	·	·	·	·	·	·	1	·	1	1	·	·	·	·	·	·
$\phi_{60,11}$	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	1	·	·	·
$\phi_{24,12}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·
$\phi_{64,13}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·
$\phi_{15,17}$	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
$\phi_{30,15}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1
$\phi_{15,16}$	·	·	·	·	·	1	·	·	·	·	·	1	·	·	1	·	·	·	·
$\phi_{20,20}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·
$\phi_{6,25}$	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
$\phi_{1,36}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·

The elementary divisors of the Cartan matrix are $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 24, 24$.

Table D. The Φ_6-blocks of H are: $\{\phi_{1,0}, \phi_{6,1}, \phi_{20,2}, \phi_{30,3}, \phi_{15,4}, \phi_{60,5}, \phi_{24,6}, \phi_{80,7}, \phi_{60,8}, \phi_{60,11}, \phi_{24,12}, \phi_{30,15}, \phi_{15,16}, \phi_{20,20}, \phi_{6,25}, \phi_{1,36}\}$ with 6-defect 2, and $\{\phi_{15,5}\}$, $\{\phi_{64,4}\}$, $\{\phi_{81,6}\}$, $\{\phi_{20,10}\}$, $\{\phi_{90,8}\}$, $\{\phi_{10,9}\}$, $\{\phi_{81,10}\}$, $\{\phi_{64,13}\}$, $\{\phi_{15,17}\}$ each with 6-defect 0. The whole decomposition matrix is given as follows:

$\phi_{1,0}$	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{6,1}$	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{20,2}$	1	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{15,5}$	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{30,3}$	·	1	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{15,4}$	1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{64,4}$	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{60,5}$	1	·	1	·	·	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·
$\phi_{24,6}$	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
$\phi_{81,6}$	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
$\phi_{20,10}$	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
$\phi_{80,7}$	·	·	1	·	1	·	·	1	1	·	·	1	·	·	·	·	·	·	·	·
$\phi_{90,8}$	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
$\phi_{60,8}$	·	·	·	·	·	1	·	1	·	·	·	·	·	1	·	·	·	·	·	·
$\phi_{10,9}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
$\phi_{81,10}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·
$\phi_{60,11}$	·	·	·	·	·	·	·	1	·	·	·	1	·	1	·	·	1	·	·	·
$\phi_{24,12}$	·	·	·	·	1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
$\phi_{64,13}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·
$\phi_{15,17}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·
$\phi_{30,15}$	·	·	·	·	·	·	·	·	1	·	·	1	·	·	·	·	·	·	·	1
$\phi_{15,16}$	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	1	·	·	·
$\phi_{20,20}$	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	1	·	·	1
$\phi_{6,25}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1
$\phi_{1,36}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·

The elementary divisors of the Cartan matrix are $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 10, 240$.

Finally, let us discuss what happens for small l, i.e., $l = 2, 3, 5$. The l-modular decomposition matrix D_l is equal to a product $D_e D'_l$, where e is such that l divides $\Phi_e(q)$. If $l = 2$, then q is odd, so we may take $e = 2$. Then we have

$$D'_2\colon \begin{array}{cccccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 4 & \cdot & 2 & \cdot & 1 \end{array}$$

If $l = 3$, then either $q \equiv 1 \bmod 3$, so that we may take $e = 3$, in which case we have

$$D'_3\colon \begin{array}{cccccccccccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 1 & \cdot & 2 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

or $q \equiv -1 \bmod 3$, so that we may take $e = 2$, in which case we have

$$D'_3\colon \begin{array}{ccccccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 2 & \cdot & 2 & 1 & \cdot & \cdot & 1 \end{array}$$

If $l = 5$, then either $q \equiv 1 \bmod 5$, so that we may take $e = 5$, or $q \equiv 2, 3 \bmod 5$, so that we may take $e = 4$, or $q \equiv 4 \bmod 5$, so that we may take $e = 2$. In all these cases, the respective matrix D'_5 is the identity matrix.

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