ON FABER POLYNOMIALS GENERATED BY AN $m$-STAR

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Abstract. In this paper, we study the Faber polynomials $F_n(z)$ generated by a regular $m$-star ($m = 3, 4, \ldots$)

$$S_m = \{x \omega^k; 0 \leq x \leq 4^{1/m}, \ k = 0, 1, \ldots, m - 1, \ \omega^m = 1\}.$$

An explicit and precise expression for $F_n(z)$ is obtained by computing the coefficients via a Cauchy integral formula. The location and limiting distribution of zeros of $F_n(z)$ are explored. We also find a class of second-order hypergeometric differential equations satisfied by $F_n(z)$. Our results extend some classical results of Chebyshev polynomials for a segment $[-2, 2]$ in the case when $m = 2$.

1. Introduction

Let $E$ be a compact set (not a single point) whose complement $C^* \setminus E$ with respect to the extended plane is simply connected. The Riemann mapping theorem asserts that there exists a conformal mapping $w = \Phi(z)$ of $C^* \setminus E$ onto the exterior of a circle $|w| = \rho_E$ in the $w$-plane. For a unique choice of $\rho_E$, we can insist that

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) = 1,$$

so that, in a neighborhood of infinity,

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots. \quad (1.1)$$

The polynomial part of $\{\Phi(z)\}^n$, denoted by $F_n(z) = z^n + \cdots$, is called the Faber polynomial of degree $n$ generated by the set $E$.

Let

$$\Psi(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \quad (1.2)$$

be the inverse function of $w = \Phi(z)$. Thus, $\Psi(w)$ maps the domain $|w| > \rho_E$ conformally onto $C^* \setminus E$. Faber [2] proved that

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \quad |w| > \rho_E, \ z \in E. \quad (1.3)$$

The Faber polynomials play an important role in approximation theory and geometric function theory. It can be shown that, under suitable conditions, a
function analytic in the inner domain of a Jordan curve $J$ can be expanded into a series of Faber polynomials that come from the mapping function of the outer domain of $J$ (cf. [1]).

The explicit construction of the Faber polynomials of a given set $E$ depends essentially on the knowledge of the mapping function $\Phi(z)$. For $E = [-2, 2]$, we know that $\Phi(z) = (z + \sqrt{z^2 - 4})/2$ with inverse $\Phi(w) = w + 1/w$. For $n \geq 1$, the polynomial part of $\{\Phi(z)\}^n$ is the same as the polynomial part of

$$\Phi^n(z) + \Phi^{-n}(z) = w^n + w^{-n},$$

which reduces to $2 \cos n \theta$, when $w = e^{i\theta}$. Thus the Faber polynomials are (apart from a multiplicative constant) the same as the classical Chebyshev polynomials $T_n(x)$ for $[-2, 2]$.

The following properties for $T_n(x)$ are well known [7]:

(i) For $n \geq 1$,

$$T_n(x) = n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(n-j-1)!}{(n-2j)!j!} x^{n-2j},$$

where

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n-1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

(ii) $T_n(x)$ satisfies the following differential equation:

$$(4 - x^2)y'' - xy' + n^2y = 0.$$  

(iii) The zeros of $T_n(x)$ are located on $(-2, 2)$ for every $n \geq 1$.

(iv) The asymptotic behavior of the zeros of $T_n(x)$ is given by the arcsine distribution

$$d\mu(t) = \frac{1}{\pi} \frac{1}{\sqrt{4 - t^2}} dt.$$  

In this paper we shall study the Faber polynomials generated by a regular $m$-star ($m = 2, 3, \ldots$)

$$S_m = \{x\omega^k ; 0 \leq x \leq 4^{1/m}, \ k = 0, 1, \ldots, m-1, \ \omega^m = 1\}.$$

Clearly, $S_m$ becomes $[-2, 2]$ when $m = 2$. It is natural to ask what can be said about Faber polynomials when $m = 3, 4, 5, \ldots$. By using the properties and characteristics of the Chebyshev polynomials as a motivation, we are able to compute an explicit representation for $F_n(z)$. In addition, the $F_n(z)$ are found to be a family of solutions to a class of second-order hypergeometric differential equations when $n \equiv 0 \pmod{m}$. We also found that the zeros of $F_n(z)$ when $n \equiv 0 \pmod{m}$, or $n \equiv \frac{m}{2} \pmod{m}$ if $m$ is even, are located on $S_m$. The asymptotic behavior of the zeros of $F_n(z)$ is also determined explicitly. Numerical results for the zeros of $F_n(z)$ will be given in §3.

2. Main results and proofs

As we shall see, the Faber polynomials for $S_m$ with $m > 2$ enjoy certain properties that are similar to those satisfied by Chebyshev polynomials. Our main results in this direction are the following:
Theorem. Let $F_n(z)$ be the Faber polynomials of $S_m$ of degree $n$. Then

(i) For $n \geq 1$,

$$F_n(z) = \frac{2n}{m} \sum_{j=0}^{\left[ \frac{n}{m} \right]} (-1)^j \frac{\Gamma(2n-j)}{\Gamma(2n-m-2j+1)} z^{n-m-j},$$

where

$$\left[ \frac{n}{m} \right] = \begin{cases} \frac{n}{m} & \text{if } n \equiv 0 \pmod{m}, \\ \frac{n-k}{m} & \text{if } n \equiv k \pmod{m}, \quad k = 0, 1, 2, \ldots, m-1. \end{cases}$$

(ii) If $n \equiv 0 \pmod{m}$, then $F_n(z)$ satisfies the following differential equation:

$$\left(4 - z^n \right) z y'' - \left[ z^m + 2m - 4 \right] y' + n^2 z^{m-1} y = 0.$$  

(iii) If $n \equiv 0 \pmod{m}$, or $n \equiv \frac{m}{2} \pmod{m}$ when $m$ is even, then the zeros of $F_n(z)$ are located on $S_m$.

(iv) For $n \geq 1$, the zero distribution of $F_n(z)$ for $S_m$ is given by

$$d\mu(t) = \frac{1}{\pi} \frac{t^{(m-2)/2}}{\sqrt{4-t^m}} dt, \quad t \in S_m.$$  

One can see that the above theorem generalizes (1.4), (1.5), and (1.6) when $m = 2$. We now proceed to prove our theorem.

Proof of (i). It is known [3, p. 395] that for $m = 2, 3, \ldots$,

$$z = \Psi(w) = w \left(1 + \frac{1}{w^m}\right)^{2/m}$$

maps $|w| > 1$ conformally onto $\mathbb{C}^* \setminus S_m$. Let $\Phi(z)$ be its inverse. It follows from the definition of Faber polynomials that $F_n(z)$ generated by $S_m$ is given by

$$F_n(z) = \sum_{j=0}^{n} c_{n-j} z^{n-j},$$

where $c_{n-j}$ is the Laurent coefficient in the expansion of $\{\Phi(z)\}^n$, that is, for $j = 0, 2, 3, \ldots, n$,

$$c_{n-j} = \int_{|z|=R} \frac{\{\Phi(z)\}^n}{z^{n-j+1}} dz,$$

with $R$ chosen sufficiently large so that $S_m$ is contained in the interior of the region bounded by the circle $|z| = R$. Alternatively, using the substitution $z = \Psi(w)$, we obtain for $j = 0, 2, 3, \ldots, n$,

$$c_{n-j} = \int_{|w|=\rho} \frac{w^n \Psi'(w)}{\{\Psi(w)\}^{n-j+1}} dw.$$  

By the symmetry of $S_m$, we see that $\Psi(w)$ is an $m$-fold symmetric mapping function (i.e., $\Psi(\exp(\frac{2k\pi i}{m}w)) = \exp(\frac{2k\pi i}{m})\Psi(w)$, $k = 0, 1, \ldots, m-1$). It is easy to see that $c_{n-j} = 0$ if $j \not\equiv 0 \pmod{m}$. Thus, $F_n(z)$ has the following form:

$$F_n(z) = \sum_{j=0}^{\left[ \frac{n}{m} \right]} c_{n-mj} z^{n-mj}.$$
By Cauchy's Theorem, we see that the coefficients $c_{n-mj}$ are the same as those of $\frac{1}{w}$ in the expansion of $w^n\Psi'(w)/[\Psi'(w)]^{n-mj+1}$. Using (2.4), we get
\[
\Psi'(w) = \left(1 - \frac{1}{w^m}\right)\left(1 + \frac{1}{w^m}\right)^{(2-m)/m}.
\]
Thus,
\[
\frac{w^n\Psi'(w)}{[\Psi'(w)]^{n-mj+1}} = w^{mj-1} \left(1 - \frac{1}{w^m}\right)^{-(m+2n-2mj)/m} \left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m}.
\]
Noticing that $|w| > 1$, we have
\[
\left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m} = \sum_{i=0}^{\infty} (-1)^i \left(\frac{m+2n-2mj}{m}\right)_i \frac{w^{-mi}}{i!},
\]
where
\[
(a)_i = a(a+1)(a+2)\cdots(a+i-1), \quad (a)_0 = 1, \quad (a)_{-1} = 0.
\]
Thus, the coefficient of $\frac{1}{w}$ is given by
\[
c_{n-mj} = (-1)^j \frac{(m+2n-2mj)_j}{j!} - (-1)^{j+1} \frac{(m+2n-2mj)(j-1)!}{j!}
\]
\[
= (-1)^j \frac{2n}{m} \frac{(2n_j + 1 - 2j)(2n_j + 1 - 2j + 1)\cdots(2n_j + 1 - 2j + j - 2)}{j!}.
\]

Proof of (ii). Let $x = z^m$ in (2.2). Then we have the following hypergeometric differential equation:
\[
(2.5) \quad x(4-x)y'' + (2-x)y' + n^2y = 0.
\]
Replacing $n$ by $nm$ in (2.1) and substituting $z^m$ for $x$, we get
\[
y := F_{nm}(z) = 2n \sum_{j=0}^{n} (-1)^j \frac{\Gamma(2n-j)}{\Gamma(2n-2j+1)!} x^{n-j}.
\]
It is easy to verify that $y$ is the polynomial solution of (2.5). Therefore, $F_{nm}(z)$ is the polynomial solution of (2.2).

Proof of (iii). If $n \equiv 0 \pmod{m}$, or $n \equiv \frac{m}{2} \pmod{m}$ ($m$ is even), then by (1.4)

\[
F_{nm}(z) = 2n \sum_{j=0}^{n} (-1)^j \frac{\Gamma(2n-j)}{\Gamma(2n-2j+1)!} z^{nm-mj},
\]

\[
F_{nm+m/2}(z) = (2n+1)z^{m/2} \sum_{j=0}^{n} (-1)^j \frac{\Gamma(2n+1-j)}{\Gamma(2n+1-2j+1)!} z^{nm-mj}.
\]
Comparing $F_{mn}(z)$ and $F_{mn+m/2}(z)$ with (1.4) yields

$$F_{mn}(z) = T_{2n}(z^{m/2}), \quad F_{mn+m/2}(z) = T_{2n+1}(z^{m/2}).$$

Since the zeros of $T_{2n}(x)$ and $T_{2n+1}(x)$ are located on $[-2, 2]$, it follows from (2.6) that the zeros of $F_{mn}(z)$ and $F_{mn+m/2}(z)$ are located on $S_m$. □

**Remark.** Using (2.4), we can derive the relations

$$\Phi(z) = \{\phi(z^{m/2})\}^{2/m} \quad \text{and} \quad \Psi(z) = \{\psi(z^{m/2})\}^{2/m},$$

where $\psi(w) = w + \frac{1}{w}$ and $\phi(z)$ is the inverse mapping function of $\psi(w)$. By the above equations and the definition of the Faber polynomials, one can also obtain (2.6).

**Proof of (iv).** In order to prove (iv), we first develop some notation. The term “capacity” means inner logarithmic capacity (cf. [8, p. 55]). For any set $E \subset \mathbb{C}$, the capacity of $E$ will be denoted by $C(E)$. If $E$ is a compact set with positive capacity, then $\mu_E$ shall denote the unique unit equilibrium measure on $E$ with the property that

$$\int_E \log |x - t| d\mu_E(t) = \log C(E)$$

quasi-everywhere (q.e.) on $E$ (cf. [8, p. 60]). A property is said to hold q.e. on a set $A$ if the subset $E$ of $A$ where it does not hold satisfies $C(E) = 0$.

To each $p_n(z) = \prod_{k=1}^n (z - z_k)$, we associate the normalized zero distribution measure $\nu(p_n)$ defined by

$$\nu(p_n) := \frac{1}{n} \sum_{k=1}^n \delta_{z_k},$$

where $\delta_{z_k}$ is the point distribution with total mass 1 at $z_k$.

We now prove the following lemma.

**Lemma.** Let $E$ be a compact subset (not a single point) whose complement $\mathbb{C}^* \setminus E$ is simply connected. Assume that $E$ has empty interior. Then the asymptotic distribution of zeros of Faber polynomials $F_n(z)$ coincides with the equilibrium measure of the set $E$. More precisely, the normalized zero distribution measures $\nu(F_n)$ converge in the weak-star topology to $\mu_E$.

**Proof.** By Theorem 2.3 in [6], we need only to prove that Faber polynomials are an asymptotically minimal norm sequence of monic polynomials, that is,

$$\limsup_{n \to \infty} \|F_n\|_{\supp(\mu_E)}^{1/n} \leq \text{cap}(E) = 1,$$

where $\mu_E$ is the equilibrium distribution of $E$. It is known from [5, p. 108] that for $r > 1$,

$$\frac{1}{2} |\Phi(z)|^n \leq |F_n(z)| \leq \frac{1}{2} |\Phi(z)|^n,$$

where $z$ is on or exterior to $C_r = \{|\Phi(z)| = r\}$. For $n$ large enough we know that

$$\|F_n\|_E \leq \|F_n\|_{C_r} \leq \frac{3}{2} r^n.$$

Taking the $n$th roots in (2.8), we get $\|F_n\|_E^{1/n} \leq (\frac{3}{2})^{1/n} r$, so we have

$$\limsup_{n \to \infty} \|F_n\|_E^{1/n} \leq r.$$

Letting $r \to 1$, we get (2.7). □
We now go back to prove (iv). Clearly, the \( m \)-star has empty interior. According to our lemma, we know that the limiting distribution of the zeros of Faber polynomials associated with \( S_m \) coincides with equilibrium measure \( d\mu^*(t) \) for \( S_m \). We now determine \( d\mu^*(t) \). Define

\[
M_k := \int_{S_m} t^k d\mu^*(t), \quad k = 0, 1, \ldots ,
\]

where \( M_k \) is the so-called \( k \)-th moment of the equilibrium distribution on \( S_m \). On the other hand, \( M_k \) can also be expressed as (cf. [4, p. 345])

\[
M_k = \frac{1}{\pi i} \int_{|z|=r} [\Psi(s)]^k \frac{ds}{s}
\]

for \( r > 1 \), where \( z = \Psi(w) = w(1 + 1/w^m)^{2/m} \) maps the exterior of \( |w| = 1 \) onto the exterior of \( S_m \) (cf. [3, p. 395]).

We now compute the \( k \)-th moment \( M_k \) from (2.10). Let

\[
t = \Psi(s) = s \left( 1 + \frac{1}{s^m} \right)^{2/m} = (s^{m/2} + s^{-m/2})^{2/m}
\]
in (2.10). Then

\[
dt = (s^{m/2} + s^{-m/2})^{2/m-1} (s^{m/2-1} - s^{-m/2-1}) ds
\]

\[
= (s^{m/2} + s^{-m/2})^{-m/2} (s^{m/2} - s^{-m/2}) \frac{ds}{s} = t^{1-m/2} i\sqrt{4 - t^m} \frac{ds}{s},
\]

and so

\[
M_k = \frac{1}{\pi i} \int_{|z|=r} [\Psi(s)]^k \frac{ds}{s} = \frac{1}{\pi} \int_{C_r} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}} dt,
\]

where \( C_r \) is the image of \( |w| = r \) under the mapping \( \Psi(w) \).

Letting \( r \rightarrow 1 \), we get

\[
M_k = \frac{1}{\pi} \int_{S_m} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}} dt.
\]

By (2.9) we have

\[
\int_{S_m} t^k d\mu^*(t) = M_k = \frac{1}{\pi} \int_{S_m} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}} dt, \quad k = 0, 1, \ldots .
\]

Thus, \( d\mu^*(t) = d\mu(t) \), where \( d\mu(t) \) is defined in (2.3). \( \Box \)

We have completed the proof of the theorem.

The situation for obtaining the locations of the zeros of \( F_n(z) \) for every \( n \) is not so favorable. However, supported by (iii) in our theorem, and numerical experiments, we formulate the following.

**Conjecture.** The zeros of Faber polynomials generated by the regular \( m \)-star \( S_m \) (\( m = 3, 4, \ldots \)) are located on \( S_m \) for every \( n \geq 1 \).

For a sequence of polynomials \( F_{nm}(z) \), and \( F_{nm+m/2}(z) \) when \( m \) is even, our theorem confirmed this conjecture. For other sequences of Faber polynomials,
we provide in §3 numerical results produced by Matlab in the cases when \( m = 3, 4, \) and \( 5. \)

3. Numerical results on the zeros of Faber polynomials

We first briefly state our algorithm for computing the zeros of Faber polynomials generated by a set \( E. \)

Let

\[
\Psi(z) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots
\]

be the mapping function as in (1.2) associated with a set \( E. \) Then

\[
\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \quad |w| > \rho_E, \ z \in E.
\]

It follows from (3.2) that \( F_n(z) \) satisfies the recurrence relation

\[
F_{n+1}(z) = (z - b_0)F_n(z) - \sum_{k=1}^{n-1} b_k F_{n-k}(z) - (n + 1)b_n,
\]

with initial condition \( F_0(z) = 1. \) This difference equation for Faber polynomials plays a very important role in computing the zeros of \( F_n(z). \) Our algorithm consists of four steps:

Step 1. Determine the coefficients \( b_k \)’s of the mapping function \( \Psi(w) \) as in (3.1).

Step 2. Compute the coefficients of \( F_n(z) \) by using (3.3).

Step 3. Find the zeros of \( F_n(z). \)

Step 4. Plot the zeros in the complex plane.

We now apply the algorithm to compute the zeros of \( F_n(z) \) for \( S_m. \) The results are shown in Figures 1–6 (see pp. 284–286).

For \( m = 2, 3, \ldots, \) we normalize the mapping function such that

\[
\Psi(w) = w \left( 1 + \frac{1}{w^m} \right)^{2/m}.
\]

Then

\[
\Psi(w) = w + \sum_{k=1}^{\infty} \frac{b_{mk-1}w^{mk-1}}{k!},
\]

where

\[
b_{mk-1} = (-1)^{k-1} \frac{2(m-2)(2m-2) \cdots ((k-1)m-2)}{m^k k!}.
\]

By the recurrence relation (3.3), we find, in particular for \( m = 3, \)

\[
F_0(z) = 1, \quad F_1(z) = z, \quad F_2(z) = z^2,
\]

\[
F_3(z) = z^3 - 2, \quad F_4(z) = z^4 - \frac{5}{3}z^2, \quad F_5(z) = z^5 - \frac{10}{3}z^3,
\]

\[
F_6(z) = z^6 - 4z^3 + 2, \quad F_7(z) = z^7 - \frac{14}{3}z^4 + \frac{35}{9}z^2, \quad F_8(z) = z^8 - \frac{16}{3}z^5 + \frac{56}{9}z^2.
\]

\[
F_9(z) = z^9 - 6z^6 + z^3 - 2, \quad F_{10}(z) = z^{10} - \frac{20}{3}z^7 + \frac{110}{9}z^4 - \frac{400}{81}z.
\]
Figure 1. Zeros of $F_n(z)$ for 3-star when $n = 31$

Figure 2. Zeros of $F_n(z)$ for 3-star when $n = 32$
Figure 3. Zeros of $F_n(z)$ for 4-star when $n = 31$

Figure 4. Zeros of $F_n(z)$ for 4-star when $n = 33$
Figure 5. Zeros of $F_n(z)$ for 5-star when $n = 31$

Figure 6. Zeros of $F_n(z)$ for 5-star when $n = 32$
Bibliography


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