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ON FABER POLYNOMIALS GENERATED BY AN m-STAR

J. BARTOLOMEO AND MATTHEW HE

ABSTRACT. In this paper, we study the Faber polynomials $F_n(z)$ generated by a regular m-star $(m = 3, 4, \ldots)$

$$S_m = \{x\omega^k;\, 0 \le x \le 4^{1/m},\ k = 0, 1, \ldots, m-1,\ \omega^m = 1\}.$$

An explicit and precise expression for $F_n(z)$ is obtained by computing the coefficients via a Cauchy integral formula. The location and limiting distribution of zeros of $F_n(z)$ are explored. We also find a class of second-order hypergeometric differential equations satisfied by $F_n(z)$. Our results extend some classical results of Chebyshev polynomials for a segment $[-2, 2]$ in the case when $m = 2$.

1. INTRODUCTION

Let E be a compact set (not a single point) whose complement $\mathbf{C}^*\backslash E$ with respect to the extended plane is simply connected. The Riemann mapping theorem asserts that there exists a conformal mapping $w = \Phi(z)$ of $\mathbf{C}^*\backslash E$ onto the exterior of a circle $|w| = \rho_E$ in the w-plane. For a unique choice of ρ_E, we can insist that

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) = 1,$$

so that, in a neighborhood of infinity,

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots. \tag{1.1}$$

The polynomial part of $\{\Phi(z)\}^n$, denoted by $F_n(z) = z^n + \cdots$, is called the *Faber polynomial* of degree n generated by the set E.

Let

$$\Psi(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \tag{1.2}$$

be the inverse function of $w = \Phi(z)$. Thus, $\Psi(w)$ maps the domain $|w| > \rho_E$ conformally onto $\mathbf{C}^*\backslash E$. Faber [2] proved that

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad |w| > \rho_E,\ z \in E. \tag{1.3}$$

The Faber polynomials play an important role in approximation theory and geometric function theory. It can be shown that, under suitable conditions, a

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function analytic in the inner domain of a Jordan curve J can be expanded into a series of Faber polynomials that come from the mapping function of the outer domain of J (cf. [1]).

The explicit construction of the Faber polynomials of a given set E depends essentially on the knowledge of the mapping function $\Phi(z)$. For $E = [-2, 2]$, we know that $\Phi(z) = (z + \sqrt{z^2 - 4})/2$ with inverse $\Phi(w) = w + 1/w$. For $n \geq 1$, the polynomial part of $\{\Phi(z)\}^n$ is the same as the polynomial part of

$$\Phi^n(z) + \Phi^{-n}(z) = w^n + w^{-n},$$

which reduces to $2\cos n\theta$, when $w = e^{i\theta}$. Thus the Faber polynomials are (apart from a multiplicative constant) the same as the classical Chebyshev polynomials $T_n(x)$ for $[-2, 2]$.

The following properties for $T_n(x)$ are well known [7]:

(i) For $n \geq 1$,

$$T_n(x) = n \sum_{j=0}^{[\frac{n}{2}]} (-1)^j \frac{(n-j-1)!}{(n-2j)!j!} x^{n-2j}, \tag{1.4}$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n-1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

(ii) $T_n(x)$ satisfies the following differential equation:

$$(4 - x^2)y'' - xy' + n^2 y = 0. \tag{1.5}$$

(iii) The zeros of $T_n(x)$ are located on $(-2, 2)$ for every $n \geq 1$.

(iv) The asymptotic behavior of the zeros of $T_n(x)$ is given by the arcsine distribution

$$d\mu(t) = \frac{1}{\pi} \frac{1}{\sqrt{4 - t^2}} dt. \tag{1.6}$$

In this paper we shall study the Faber polynomials generated by a regular m-star $(m = 2, 3, \ldots)$

$$S_m = \{x\omega^k ; 0 \leq x \leq 4^{1/m}, \ k = 0, 1, \ldots, m-1, \ \omega^m = 1\}. \tag{1.7}$$

Clearly, S_m becomes $[-2, 2]$ when $m = 2$. It is natural to ask what can be said about Faber polynomials when $m = 3, 4, 5, \ldots$. By using the properties and characteristics of the Chebyshev polynomials as a motivation, we are able to compute an explicit representation for $F_n(z)$. In addition, the $F_n(z)$ are found to be a family of solutions to a class of second-order hypergeometric differential equations when $n \equiv 0 \pmod m$. We also found that the zeros of $F_n(z)$ when $n \equiv 0 \pmod m$, or $n \equiv \frac{m}{2} \pmod m$ if m is even, are located on S_m. The asymptotic behavior of the zeros of $F_n(z)$ is also determined explicitly. Numerical results for the zeros of $F_n(z)$ will be given in §3.

2. Main results and proofs

As we shall see, the Faber polynomials for S_m with $m > 2$ enjoy certain properties that are similar to those satisfied by Chebyshev polynomials. Our main results in this direction are the following:

Theorem. *Let $F_n(z)$ be the Faber polynomials of S_m of degree n. Then*

(i) *For $n \geq 1$,*

$$F_n(z) = \frac{2n}{m} \sum_{j=0}^{[\frac{n}{m}]} (-1)^j \frac{\Gamma(\frac{2n}{m} - j)}{\Gamma(\frac{2n}{m} - 2j + 1) j!} z^{n-mj}, \tag{2.1}$$

where

$$\left[\frac{n}{m}\right] = \begin{cases} \dfrac{n}{m} & \text{if } n \equiv 0 \pmod{m}, \\ \dfrac{n-k}{m} & \text{if } n \equiv k \pmod{m},\ k = 0, 1, 2, \dots, m-1. \end{cases}$$

(ii) *If $n \equiv 0 \pmod{m}$, then $F_n(z)$ satisfies the following differential equation:*

$$(4 - z^m) z y'' - [z^m + 2m - 4] y' + n^2 z^{m-1} y = 0. \tag{2.2}$$

(iii) *If $n \equiv 0 \pmod{m}$, or $n \equiv \frac{m}{2} \pmod{m}$ when m is even, then the zeros of $F_n(z)$ are located on S_m.*

(iv) *For $n \geq 1$, the zero distribution of $F_n(z)$ for S_m is given by*

$$d\mu(t) = \frac{1}{\pi} \frac{t^{(m-2)/2}}{\sqrt{4 - t^m}} dt, \qquad t \in S_m. \tag{2.3}$$

One can see that the above theorem generalizes (1.4), (1.5), and (1.6) when $m = 2$. We now proceed to prove our theorem.

Proof of (i). It is known [3, p. 395] that for $m = 2, 3, \dots$,

$$z = \Psi(w) = w \left(1 + \frac{1}{w^m}\right)^{2/m} \tag{2.4}$$

maps $|w| > 1$ conformally onto $\mathbf{C}^* \backslash S_m$. Let $\Phi(z)$ be its inverse. It follows from the definition of Faber polynomials that $F_n(z)$ generated by S_m is given by

$$F_n(z) = \sum_{j=0}^{n} c_{n-j} z^{n-j},$$

where c_{n-j} is the Laurent coefficient in the expansion of $\{\Phi(z)\}^n$, that is, for $j = 0, 2, 3, \dots, n$,

$$c_{n-j} = \int_{|z|=R} \frac{\{\Phi(z)\}^n}{z^{n-j+1}} dz,$$

with R chosen sufficiently large so that S_m is contained in the interior of the region bounded by the circle $|z| = R$. Alternatively, using the substitution $z = \Psi(w)$, we obtain for $j = 0, 2, 3, \dots, n$,

$$c_{n-j} = \int_{|w|=r>\rho} \frac{w^n \Psi'(w)}{\{\Psi(w)\}^{n-j+1}} dw.$$

By the symmetry of S_m, we see that $\Psi(w)$ is an m-fold symmetric mapping function (i.e., $\Psi(\exp(\frac{2k\pi i}{m} w)) = \exp(\frac{2k\pi i}{m}) \Psi(w)$, $k = 0, 1, \dots, m-1$). It is easy to see that $c_{n-j} = 0$ if $j \not\equiv 0 \pmod{m}$. Thus, $F_n(z)$ has the following form:

$$F_n(z) = \sum_{j=0}^{[\frac{n}{m}]} c_{n-mj} z^{n-mj}.$$

By Cauchy's Theorem, we see that the coefficients c_{n-mj} are the same as those of $\frac{1}{w}$ in the expansion of $w^n\Psi'(w)/[\Psi(w)]^{n-mj+1}$. Using (2.4), we get

$$\Psi'(w) = \left(1 - \frac{1}{w^m}\right)\left(1 + \frac{1}{w^m}\right)^{(2-m)/m}.$$

Thus,

$$\begin{aligned}\frac{w^n\Psi'(w)}{[\Psi(w)]^{n-mj+1}} &= w^{mj-1}\left(1 - \frac{1}{w^m}\right)\left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m} \\ &= w^{mj-1}\left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m} \\ &\quad - w^{mj-m-1}\left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m}.\end{aligned}$$

Noticing that $|w| > 1$, we have

$$\left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m} = \sum_{i=0}^{\infty}(-1)^i\left(\frac{m+2n-2mj}{m}\right)_i \frac{w^{-mi}}{i!},$$

where

$$(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1), \qquad (\alpha)_0 = 1, \qquad (\alpha)_{-1} = 0.$$

Thus, the coefficient of $\frac{1}{w}$ is given by

$$\begin{aligned}c_{n-mj} &= (-1)^j\frac{(\frac{m+2n-2mj}{m})_j}{j!} - (-1)^{j-1}\frac{(\frac{m+2n-2mj}{n})_{j-1}}{(j-1)!} \\ &= (-1)^j\frac{2m}{n}\frac{(\frac{2n}{m}+1-2j)(\frac{2n}{m}+1-2j+1)\cdots(\frac{2n}{m}+1-2j+j-2)}{j!} \\ &= (-1)^j\frac{2n}{m}\frac{\Gamma(\frac{2n}{m}-j)}{\Gamma(\frac{2n}{m}-2j+1)j!}. \quad \square\end{aligned}$$

Proof of (ii). Let $x = z^m$ in (2.2). Then we have the following hypergeometric differential equation:

$$x(4-x)y'' + (2-x)y' + n^2y = 0. \tag{2.5}$$

Replacing n by nm in (2.1) and substituting z^m for x, we get

$$y := F_{nm}(z) = 2n\sum_{j=0}^{n}(-1)^j\frac{\Gamma(2n-j)}{\Gamma(2n-2j+1)j!}x^{n-j}.$$

It is easy to verify that y is the polynomial solution of (2.5). Therefore, $F_{nm}(z)$ is the polynomial solution of (2.2). $\square$

Proof of (iii). If $n \equiv 0 \pmod m$, or $n \equiv \frac{m}{2} \pmod m$ (m is even), then by (1.4)

$$F_{nm}(z) = 2n\sum_{j=0}^{n}(-1)^j\frac{\Gamma(2n-j)}{\Gamma(2n-2j+1)j!}z^{nm-mj},$$

$$F_{nm+m/2}(z) = (2n+1)z^{m/2}\sum_{j=0}^{n}(-1)^j\frac{\Gamma(2n+1-j)}{\Gamma(2n+1-2j+1)j!}z^{nm-mj}.$$

Comparing $F_{mn}(z)$ and $F_{mn+m/2}(z)$ with (1.4) yields

$$F_{mn}(z) = T_{2n}(z^{m/2}), \qquad F_{mn+m/2}(z) = T_{2n+1}(z^{m/2}). \tag{2.6}$$

Since the zeros of $T_{2n}(x)$ and $T_{2n+1}(x)$ are located on $[-2, 2]$, it follows from (2.6) that the zeros of $F_{mn}(z)$ and $F_{mn+m/2}(z)$ are located on S_m. □

Remark. Using (2.4), we can derive the relations

$$\Phi(z) = \{\phi(z^{m/2})\}^{2/m} \quad \text{and} \quad \Psi(z) = \{\psi(z^{m/2})\}^{2/m},$$

where $\psi(w) = w + \frac{1}{w}$ and $\phi(z)$ is the inverse mapping function of $\psi(w)$. By the above equations and the definition of the Faber polynomials, one can also obtain (2.6).

Proof of (iv). In order to prove (iv), we first develop some notation. The term "capacity" means inner logarithmic capacity (cf. [8, p. 55]). For any set $E \subset \mathbf{C}$, the capacity of E will be denoted by $C(E)$. If E is a compact set with positive capacity, then μ_E shall denote the unique unit equilibrium measure on E with the property that

$$\int_E \log|x - t|\, d\mu_E(t) = \log C(E)$$

quasi-everywhere (q.e.) on E (cf. [8, p. 60]). A property is said to hold q.e. on a set A if the subset E of A where it does not hold satisfies $C(E) = 0$.

To each $p_n(z) = \prod_{k=1}^n (z - z_k)$, we associate the *normalized zero distribution measure* $\nu(p_n)$ defined by

$$\nu(p_n) := \frac{1}{n}\sum_{k=1}^n \delta_{z_k},$$

where δ_{z_k} is the point distribution with total mass 1 at z_k.

We now prove the following lemma.

Lemma. *Let E be a compact subset (not a single point) whose complement $\mathbf{C}^*\backslash E$ is simply connected. Assume that E has empty interior. Then the asymptotic distribution of zeros of Faber polynomials $F_n(z)$ coincides with the equilibrium measure of the set E. More precisely, the normalized zero distribution measures $\nu(F_n)$ converge in the weak-star topology to μ_E.*

Proof. By Theorem 2.3 in [6], we need only to prove that Faber polynomials are an *asymptotically minimal norm sequence* of monic polynomials, that is,

$$\limsup_{n\to\infty} \|F_n\|_{\mathrm{supp}(\mu_E)}^{1/n} \le \mathrm{cap}(E) = 1, \tag{2.7}$$

where μ_E is the equilibrium distribution of E. It is known from [5, p. 108] that for $r > 1$,

$$\tfrac{1}{2}|\Phi(z)|^n \le |F_n(z)| \le \tfrac{3}{2}|\Phi(z)|^n,$$

where z is on or exterior to $C_r = \{|\Phi(z)| = r\}$. For n large enough we know that

$$\|F_n\|_E \le \|F_n\|_{C_r} \le \tfrac{3}{2}r^n. \tag{2.8}$$

Taking the nth roots in (2.8), we get $\|F_n\|_E^{1/n} \le (\frac{3}{2})^{1/n} r$, so we have

$$\limsup_{n\to\infty} \|F_n\|_E^{1/n} \le r.$$

Letting $r \to 1$, we get (2.7). □

We now go back to prove (iv). Clearly, the m-star has empty interior. According to our lemma, we know that the limiting distribution of the zeros of Faber polynomials associated with S_m coincides with equilibrium measure $d\mu^*(t)$ for S_m. We now determine $d\mu^*(t)$. Define

$$M_k := \int_{S_m} t^k \, d\mu^*(t), \qquad k = 0, 1, \ldots, \tag{2.9}$$

where M_k is the so-called kth moment of the equilibrium distribution on S_m. On the other hand, M_k can also be expressed as (cf. [4, p. 345])

$$M_k = \frac{1}{\pi i} \int_{|s|=r} [\Psi(s)]^k \frac{ds}{s} \tag{2.10}$$

for $r > 1$, where $z = \Psi(w) = w(1 + 1/w^m)^{2/m}$ maps the exterior of $|w| = 1$ onto the exterior of S_m (cf. [3, p. 395]).

We now compute the kth moment M_k from (2.10). Let

$$t = \Psi(s) = s\left(1 + \frac{1}{s^m}\right)^{2/m} = (s^{m/2} + s^{-m/2})^{2/m}$$

in (2.10). Then

$$\begin{aligned} dt &= (s^{m/2} + s^{-m/2})^{2/m-1}(s^{m/2-1} - s^{-m/2-1})\, ds \\ &= (s^{m/2} + s^{-m/2})^{-(m-2)/m}(s^{m/2} - s^{-m/2})\frac{ds}{s} = t^{1-m/2} i\sqrt{4 - t^m}\frac{ds}{s}, \end{aligned}$$

and so

$$M_k = \frac{1}{\pi i} \int_{|s|=r} [\Psi(s)]^k \frac{ds}{s} = \frac{1}{\pi} \int_{C_r} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}}\, dt,$$

where C_r is the image of $|w| = r$ under the mapping $\Psi(w)$.

Letting $r \to 1$, we get

$$M_k = \frac{1}{\pi} \int_{S_m} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}}\, dt.$$

By (2.9) we have

$$\int_{S_m} t^k \, d\mu^*(t) = M_k = \frac{1}{\pi} \int_{S_m} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}}\, dt, \qquad k = 0, 1, \ldots.$$

Thus, $d\mu^*(t) = d\mu(t)$, where $d\mu(t)$ is defined in (2.3). □

We have completed the proof of the theorem.

The situation for obtaining the locations of the zeros of $F_n(z)$ for every n is not so favorable. However, supported by (iii) in our theorem, and numerical experiments, we formulate the following.

Conjecture. *The zeros of Faber polynomials generated by the regular m-star S_m ($m = 3, 4, \ldots$) are located on S_m for every $n \geq 1$.*

For a sequence of polynomials $F_{nm}(z)$, and $F_{nm+m/2}(z)$ when m is even, our theorem confirmed this conjecture. For other sequences of Faber polynomials, we provide in §3 numerical results produced by Matlab in the cases when $m = 3, 4$, and 5.

3. Numerical results on the zeros of Faber polynomials

We first briefly state our algorithm for computing the zeros of Faber polynomials generated by a set E.

Let

$$\Psi(z) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \tag{3.1}$$

be the mapping function as in (1.2) associated with a set E. Then

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad |w| > \rho_E, \ z \in E. \tag{3.2}$$

It follows from (3.2) that $F_n(z)$ satisfies the recurrence relation

$$F_{n+1}(z) = (z - b_0)F_n(z) - \sum_{k=1}^{n-1} b_k F_{n-k}(z) - (n+1)b_n, \tag{3.3}$$

with initial condition $F_0(z) = 1$. This difference equation for Faber polynomials plays a very important role in computing the zeros of $F_n(z)$. Our algorithm consists of four steps:

Step 1. Determine the coefficients b_k's of the mapping function $\Psi(w)$ as in (3.1).

Step 2. Compute the coefficients of $F_n(z)$ by using (3.3).

Step 3. Find the zeros of $F_n(z)$.

Step 4. Plot the zeros in the complex plane.

We now apply the algorithm to compute the zeros of $F_n(z)$ for S_m. The results are shown in Figures 1–6 (see pp. 284–286).

For $m = 2, 3, \ldots$, we normalize the mapping function such that

$$\Psi(w) = w\left(1 + \frac{1}{w^m}\right)^{2/m}.$$

Then

$$\Psi(w) = w + \sum_{k=1}^{\infty} \frac{b_{mk-1}}{w^{mk-1}},$$

where

$$b_{mk-1} = (-1)^{k-1}\frac{2(m-2)(2m-2)\cdots((k-1)m-2)}{m^k k!}.$$

By the recurrence relation (3.3), we find, in particular for $m = 3$,

$$F_0(z) = 1, \quad F_1(z) = z, \quad F_2(z) = z^2,$$
$$F_3(z) = z^3 - 2, \quad F_4(z) = z^4 - \tfrac{8}{3}z, \quad F_5(z) = z^5 - \tfrac{10}{3}z^2,$$
$$F_6(z) = z^6 - 4z^3 + 2, \quad F_7(z) = z^7 - \tfrac{14}{3}z^4 + \tfrac{35}{9}z, \quad F_8(z) = z^8 - \tfrac{16}{3}z^5 + \tfrac{56}{9}z^2,$$
$$F_9(z) = z^9 - 6z^6 + z^3 - 2, \quad F_{10}(z) = z^{10} - \tfrac{20}{3}z^7 + \tfrac{110}{9}z^4 - \tfrac{400}{81}z.$$

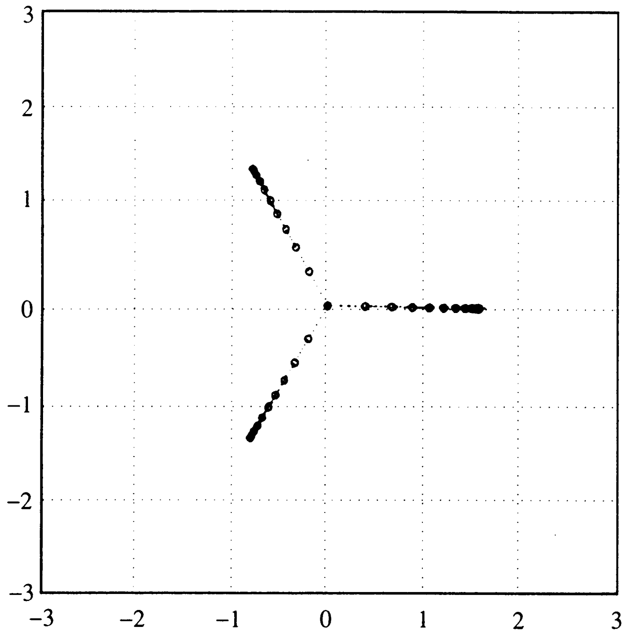

FIGURE 1. Zeros of $F_n(z)$ for 3-star when $n = 31$

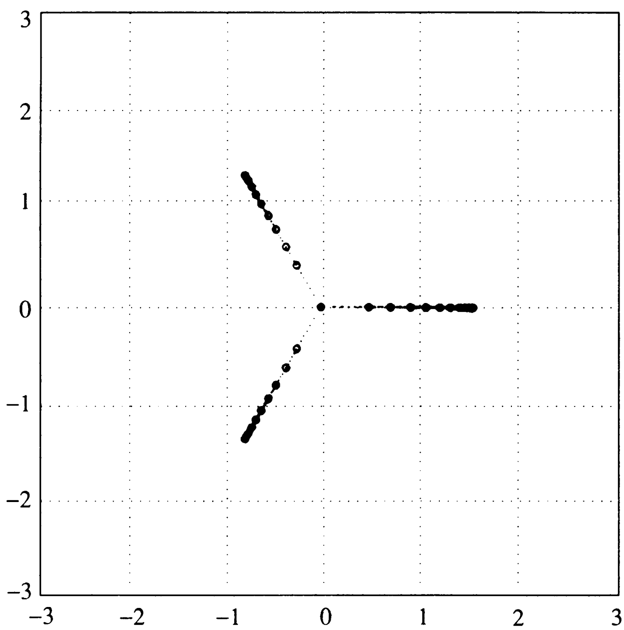

FIGURE 2. Zeros of $F_n(z)$ for 3-star when $n = 32$

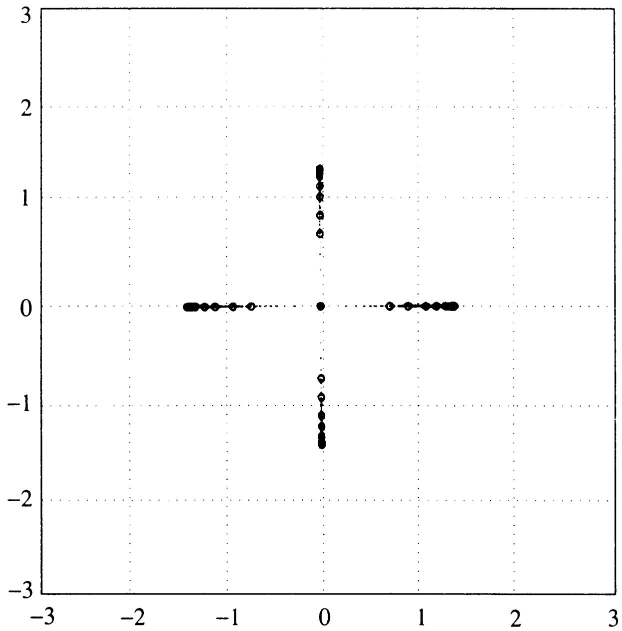

FIGURE 3. Zeros of $F_n(z)$ for 4-star when $n = 31$

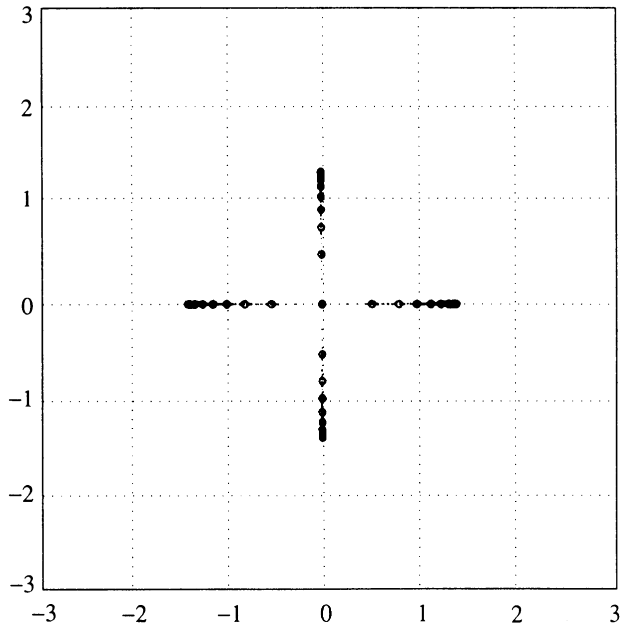

FIGURE 4. Zeros of $F_n(z)$ for 4-star when $n = 33$

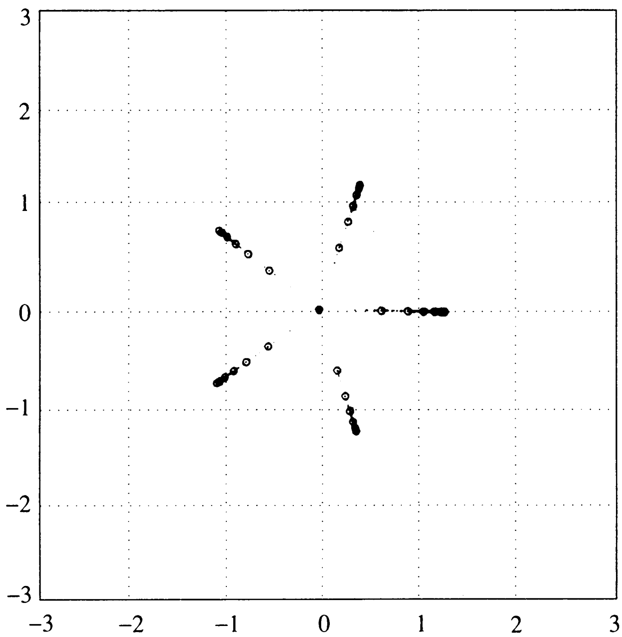

FIGURE 5. Zeros of $F_n(z)$ for 5-star when $n = 31$

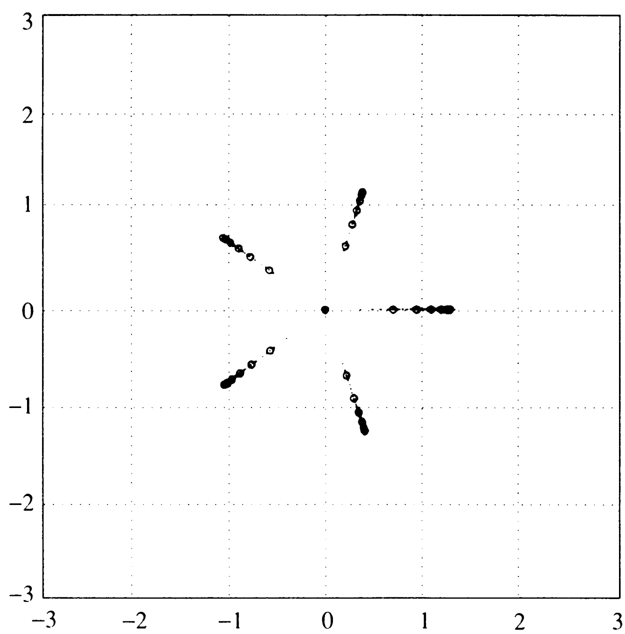

FIGURE 6. Zeros of $F_n(z)$ for 5-star when $n = 32$

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DEPARTMENT OF MATHEMATICS, NOVA UNIVERSITY, FT. LAUDERDALE, FLORIDA 33314
E-mail address, M. He: hem@polaris.nova.edu