ERROR BOUNDS FOR GAUSS-KRONROD QUADRATURE FORMULAE

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Abstract. The Gauss-Kronrod quadrature formula \( Q_{2n+1}^{GK} \) is used for a practical estimate of the error \( R_n^G \) of an approximate integration using the Gaussian quadrature formula \( Q_n^G \). Studying an often-used theoretical quality measure, for \( \kappa \), we prove best presently known bounds for the error constants

\[ c_s(R_{2n+1}^{GK}) = \sup_{\| f \|_{\infty} \leq 1} |R_{2n+1}^{GK}[f]| \]

in the case \( s = 3n + 2 + \kappa \), \( \kappa = \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \). A comparison with the Gaussian quadrature formula \( Q_{2n+1}^G \) shows that there exist quadrature formulae using the same number of nodes but having considerably better error constants.

1. Introduction and statement of the main results

For a given nonnegative and integrable weight function \( w \) on \( [-1, 1] \), a quadrature formula \( Q_n \) and the corresponding remainder \( R_n \) of (precise) degree of exactness \( \text{deg}(R_n) = s \) are linear functionals defined by

\[ Q_n[f] = \sum_{\nu=1}^{n} a_{\nu,n} f(x_{\nu,n}), \quad R_n[f] = \int_{-1}^{1} w(x)f(x)\,dx - Q_n[f], \]

\[ \text{deg}(R_n) = s \iff R_n[p_{\nu}] = 0, \quad \nu = 0, 1, \ldots, s, \]

\[ \neq 0, \quad \nu = s + 1, \]

with nodes \( -\infty < x_{1,n} < \cdots < x_{n,n} < \infty \) and weights \( a_{\nu,n} \in \mathbb{R} \). It is well known that the Gaussian quadrature formula \( Q_n^G[f] = \sum_{\nu=1}^{n} a_{\nu}^G f(x_{\nu}^G) \) having the highest possible degree of exactness \( \text{deg}(R_n^G) = 2n-1 \) exists uniquely under these assumptions.

In order to obtain an estimate for \( R_n[f] \) in practice, often a second quadrature formula is used whose nodes, for economical reasons, include \( x_{1,n}^G, \ldots, x_{n,n}^G \). If there exist \( n+1 \) further real and distinct nodes \( \xi_1, 2n+1, \ldots, \xi_{n+1}, 2n+1 \) and weights \( \beta_{1,2n+1}^{(1)}, \ldots, \beta_{n,2n+1}^{(1)}, \beta_{1,2n+1}^{(2)}, \ldots, \beta_{n,2n+1}^{(2)} \) such that the quadrature formula

\[ Q_{2n+1}^{GK}[f] := \sum_{\nu=1}^{n} \beta_{\nu,2n+1}^{(1)} f(x_{\nu}^{G,n}) + \sum_{\mu=1}^{n+1} \beta_{\mu,2n+1}^{(2)} f(\xi_{\mu,2n+1}) \]

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satisfies $\deg(R_{2n+1}^{GK}) \geq 3n+1$, then $Q_{2n+1}^{GK}$ is called a Gauss-Kronrod quadrature formula. Considering $Q_{2n+1}^{GK}[f]$ as a much better approximation than $Q_n^G[f]$, their difference serves as an estimate for $R_{2n+1}^G[f]$.

For surveys on this method, cf. Gautschi [5] and Monegato [8], while for existence results with respect to special weight functions $w$ cf., e.g., Szegő [14] and the recent results of Notaris [9] and Peherstorfer [10].

The Gauss-Kronrod method is basic for several practical integration routines, e.g. in QUADPACK [11], and hence one of the most often used methods for approximate integration with practical error estimate. Yet, there is still a need for a theoretical study of $R_{2n+1}^{GK}$ which could justify the important role Gauss-Kronrod quadrature plays in practical numerical computation (cf. [8, Part II.2]).

As a basis of a systematic study, and as an often-used quality measure, we define the error constants $c_s(R_{2n+1}^{GK})$ by

$$c_s(R_{2n+1}^{GK}) = \sup_{\|f^{(s)}\|_\infty \leq 1} |R_{2n+1}^{GK}[f]|,$$

where $\|g\|_\infty := \sup_{x \in [-1, 1]} |g(x)|$ for $g : [-1, 1] \to \mathbb{R}$. By definition, the error constants $c_s(R_{2n+1}^{GK})$ are the smallest real values independent of $f$ satisfying the standard error bounds

$$|R_{2n+1}^{GK}[f]| \leq c_s(R_{2n+1}^{GK})\|f^{(s)}\|_\infty$$

and are well known to exist whenever $\deg(R_{2n+1}^{GK}) \geq s - 1$ (cf. [1]).

Brass and Förster [2] proved

$$\frac{c_{2n}(R_{2n+1}^{GK})}{c_{2n}(R_n^G)} \leq \text{const} \sqrt{n} \left( \frac{1}{3.493 \ldots} \right)^n$$

and considered this result as a theoretical explanation for the significant superiority of $Q_{2n+1}^{GK}$ over $Q_n^G$.

Only very little is known about the quality of $Q_{2n+1}^{GK}$ itself. Rabinowitz [12] proved the existence of $c_s(R_{2n+1}^{GK})$ for $s = 1, \ldots, 3n+2+\kappa$, $\kappa = \lceil \frac{n+1}{2} \rceil - \lfloor \frac{n}{2} \rfloor$ and nonexistence for $s > 3n+2+\kappa$, showing that $\deg(R_{2n+1}^{GK}) = 3n+1+\kappa$; cf. also Rabinowitz [13] for a proof of the nondefiniteness of $R_{2n+1}^{GK}$. Brass and Förster [2], Brass and Schmeisser [3], and Monegato [7] proved upper bounds for $c_{3n+2+\kappa}(R_{2n+1}^{GK})$. In the following theorem, we give lower bounds as well as new upper bounds for $c_{3n+2+\kappa}(R_{2n+1}^{GK})$ in the case $w \equiv 1$ that improve the hitherto best-known bounds.

**Theorem.** Let $w \equiv 1$, $n \geq 4$, $\kappa = \lceil \frac{n+1}{2} \rceil - \lfloor \frac{n}{2} \rfloor$. Then there holds

$$c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq \bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK}),$$

where

$$\bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK}) := \frac{1}{(3n+2+\kappa)!2^{3n+\kappa}} \left\{ \frac{17}{10\sqrt{3n-3(2+\kappa)}} + \frac{2\sqrt{2}}{\sqrt{(6n+3+2\kappa)(6n+5+2\kappa)}} \right\}.$$
Furthermore, for \( n \) even there holds
\[
c_{3n+2}(R_{2n+1}^{GK}) \geq \frac{1}{3e\sqrt{\pi}} \frac{2^{n+3}}{(3n+2)!} \frac{[n!]^4}{(2n)!(2n+1)!} \cdot \sqrt{\frac{3n-2}{n^2-2n(n-2)^2(n+2)(3n-1)^3(3n+1)^2}},
\]
while for \( n \) odd there holds
\[
c_{3n+3}(R_{2n+1}^{GK}) \geq \frac{1}{27e\sqrt{\pi}} \frac{2^{n+3}}{(3n+3)!} \frac{[n!]^4}{(2n)!(2n+1)!} \cdot \sqrt{\frac{3n-1}{n^2-3n(n-1)^2n^2(n+2)(n-3)^2(3n-3)(3n-2)^2}}.
\]

Remark. While the upper bound in the theorem improves known results, it may still be sharpened. However, an improvement can only be obtained by a polynomial factor, since it follows that
\[
\frac{c_{3n+2+\kappa}(R_{2n+1}^{GK})}{c_{3n+2+\kappa}(R_{2n+1}^{GK})} = O(n^5)
\]
if we replace \( c_{3n+2+\kappa}(R_{2n+1}^{GK}) \) by the lower bounds (1). A result of Brass and Schmeisser [3, Theorem 7] implies that amongst all quadrature formulae \( Q_m \) with positive weights and degree of exactness \( \deg(Q_m) \geq 3n+1+\kappa \), the Lobatto formula \( Q_{m^*}^{L_0} \) for \( m^* = \frac{3n+2+\kappa}{2} \) is worst with respect to the error constant \( c_{3n+2+\kappa}(R_{2n+1}^{GK}) \), i.e. (cf. [1, p. 149])
\[
c_{3n+2+\kappa}(R_{m^*}^{L_0}) = \pi 2^{-(3n+2+\kappa)}(1 + o(1)) \quad \text{as} \quad n \to \infty.
\]
The lower bounds (1) now prove that \( Q_{2n+1}^{GK} \) can be better than this upper bound only by a polynomial factor \( O(n^{5.5})^{-1} \). Note that Brass and Schmeisser [3, Remark 4] explicitly construct a positive quadrature formula \( Q_{3n+2+\kappa}^{BS} \) that satisfies
\[
c_{3n+2+\kappa}(R_{3n+2+\kappa}^{BS}) = \pi 4^{-(3n+2+\kappa)}(1 + o(1)) \quad \text{as} \quad n \to \infty.
\]
In the following corollary we will compare \( Q_{2n+1}^{GK} \) and \( Q_{2n+1}^{G} \) in order to show that there also exist quadrature formulae using the same number of nodes but having considerably better error constants.

Corollary. Let \( w \equiv 1, \ n \geq 1 \). Then \( c_{3n+2+\kappa}(R_{2n+1}^{G}) \leq c_{3n+2+\kappa}(R_{2n+1}^{GK}) \), and we have equality only in the case \( n = 1 \), where the Gaussian formula and the Gauss-Kronrod formula are identical. Furthermore, for \( n \geq 15 \) there holds the sharper bound
\[
\frac{c_{3n+2+\kappa}(R_{2n+1}^{GK})}{c_{3n+2+\kappa}(R_{2n+1}^{G})} \leq 3^{-n+1},
\]
while asymptotically we find
\[
\lim_{n \to \infty} \sqrt[n]{\frac{c_{3n+2+\kappa}(R_{2n+1}^{GK})}{c_{3n+2+\kappa}(R_{2n+1}^{G})}} = \sqrt[77]{\frac{6^6}{4.2013\ldots}}.
\]
2. Proofs of the results

In the sequel let \( m = \left\lfloor \frac{n+1}{2} \right\rfloor \), \( \kappa = m - \left\lfloor \frac{n}{2} \right\rfloor \), and let \( \mathcal{P}_s \) denote the space of polynomials of degree less than or equal to \( s \).

Proof of the Theorem. We will first prove the lower bounds (1). Let \( E_{n+1} \in \mathcal{P}_{n+1} \) be defined by

\[
E_{n+1} = \sum_{\nu=0}^{m-1} \alpha_{\nu,n} T_{n+1-2\nu} + \begin{cases} \alpha_{m,n} T_1, & n \text{ even}, \\ \frac{1}{2} \alpha_{m,n}, & n \text{ odd}, \end{cases}
\]

where \( T_\nu \) denotes the \( \nu \)th Chebyshev Polynomial of the first kind and

\[
\alpha_{0,n} = 1, \quad \alpha_{\nu,n} = -f_{\nu}, n - \sum_{\mu=1}^{\nu-1} f_{\mu,n} \alpha_{\nu-\mu,n}, \quad \nu = 2, 3, \ldots,
\]

(2)

\[
f_{0,n} = 1, \quad f_{\nu,n} = \sigma_{\nu,n} f_{\nu-1,n}, \quad \sigma_{\nu,n} = \left(1 - \frac{1}{2\nu}\right) \left(1 - \frac{1}{2n+2\nu+1}\right).
\]

For reasons of simplicity, when no ambiguity arises, we do not indicate the dependence on \( n \), i.e., \( \alpha_{\nu} := \alpha_{\nu,n} \), \( f_{\nu} := f_{\nu,n} \), and \( \sigma_{\nu} := \sigma_{\nu,n} \). The zeros of \( E_{n+1} \) are the additionally chosen nodes \( \xi_{1,2n+1}, \ldots, \xi_{n+1,2n+1} \) of \( Q^{2n+K} \) (cf. [7]). Rabinowitz [12] showed that with \( g_K = P_n E_{n+1} P_n + K \) (\( P_\nu \) denoting the \( \nu \)th Legendre Polynomial) there holds, for \( n \) even,

\[
R^{GK}_{2n+1}(g_0) = \frac{1}{2n+2}(\alpha_m - \alpha_{m+1}),
\]

while for \( n \) odd, there holds

\[
R^{GK}_{2n+1}(g_1) = \frac{2n+3}{(2n+2)(2n+4)}(\alpha_{m-1} - \alpha_{m+1}).
\]

From the definition of the error constants \( c_s(\mathcal{R}^{GK}_{2n+1}) \) we get

\[
c_s(\mathcal{R}^{GK}_{2n+1}) \geq \frac{R^{GK}_{2n+1}(f)}{\|f(s)\|_\infty} \quad \text{for all } f \in C^s[-1, 1]\setminus \mathcal{P}_{s-1}.
\]

Defining \( p_\nu(x) := x^\nu \), we conclude

\[
c_{3n+2+K}(\mathcal{R}^{GK}_{2n+1}) \geq \frac{R^{GK}_{2n+1}[p_{3n+2+K}]}{(3n+2+K)!}.
\]

Taking into account the leading coefficients of \( p_\nu \) and \( E_{n+1} \), one readily verifies

\[
g_K(x) = \frac{(2n)!}{2^n(n)!^2} 2^n \frac{(2n + 2 + 2\kappa)!}{2n+1+K[(n + 1 + \kappa)!]^2} x^{3n+2+\kappa} + p(x), \quad p \in \mathcal{P}_{3n+1+\kappa},
\]

and, using the linearity and the degree of exactness of \( R^{GK}_{2n+1} \),

\[
R^{GK}_{2n+1}[p_{3n+2+K}] = \frac{2^{n-1-\kappa}[n!]^4}{(2n)!}(\alpha_{m-K} - \alpha_{m+1}).
\]

We will now derive lower bounds for the difference \( |\alpha_{m-K} - \alpha_{m+1}| \). Szegö [14] proved that the sequence \((-\alpha_{\nu+1})\) is positive throughout, and completely monotonic, i.e.,

\[
(-1)^{\nu+1} \Delta^\nu \alpha_{\nu+1} > 0, \quad \nu = 0, 1, 2, \ldots,
\]

while for its sum he proved \( \sum_{\nu=0}^{\infty} \alpha_{\nu+1} = -1 \). In the following Lemma 1 we state some further properties of \( (\alpha_{\nu}) \) needed here; the proof of Lemma 1 will be given later.
Lemma 1. The sequence \((\alpha_\nu)\) satisfies
\[
\alpha_0 = 1, \quad \alpha_1 = -f_1, \quad \alpha_\nu = \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1-\mu}(\sigma_\nu - \sigma_{\nu-\mu}), \quad \nu \geq 2.
\]

Bounds for \((\alpha_\nu)\) are given by
\[
f_{\nu-1}(\sigma_\nu - \sigma_{\nu-1}) < |\alpha_\nu| < f_{\nu-1}(\sigma_\nu - \sigma_1).
\]

Lower bounds for the differences of \((\alpha_\nu)\) are, for every \(p \in \mathbb{N}\),
\[
p(\alpha_{\nu+1} - \alpha_{\nu}) \geq \alpha_{\nu+p} - \alpha_{\nu},
p(\alpha_{\nu+1} - \alpha_{\nu-1}) \geq \alpha_{\nu+2p+1} - \alpha_{\nu-1}.
\]

According to the lower bounds for the differences in Lemma 1, we are now looking for \(p = p(n)\) such that the right-hand sides of
\[
|\alpha_m - \alpha_{m+1}| \geq \frac{1}{p(n)} (f_{m-1}(\sigma_m - \sigma_{m-1}) - f_{m+p(n)-1}(1 - \sigma_1))
\]
for \(n\) even, and
\[
|\alpha_{m-1} - \alpha_{m+1}| \geq \frac{1}{p(n)} (f_{m-2}(\sigma_{m-1} - \sigma_{m-2}) - f_{m+2p(n)-1}(1 - \sigma_1))
\]
for \(n\) odd, be positive and as big as possible. (Recall the definition of \(m\) at the beginning of this section.) By (2) we calculate \(1 - \sigma_i = \frac{n+2}{2n+3}\), and for even \(n > 2\),
\[
\sigma_m - \sigma_{m-1} = \frac{4(5n - 3)}{(n-2)(3n-1)(3n+1)}
\]
while for odd \(n > 3\),
\[
\sigma_{m-1} - \sigma_{m-2} = \frac{4(5n^2 - 7n + 3)}{3n(n-3)(n-1)(3n-2)}.
\]

Now let \(n\) be even. The following representation of the \(f_\nu\) can easily be proved by induction:
\[
f_\nu = \frac{(2\nu)!}{[\nu!]^2} \left[ \frac{(n + \nu)!}{n!} \right]^2 \left\{ \frac{(2n+1)!}{(2n+2\nu+1)!} \right\}.
\]

We determine \(p = p(n)\) such that
\[
f_{m-1} \frac{4(5n - 3)}{(n-2)(3n-1)(3n+1)} - f_{m+p(n)-1} \frac{n+2}{2n+3} > 0
\]
holds, which is equivalent to
\[
\frac{4(2n+3)(5n-3)}{(n-2)(n+2)(3n-1)(3n+1)} > \frac{[(m+2p(n)-2)!((m-1)!)^2(2n+2m-2)!}{[(m+p(n)-1)!]^2(2m-2)!((n+m-1)!)^2}
\]
\[
\cdot \frac{(n+m+p(n)-1)!^2}{(2n+2m+2p(n)-2)!}.
\]
For the right side of the inequality we find by some elementary calculation, using Stirling's formula,
\[
\frac{(2m + 2p(n) - 2)!}{[(m + p(n) - 1)!]^2} \frac{(2n + 2m - 2)!}{[(m + p(n) - 1)]^2} \frac{(n + m + p(n) - 2)!}{(2n + 2m + 2p(n) - 2)!} < \frac{\sqrt{e}}{2n + 2m + 2p(n) - 1}.
\]
Replacing the right side of (5) by this simpler estimate, the following condition for \( p(n) \) arises:
\[
2p(n) \geq \frac{\sqrt{e}(n - 2)(n + 2)(3n - 1)^2(3n + 1)}{4(5n - 3)(2n + 3) - (3n - 1)}.
\]
We may now choose \( p(n) \) for \( n \) even as
\[
p(n) := \left[ c \cdot \frac{\sqrt{e}(n - 2)(n + 2)(3n - 1)^2(3n + 1)}{8(2n + 3)(5n - 3)} - \frac{3n - 1}{2} \right], \quad c \geq 1,
\]
leading to
\[
|\alpha_m - \alpha_{m+1}| > \frac{1}{p(n)} \frac{4(5n - 3)}{(n - 2)(3n - 1)(3n + 1)} \left( 1 - \frac{1}{c} \right),
\]
where for \( n \geq 4 \) from (4), using Stirling's formula, we can show
\[
f_{m-1} \geq \frac{2}{3\sqrt{e\pi}} \sqrt{\frac{3n - 2}{n(n - 2)}}.
\]
Since by definition of \( p(n) \),
\[
p(n) \leq c \cdot \frac{\sqrt{e}(n - 2)(n + 2)(3n - 1)^2(3n + 1)}{8(5n - 3)(2n + 3)},
\]
we readily conclude
\[
|\alpha_m - \alpha_{m+1}| > \frac{64}{3e\sqrt{\pi}} \frac{3n - 2}{n(n - 2)} (n - 2)(n + 2)(3n - 1)^3(3n + 1)^2 \left( \frac{c - 1}{c^2} \right).
\]
Maximizing the right side with respect to \( c \geq 1 \) leads to \( c = 2 \) and therefore to the asserted inequality for \( n \) even. For the proof of the lower bound for \( |\alpha_{m-1} - \alpha_{m+1}|, \ n \ odd \), we can proceed in an analogous way. This time, a sufficient condition for the \( p(n) \) is
\[
\frac{4(2n + 3)(5n^2 - 7n + 3)}{3(n - 3)(n - 1)(n + 2)(3n - 2)} > \frac{2n + 2m - 4}{2n + 2m + 4p(n) + 1}.
\]
For \( n \) odd, we then choose
\[
p(n) := \left[ c \cdot \frac{\sqrt{e}(n - 3)(n - 1)(n + 2)(3n - 3)(3n - 2)}{16(2n + 3)(5n^2 - 7n + 3)} - \frac{3n + 2}{4} \right], \quad c \geq 1.
\]
Using the inequality
\[
f_{m-2} \geq \frac{2}{3\sqrt{e\pi}} \sqrt{\frac{3n - 1}{n(n - 3)}}
\]
leads again, after maximization with respect to \( c \), to the lower bound for \( n \) odd.

We will now prove the upper bounds stated in the Theorem with the help of the following lemma, which is contained in [2, Theorem 1].
Lemma 2. Let \( R \) be a continuous, linear functional on \( C[-1, 1] \) satisfying \( R[\mathcal{P}_m] = 0 \), and let

\[
K_{m,s}(x) := \frac{2}{\pi} \frac{m!2^m}{(2m)!} \sum_{\mu=0}^{s-1} \frac{1 - x^2}{(m - 1/2)R[T_{m+\mu}]P_{\mu}^{(m)}(x)} P_{\mu}^{(m)}(1),
\]

where \( P_{\mu}^{(m)} \) denotes the \( \mu \)th ultraspherical polynomial of order \( m \). The limit

\[
K_m(x) := \lim_{s \to \infty} K_{m,s}(x), \quad x \in [-1, 1],
\]

exists. If \( f \in C^m[-1, 1] \), then

\[
R[f] = \int_{-1}^{1} f^{(m)}(x)K_m(x) \, dx.
\]

If \( s = 0, 1, 2, \ldots \), then

\[
c_m(R) = \int_{-1}^{1} |K_{m,s}(x)| \, dx + \rho_s,
\]

where

\[
|\rho_s| \leq \frac{1}{m!2^{m-1}} \left\{ \frac{(2m)!s!}{(2m+s)!2m-1} \right\}^{1/2} \sup_{\mu \geq s} |R[T_{\mu+m}]|.
\]

Using (3) and \( T_\nu(x) = 2^{\nu-1}x^\nu + p(x), \ p \in \mathcal{P}_{\nu-1} \), we find

\[
R^G_{2n+1}[T_{3n+2+k}] = \frac{24^n[n!]^4}{(2n)!(2n+1)!}(\alpha_{m-k} - \alpha_{m+1}).
\]

Taking advantage of the monotonicity of \((-\alpha_{\nu+1})\), from Lemma 1 we get

\[
|\alpha_{m-k} - \alpha_{m+1}| < |\alpha_{m-k}| < f_{m-1-k}(\sigma_{m-k} - \sigma_1).
\]

Using (4), we obtain by the use of Stirling's formula

\[
f_{m-1-k} \leq \frac{2n+2-k}{3n} \sqrt{\frac{3n-2+k}{n(n-2-k)}}.
\]

According to (2) we get for \( n \) even

\[
\sigma_m - \sigma_1 = \frac{3n^2 - n - 10}{(2n+3)(3n+1)}
\]

and for \( n \) odd

\[
\sigma_{m-1} - \sigma_1 = \frac{3n^3 - 5n^2 - 14n + 6}{(n-1)(2n+3)(3n-2)}.
\]

Lemma 2 implies for \( R = R^G_{2n+1}, \ s = 1, \) and \( m = 3n + 2 + k \)

\[
c_{3n+2+k}(R^G_{2n+1}) \leq |R^G_{2n+1}[T_{3n+2+k}]] \int_{-1}^{1} \left| \frac{2}{\pi} \frac{(3n+2+k)!2^{3n+2+k}}{(6n+4+2k)!} \right| (1 - x^2)^{3n+2+k-1/2} \, dx
\]

\[
+ \frac{1}{(3n+2+k)!2^{3n+1+k}} \left\{ \frac{2(6n+4+2k)!6n+6+2k}{(6n+6+2k)!6n+3+2k} \right\}^{1/2}
\]

\[
\cdot \sup_{\mu \geq 2} |R^G_{2n+1}[T_{3n+2+k+\mu}]|.
\]
Since
\[
\int_{-1}^{1} \left| (1 - x^2)^{3n+2+\kappa - 1/2} \right| dx = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(3n + 2 + \kappa + \frac{1}{2})}{\Gamma(3n + 3 + \kappa)}
\]
and \(\sup_{\mu \geq 2} |R_{2n+1}^{GK}[T_{3n+2+\mu}]| \leq 4\), we find
\[
c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq \frac{1}{(3n + 2 + \kappa)!(23n+1+\kappa)} \cdot \left( |R_{2n+1}^{GK}[T_{3n+2+\kappa}]| + \frac{4\sqrt{2}}{\sqrt{(6n + 3 + 2\kappa)(6n + 5 + 2\kappa)}} \right).
\]
Using (6), (7), (8), (9), and (10), we conclude the result after some simplifying calculation. \(\square\)

Proof of Lemma 1. The recursion formula (2) for \(\alpha_\nu\) and \(\alpha_{\nu-1}\), respectively, is
\[
\alpha_\nu = f_\nu - \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-\mu}, \quad 0 = f_{\nu-1} - \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1-\mu}.
\]
Multiplication of the first by \(f_{\nu-1}\) and the second by \(-f_\nu\), and summation, yields
\[
f_{\nu-1}\alpha_\nu = \sum_{\mu=1}^{\nu-1} \alpha_\mu [f_{\nu-1-\mu} f_\nu - f_\nu f_{\nu-1}] = \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1-\mu} (\sigma_\nu - \sigma_{\nu-\mu})
\]
\[
= f_{\nu-1} \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1-\mu} (\sigma_\nu - \sigma_{\nu-\mu}),
\]
leading to the first assertion (see also [6] for this method). Using \(f_\nu > 0\) and \(\sigma_\nu - \sigma_{\nu-\mu} > 0\) for \(\mu = 1, \ldots, \nu - 1\), we find that all terms in the right-hand sum have the same sign, and we conclude the second assertion, using the monotonicity of \(\sigma_\nu\). Since \((-\alpha_{\nu+1})\) is completely monotonic, the last two inequalities of Lemma 1 follow readily. \(\square\)

Proof of the Corollary. Theorem 4 of [2] states that
\[
c_{3n+2+\kappa}(R_{2n+1}^{G}) \leq \overline{c}_{3n+2+\kappa}(R_{2n+1}^{G}),
\]
where
\[
\overline{c}_{3n+2+\kappa}(R_{2n+1}^{G}) := \frac{\pi}{(3n + 2 + \kappa)!23n+2+\kappa} \cdot \left\{ \frac{(n - \kappa)! (6n + 4 + 2\kappa)! 7n + 4 + \kappa}{(7n + 4 + \kappa)! 8n + 4} \cdot \left( 1 + \frac{5}{3} \frac{(n + 1 - \kappa)(n + 2 - \kappa)}{(6n + 3 + 2\kappa)(7n + 5 + \kappa)} \right) \right\}^{1/2}.
\]
Using the lower bounds (1) for \(c_{3n+2+\kappa}(R_{2n+1}^{GK})\), the following inequality can be proved for \(n \geq 4\) after some (elementary) calculation involving Stirling's formula:
\[
\frac{c_{3n+2+\kappa}(R_{2n+1}^{G})}{c_{3n+2+\kappa}(R_{2n+1}^{GK})} < c \cdot n^{3/4} \left( \frac{66}{77} \right)^{n},
\]
(11)
with \( c = 0.62 \). Inequality (11) yields \( c_{3n+2+\kappa}(R_{2n+1}^G) \leq c_{3n+2+\kappa}(R_{2n+1}^{GK}) \) for \( n \geq 10 \) and \( c_{3n+2+\kappa}(R_{2n+1}^G)/c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq 3^{-n+1} \) for \( n \geq 68 \), where for \( 1 \leq n \leq 9 \) and \( 15 \leq n \leq 68 \), respectively, the results can be proved numerically (cf. [4]). In [2, Theorem 4] it is stated that the upper bound \( c_{3n+2+\kappa}(R_{2n+1}^G) \) can only be improved by \( O(n^{3/4}) \). Using this result and the remark in §1 to obtain also a lower bound for the ratio in (11), the corollary follows. □

3. Further remarks

1. Using similar methods as described above, respective results can be obtained in the more general case of an ultraspherical weight function (cf. [4]).

For \( w(x) = (1-x^2)^{\lambda-1/2}, \lambda \in (0, 1), n \geq 4, \kappa = \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{\lambda}{2} \rfloor \) there holds

\[
c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq \bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK}),
\]

where

\[
c_{3n+2+\kappa}(R_{2n+1}^{GK}) := \frac{1}{(3n + 2 + \kappa)!2^{3n+1+\kappa}} \left( \pi^\lambda \frac{9n+4\lambda}{2^\lambda-1\Gamma(1-\lambda)} \frac{9n-3\kappa+4\lambda}{(n+\lambda-1+\kappa)(9n+4\lambda)} \right)^\lambda \left\{ \frac{6(2n+2+\lambda)}{(n+\lambda-1+\kappa)(9n+4\lambda)} \right\}^{\lambda} + \frac{2\sqrt{\pi}(\lambda+(1+\kappa)/4)^{\lambda-1}}{\sqrt{(6n+3+2\kappa)(6n+5+2\kappa)}}.
\]

Lower bounds for \( c_{3n+2+\kappa}(R_{2n+1}^{GK}) \) can be found in [4], which for the quality of the upper bounds yield

\[
\frac{\bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK})}{c_{3n+2+\kappa}(R_{2n+1}^{GK})} = O(n^{3+1/\lambda}).
\]

The corresponding error constants of \( Q_{2n+1}^G \) are again significantly smaller (cf. [4]).

2. Using the methods derived in the proof of the theorem in §1, we can also prove bounds for the error constants \( c_{3n+2+\kappa-s}(R_{2n+1}^{GK}) \) of nonmaximum order (cf. [4] for details). In the case of constant \( s \in \mathbb{N} \), these bounds are again better than the hitherto best-known bounds. For their quality we can prove

\[
\frac{\bar{c}_{3n+2+\kappa-s}(R_{2n+1}^{GK})}{c_{3n+2+\kappa-s}(R_{2n+1}^{GK})} = O(n^{s+(s/2)}).
\]

In the case \( s = s(n) \in \mathbb{N} \), \( \lim_{n \to \infty} \frac{3n+2+s-n}{c_{3n+2+\kappa-s}(R_{2n+1}^{GK})} = A > 0 \), the new bounds are of quality

\[
\lim_{n \to \infty} \sqrt{\frac{\bar{c}_{3n+2+\kappa-s}(R_{2n+1}^{GK})}{c_{3n+2+\kappa-s}(R_{2n+1}^{GK})}} = \frac{A^A(3-A)(3-A/2)(3+A)(3+A/2)}{3^3(2A)^4}.
\]

The corresponding error constants of \( Q_{2n+1}^G \) can be proved to be significantly smaller for \( 2 < A \leq 3 \).

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