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ON UNIT GROUPS AND CLASS GROUPS OF QUARTIC FIELDS OF SIGNATURE (2, 1)

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ABSTRACT. This is the third and last paper of a series, now completing the description of the unit group and class group of all quartic number fields F of discriminant d_F with $|d_F| < 10^6$.

1. INTRODUCTION

In this paper we consider quartic fields F with two real conjugates. Using the tables of David Ford and the first two authors [2], we computed unit groups $U_F = \langle -1\rangle \times \langle \varepsilon_1\rangle \times \langle \varepsilon_2\rangle$ and class groups Cl_F of all 90671 number fields F whose discriminant d_F is bounded in absolute value by one million. A comparison shows that our results are not in precise agreement with the predictions of Cohen and Martinet [4]. However, this was not to be expected because of the relatively small range of discriminants under consideration.

After [3] and [5], this paper completes the description of the most important invariants of all quartic number fields F with $|d_F| < 10^6$.

2. UNIT GROUPS

The fundamental units were computed by using the generalized Voronoi algorithm [1]. The algorithm operates as follows: For $\alpha \in F$, let $\alpha^{(1)}, \alpha^{(2)}$ be the real conjugates of α, and $\alpha^{(3)}$, $\alpha^{(4)}$ the nonreal, complex conjugates; i.e., $\alpha^{(4)}$ is complex conjugate to $\alpha^{(3)}$. We set

$$|\alpha|_1 := |\alpha^{(1)}|,$$
$$|\alpha|_2 := |\alpha^{(2)}|,$$
$$|\alpha|_3 := |\alpha^{(3)}|^2.$$

A fractional ideal $\mathbf{a}$ in F is called *reduced* if

$$1 \in \mathbf{a} \quad \text{and} \quad \{\alpha \in \mathbf{a} : |\alpha|_i < 1,\ 1 \le i \le 3\} = \{0\}.$$

For $i \in \{1, 2, 3\}$ the i-neighbor of a reduced ideal $\mathbf{a}$ is defined as the ideal $\mathbf{b} = \frac{1}{\alpha}\mathbf{a}$ for $\alpha \in \mathbf{a}$ subject to

$$|\alpha|_i > 1, \qquad |\alpha|_j < 1 \quad \text{for } j \in \{1, 2, 3\} \setminus \{i\}$$

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and

$$\{\alpha' \in \mathbf{a} : |\alpha'|_i < \max\{1, |\alpha|_i\}\} = \{0\}.$$

In [1] it is proved that a system of fundamental units can be obtained as follows:

Algorithm.

Input: maximal order O_F of F.

Output: system $\{\varepsilon_1, \varepsilon_2\}$ of fundamental units.

found = false, $k = 1$, $\mathbf{a}_1 = o_F$.

While not found **do**

{

Compute 1-neighbor $\mathbf{a}_{k+1} = \frac{1}{\alpha_k}\mathbf{a}_k$ of $\mathbf{a}_k$.

If there is $l_1 < k$ with $\mathbf{a}_{l_1} = \mathbf{a}_{k+1}$

then found = true,

else $k = k + 1$.

}

found = false, $i = 1$, $\mathbf{b}_1 = \mathbf{a}_{l_1}$.

While not found **do**

{

Compute 2-neighbor $\mathbf{b}_{i+1} = \frac{1}{\beta_i}\mathbf{b}_i$ of $\mathbf{b}_i$.

If there is $l_2 \in \{l_1, \dots, k\}$ with $\mathbf{a}_{l_2} = \mathbf{b}_{i+1}$

then found = true,

else $i = i + 1$.

}

Set

$$\varepsilon_1 = \prod_{j=l_1}^{k} \alpha_j \quad \text{and} \quad \varepsilon_2 = \prod_{j=1}^{i} \beta_j / \prod_{j=l_1}^{l_2-1} \alpha_j.$$

End.

In the following table we show the magnitudes of the regulators

$$R_F := \left|\det\begin{pmatrix} \log|\varepsilon_1^{(1)}| & \log|\varepsilon_2^{(1)}| \\ \log|\varepsilon_1^{(2)}| & \log|\varepsilon_2^{(2)}| \end{pmatrix}\right|$$

in dependence on the Galois group structure:

	D4		S4		#/frequency	
#/frequency	9772	10.78%	80899	89.22%	90671	100%
$0 < R_F < 1$	6	0.06%	9	0.01%	15	0.02%
$1 \le R_F < 5$	485	4.96%	347	0.43%	832	0.92%
$5 \le R_F < 10$	1438	14.72%	1719	2.12%	3157	3.48%
$10 \le R_F < 20$	2624	26.85%	6430	7.95%	9054	9.99%
$20 \le R_F < 50$	3428	35.08%	23943	29.60%	27371	30.19%
$50 \le R_F$	1791	18.33%	48451	59.89%	50242	55.41%

3. Class groups

Since for the class group computation the same algorithm as in [3, 5] was used, we will not present details. The main idea is to compute all prime ideals with norm below the Zimmert bound [9]

$$M_F = \left\lfloor \frac{1}{6.792}\sqrt{-d_F} \right\rfloor$$

and to compute sufficiently many relations between those ideals (see [6, 8]).

We first show the distribution of noncyclic class groups:

D4	S4	$\sum$
244	125	369
0.25%	0.15%	0.41%

The following table shows the distribution of class numbers:

	D4		S4		#/frequency	
#/frequency	9772	10.78%	80899	89.22%	90671	100%
$h_F = 1$	5199	53.20%	68533	84.71%	73732	81.32%
$h_F = 2$	2839	29.05%	9270	11.46%	12109	13.35%
$h_F = 3$	501	5.13%	1462	1.81%	1963	2.16%
$h_F = 4$	769	7.87%	1037	1.28%	1806	1.99%
$5 \le h_F < 10$	420	4.30%	574	0.71%	994	1.10%
$10 \le h_F < 20$	42	0.43%	23	0.03%	65	0.07%
$20 \le h_F$	2	0.02%	0	0.00%	2	0.00%

We finally present the frequency of each class group structure and the corresponding minimal field discriminant (if greater than -10^6):

h_F	Cl_F	D4	S4	#
1	1	5199 (−275)	68533 (−283)	73732
2	2	2839 (−7975)	9270 (−6848)	12109
3	3	501 (−20975)	1462 (−25471)	1963
4	4	547 (−51207)	916 (−54764)	1463
4	2, 2	222 (−83600)	121 (−132800)	343
5	5	135 (−82975)	311 (−69128)	446
6	6	154 (−190400)	110 (−137300)	264
7	7	48 (−165744)	75 (−169924)	123
8	8	47 (−218975)	54 (−273491)	101
8	2, 4	21 (−319424)	4 (−804875)	25
9	9	15 (−323975)	20 (−326111)	35
10	10	14 (−451975)	9 (−645700)	23
11	11	9 (−593856)	14 (−436227)	23
12	12	11 (−367975)	—	11
12	2, 6	1 (−995600)	—	1
13	13	3 (−645056)	—	3
14	14	2 (−788975)	—	2
16	16	2 (−328975)	—	2
20	20	1 (−302975)	—	1
23	23	1 (−616475)	—	1
#		9772	80899	90671

The computations were done on Apollo workstations (CPU Motorola 68030), using the number-theoretic program package KANT (see [7]). All data can be obtained from the second author.

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