MATHEMATICS OF COMPUTATION
VOLUME 62, NUMBER 206
APRIL 1994, PAGES 923–930

A SPECIAL EXTENSION OF WIEFERICH'S CRITERION

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ABSTRACT. The following theorem is proved in this paper: "If the first case of Fermat's Last Theorem does not hold for sufficiently large prime l, then

$$\sum_x x^{l-2}\left[\frac{kl}{N} < x < \frac{(k+1)l}{N}\right] \equiv 0 \pmod{l}$$

for all pairs of positive integers N, k, $N \le 94$, $0 \le k \le N-1$." The proof of this theorem is based on a recent paper of Skula and uses computer techniques.

0. INTRODUCTION

The first case of Fermat's Last Theorem states that for each odd prime l the equation

$$x^l + y^l + z^l = 0$$

has no integral solution x, y, z with $l \nmid xyz$.

One of many methods investigating this problem was introduced by A. Wieferich. This method is connected with the Fermat quotients $q_l(a)$,

$$q_l(a) = \frac{a^{l-1}-1}{l},$$

defined for each integer a such that a is not divisible by l.

Let us assume in this paragraph that l is an odd prime which does not satisfy the first case of Fermat's Last Theorem.

In 1909, Wieferich [7] published the following important result:

$$q_l(2) \equiv 0 \pmod{l}.$$

Many mathematicians have extended this Wieferich criterion. The latest result is due to A. Granville and B. Monagan [1] and states $q_l(p) \equiv 0 \pmod{l}$ for each prime p such that $p \le 89$.

These considerations have been generalized by L. Skula. He studied the sums

$$s(k, N) = \sum_x x^{l-2}\left(\frac{kl}{N} < x < \frac{(k+1)l}{N}\right)$$

Received by the editor May 6, 1991 and, in revised form, November 25, 1991 and November 5, 1992.

1991 *Mathematics Subject Classification*. Primary 11D41.

Key words and phrases. The first case of Fermat's Last Theorem, Fermat quotient, Bernoulli numbers, Bernoulli polynomials.

for integers N, k, $1 \leq N \leq l-1$, $0 \leq k \leq N-1$. These sums are connected with the Fermat quotients by a formula introduced essentially by M. Lerch [2]:

$$q_l(N) \equiv N^{l-2} \sum_{k=0}^{N-1} ks(k, N) \pmod{l}.$$

Skula [4] proved

$$s(k, N) \equiv 0 \pmod{l}, \qquad 0 \leq k \leq N-1,$$

for each $N \in \{2, 3, \ldots, 10\} \cup \{12\}$.

In this paper, Skula's result is improved for integers $N \leq 94$ (Main Theorem 3.2), but only for sufficiently large primes l.

Remark. It is easy to prove that the statement

$$s(k, N) \equiv 0 \pmod{l}, \qquad 0 \leq k \leq N-1,$$

is equivalent to the statement

$$B_{l-1}\left(\frac{j}{N}\right) - B_{l-1} \equiv 0 \pmod{l}, \qquad 0 \leq j \leq N,$$

where B_n, $B_n(x)$ are the nth Bernoulli number and Bernoulli polynomial, respectively. Therefore, our result implies that the polynomial $B_{l-1}(t) - B_{l-1}$ has at least $1 + \sum_{N=1}^{94} \varphi(N) = 2703$ distinct zeros modulo l for sufficiently large prime l, where l does not satisfy the first case of Fermat's Last Theorem.

1. Basic notions and assertions

We will assume in this section that there is an odd prime l which does not satisfy the first case of Fermat's Last Theorem, briefly $(\mathrm{FLTI})_l$ fails; i.e., there exist integers x, y, z such that

$$x^l + y^l + z^l = 0, \qquad l \nmid xyz.$$

1.1. **Definition.** Let $\tau_1, \ldots, \tau_6$ denote the integers satisfying

$$\begin{aligned} x\tau_1 &\equiv -y \pmod{l}, & x\tau_3 &\equiv -z \pmod{l}, & y\tau_5 &\equiv -z \pmod{l}, \\ y\tau_2 &\equiv -x \pmod{l}, & z\tau_4 &\equiv -x \pmod{l}, & z\tau_6 &\equiv -y \pmod{l}. \end{aligned}$$

The definition of $\tau_1, \ldots, \tau_6$ implies

1.2. **Lemma.** *The integers $\tau_1, \ldots, \tau_6$ satisfy the following congruences*:

$$\begin{gathered} \tau_1\tau_2 \equiv \tau_3\tau_4 \equiv \tau_5\tau_6 \equiv 1 \pmod{l}, \\ \tau_1 + \tau_3 \equiv \tau_2 + \tau_5 \equiv \tau_4 + \tau_6 \equiv 1 \pmod{l}, \\ 0 \not\equiv \tau_i \not\equiv 1 \pmod{l}, \qquad 1 \leq i \leq 6. \end{gathered}$$

According to the results of Pollaczek ([3], See [1, Lemma 15]) we have

1.3. **Lemma.** *Let $r_1, \ldots, r_6$ denote the orders of the integers $\tau_1, \ldots, \tau_6$* mod l. *Then $r_1 = r_2$, $r_3 = r_4$, $r_5 = r_6$, and each of the products r_1r_3, r_3r_5, r_1r_5 is greater than or equal to*

$$\frac{3\log(l)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

1.4. **Definition.** Pollaczek introduced a matrix $A_s(t)$ of size $2\varphi(s) \times \varphi(s)$ (φ Euler's function) for integers $s \geq 2$ and variable t in [3]. Let $r(s, t)$ denote the rank of the matrix $A_s(t)$ over the finite $\mathbf{Z}/l\mathbf{Z}$.

According to the results from [1, Table 1] (see also [4, 5.1.1]) we obtain

1.5. **Lemma.** *Let s, t be integers, $2 \leq s \leq 46$ and the order of t modulo l be greater than* 44. *Then* $r(s, t) = \varphi(s)$.

1.6. **Definition.** Skula ([4, Definition 4.13]) has introduced the following square matrix $D_N = D_N(t)$ of order $\frac{\varphi(N)}{2}$ for integers $N \geq 3$ and variable t by the formula

$$D_N = D_N(t) = [t^{z(u,v)-1} + t_{u,v}^{N-1-z(u,v)}],$$
$$1 \leq u, v \leq \frac{N}{2}, \ \gcd(u, N) = \gcd(v, N) = 1,$$

where $z(u, v)$ is the integer such that $1 \leq z(u, v) \leq N-1$, $v \equiv uz(u, v) \pmod{N}$.

Let us denote $d_N(t) = \det D_N(t)$.

The next theorem follows from Skula's results ([4, Main Theorem 4.14, 5.4.2]).

1.7. **Theorem.** *Let N be an integer, $N \geq 2$, $\frac{(N-2)(N-1)}{2} < l$, and $\tau_1, \dots, \tau_6$ be the integers from* 1.1. *Assume that there exists $1 \leq a \leq 6$ such that the following conditions are satisfied*:

(a) *$d_M(\tau_a) \not\equiv 0 \pmod{l}$ for each integer $M \geq 3$, $M|N$*;

(b) *$r(s, \tau_a) = \varphi(s)$ for each integer s, $2 \leq s < \frac{N}{2}$.*

Then $s(k, N) \equiv 0 \pmod{l}$ for each $0 \leq k \leq N-1$.

2. Some auxiliary statements

2.1. **Lemma.** *Let p be a prime, $f(t)$, $g(t)$ be polynomials over $\mathbf{Z}$, the leading coefficients of which are not divisible by p. If f, g are relatively prime over the finite field $\mathbf{Z}/p\mathbf{Z}$, then f, g are relatively prime over $\mathbf{Q}$.*

Proof. It is sufficient to prove that $\gcd(f, g)$ over $\mathbf{Z}$ is a constant. Assume on the contrary that there exist polynomials h, u, v over $\mathbf{Z}$ such that

$$f = hu, \quad g = hv, \quad \deg(h) > 0. \tag{1}$$

We can consider f, g, h, u, v as polynomials over $\mathbf{Z}/p\mathbf{Z}$. Their degrees do not change because p does not divide the leading coefficients of these polynomials. Then the equation (1) holds also over $\mathbf{Z}/p\mathbf{Z}$, and this is a contradiction. □

2.2. **Theorem.** *Let m be a positive integer. There is an integer $L_0 = L_0(m)$ with the following property*:

Let $l > L_0$ be a prime for which $(FLTI)_l$ fails. Then there exist two different integers a, b, $1 \leq a$, $b \leq 6$, such that

$$\tau_a + \tau_b \equiv 1 \pmod{l}, \qquad r_a > m, r_b > m,$$

where r_a, r_b are the orders of the integers τ_a, τ_b modulo l.

Proof. Let L_0 be the smallest integer greater than

$$\left(\frac{1+\sqrt{5}}{2}\right)^{m^2/3}.$$

The proof then easily follows from Pollaczek's Lemmas 1.3. and 1.2. □

2.3. **Theorem.** *Let N be an integer, $2 \le N \le 94$, $d(t)$ be any common multiple of the polynomials $d_M(t)$, $3 \le M$, $M|N$. Let $g(t)$ be a polynomial such that:*

(a) *$g(t)$ is a product of some cyclotomic polynomials,*

(b) *$g(t)|d(t)$ over the ring $\mathbf{Z}[t]$ (we allow $g(t) = 1$).*

Let the polynomial $f(t) = \frac{d(t)}{g(t)}$ satisfy

$$\gcd(f(t), f(1-t)) = 1 \quad \textit{over } \mathbf{Q}. \tag{2}$$

Then there exists a positive integer L such that

$$s(k, N) \equiv 0 \pmod{l}, \qquad 0 \le k \le N-1,$$

for each prime $l > L$ for which $(FLTI)_l$ fails.

Proof. Suppose $f(t)$, $f(1-t)$ are relatively prime over the field $\mathbf{Q}$. Then there exist an integer c and integral polynomials $u(t)$, $v(t)$ such that

$$f(t)u(t) + f(1-t)v(t) = c. \tag{3}$$

Let c be the smallest integer with this property.

Let us put $n_0 = \max\{n, \Phi_n(t)|g(t)\}$ (Φ_n is the nth cyclotomic polynomial), $m = \max\{n_0, 45\}$, $L_0 = L_0(m)$ the integer from 2.2.

Let l be a prime, $l > L_0$, $l \nmid c$, for which $(\mathrm{FLTI})_l$ fails. According to 2.2 there exist different integers a, b, $1 \le a$, $b \le 6$, such that

$$\tau_a + \tau_b \equiv 1 \pmod{l}, \qquad r_a > m, r_b > m.$$

By (3) we have $f(\tau_a) \not\equiv 0 \pmod{l}$ or $f(\tau_b) \not\equiv 0 \pmod{l}$. Therefore, we can assume

$$f(\tau_a) \not\equiv 0 \pmod{l}. \tag{4}$$

Since $r_a > m \ge n_0$, we have

$$\Phi_n(\tau_a) \not\equiv 0 \pmod{l}, \qquad 1 \le n \le n_0,$$

and it follows that

$$g(\tau_a) \not\equiv 0 \pmod{l}. \tag{5}$$

Putting (4) and (5) together, we obtain

$$f(\tau_a)g(\tau_a) = d(\tau_a) \not\equiv 0 \pmod{l};$$

therefore,

$$d_M(\tau_a) \not\equiv 0 \pmod{l}$$

for all integers M, $3 \le M \le N$, $M|N$.

We can see that the integer a satisfies the first condition of Theorem 1.7. The second condition is satisfied according to 1.5. The proof now immediately follows from Theorem 1.7. □

What follows is useful for practical computer calculation. Instead of dealing with polynomials $d_M(t) = \det D_M(t)$, it allows to work with polynomials of lower degrees. These assertions follow from Washington's book [6, (4.5.26)]. For the convenience of readers we include proofs of these assertions.

Let χ be an even Dirichlet's character $\bmod M$. Let $f_\chi(t)$ be a polynomial of form

$$f_\chi(t) = \sum_i \chi(i) t^{i-1}, \qquad 1 \le i \le M, \ \gcd(i, M) = 1.$$

2.4. **Lemma.** *Let M be an integer, $M \ge 3$. Then*

$$\det D_M(t) = \pm \prod_\chi f_\chi(t),$$

where the product is over all even Dirichlet's characters $\bmod M$.

Proof. Let $\langle\alpha\rangle$ denote the fractional part of a real number α. It is easy to see that

$$x \equiv M\left\langle \frac{x}{M}\right\rangle \pmod{M}$$

for each integer x.

According to 1.6 we have

$$D_M(t) = [t^{-1}(t^{z(u,v)} + t^{M-z(u,v)})]_{v,u}, \qquad 1 \le u,\ v \le \frac{M}{2},$$

$$\gcd(u, M) = \gcd(v, M) = 1,$$
$$1 \le z(u, v) \le M-1,\ v \equiv uz(u, v) \pmod{M}.$$

Putting $i \equiv \pm u^{-1} \pmod{l}$, so that $1 \le i \le \frac{M}{2}$, we get

$$d_M(t) = \pm t^{-\varphi(M)/2} \det A,$$

where A is a matrix of the form

$$A = [t^{M\langle iv/M\rangle} + t^{M\langle -iv/M\rangle}]_{i,v}, \qquad 1 \le i,\ v \le \frac{m}{2},\ \gcd(i, M) = \gcd(v, M) = 1.$$

Now it is sufficient to show that

$$\det A = \pm t^{\varphi(M)/2} \prod_\chi f_\chi(t),$$

where the product is over all even characters $\bmod M$.

Let B be the square matrix

$$B = [\chi(i)]_{\chi, i},$$

χ an even Dirichlet's character $\bmod M$, $1 \le i \le \frac{M}{2}$, $\gcd(i, M) = 1$. It is easy to prove that this matrix is nonsingular (see, e.g., Van der Waerden [5, §§124–126]), and we have

$$BA = \left[\sum_i \chi(i) t^{M\langle iv/M\rangle}\right] = \left[\chi^{-1}(v) \sum_i \chi(i) t^i\right]_{\chi, v}$$
$$(1 \le i \le M, \gcd(i, M) = 1);$$

hence

$$\det B \det A = \pm t^{\varphi(M)/2} \left(\prod_\chi f_\chi(t)\right) \det B.$$

This completes the proof. □

2.5. **Lemma.** *Let* χ *be an even character* $\bmod M$ *of order* $n \geq 1$. *Then the polynomial*

$$F_\chi(t) = \prod_a f_{\chi^a}(t), \qquad 1 \leq a \leq n, \ \gcd(a, n) = 1,$$

is a polynomial with integer coefficients.

Proof. The polynomial $f_\chi(t)$ is polynomial over the field $\mathbf{Q}(\xi_n)$, $\xi_n = e^{2\pi i/n}$. Let us consider the Galois group G of the extension $\mathbf{Q}(\xi_n)/\mathbf{Q}$. It is well known that

$$G = \{\sigma_s, \ s \in \mathbf{Z}, \ 1 \leq s \leq n, \ \gcd(s, n) = 1, \ \sigma_s(\xi_n) = \xi_n^s\}.$$

Every isomorphism σ_s can be extended in the natural way on the ring $\mathbf{Q}(\xi_n)[t]$, and obviously

$$\sigma_s(F_\chi(t)) = F_\chi(t).$$

Since $F_\chi(t)$ is an element of $\mathbf{Z}(\xi_n)[t]$, we have $F_\chi(t) \in \mathbf{Z}[t]$. □

3. Main results

Let N be an integer, $3 \leq N \leq 94$. By 2.4, 2.5 we can express the polynomial $d_N(t)$ as a product of integers polynomials $F_\chi(t)$. Let K_N denote the number of these polynomials. We will enumerate them (for example according to the values of their degrees) and add the index N so we have

$$d_N(t) = \prod_{i=1}^{K_N} F_{N,i}(t).$$

Let $g_{N,i}$ be the product of all cyclotomic polynomials dividing $F_{N,i}$, and put $f_{N,i} = F_{N,i}/g_{N,i}$ for each $1 \leq i \leq K_N$. According to 2.1 the condition (2) holds if we find a prime $p = p(L, M, i, j)$ for each set of integers L, M, i, j, $3 \leq L$, M, $L|N$, $M|N$, $1 \leq i \leq K_L$, $1 \leq j \leq K_M$ such that

$$\gcd(f_{L,i}(t), f_{M,j}(1-t)) = 1 \quad \text{over } \mathbf{Z}/p\mathbf{Z}. \tag{6}$$

This was done using a personal computer. In most cases, (6) holds for polynomials $F_{L,i}(t)$, $F_{M,j}(1-t)$, and some prime $p \leq 17$, so it is sufficient to compute only polynomials $F_{N,i}(t)$, $F_{N,i}(1-t)$ modulo small primes. The calculation of polynomials $F_{N,i}(t), g_{N,i}(t)$, $f_{N,i}(t)$, and $f_{N,i}(1-t)$ over $\mathbf{Z}$ is necessary only in a few cases (for example, if $\Phi_3(t)|F_{N,i}(t)$, because $\Phi_3(t) = \Phi_3(1-t)$). The relation (6) also holds in these cases for some prime p, $p \leq 17$.

Therefore, from our computation we obtain the following lemma.

3.1. **Lemma.** *Let* L, M, i, j *be integers,* $3 \leq L$, M, $\operatorname{lcm}[L, M] \leq 94$, $1 \leq i \leq K_L$, $1 \leq j \leq K_M$. *Then there exists a prime* $p \in \{2, 3, 5, 7, 11, 17\}$ *such that the polynomials* $f_{L,i}(t)$, $f_{M,j}(1-t)$ *are relatively prime over* $\mathbf{Z}/p\mathbf{Z}$.

The Main Theorem follows now immediately from 3.1, 2.6, 2.1, and 1.6.

3.2. **Theorem.** *Let N be an integer, $2 \le N \le 94$. There exists an integer L such that*

$$s(k, N) \equiv 0 \pmod{l}, \qquad 0 \le k \le N-1,$$

for each prime $l > L$ for which the first case of Fermat's Last Theorem is false for prime exponent l.

3.3. *Remark.* Let us try to find a value for the number L in the last theorem. In our calculations we shall suppose that the polynomials $g_{n,i}$ have not been divided by cyclotomic polynomials $\Phi_n(t)$, $n > 45$. According to the proofs of 2.3 and 2.2, the first condition for the number L is that

$$L > \left(\frac{1+\sqrt{5}}{2}\right)^{45^2/3}.$$

The second condition is that L is greater than the largest prime dividing the number c in (3). This certainly holds if L is greater than the resultant of the polynomials $f(t)$, $f(1-t)$ (it is known that the number c divides this resultant—see [1, Lemma 20]).

We will find the rough upper bound of this resultant for the cases N being a prime. In these cases we have

$$f(t) = \frac{d_N(t)}{g(t)},$$

$$k = \deg f(t) = \deg f(1-t) \deg d_N(t) = \frac{\varphi(N)(N-2)}{2} = \frac{(N-1)(N-2)}{2}.$$

Let $f(t) = (t-\alpha_1)\cdots(t-\alpha_k)$ over the field of complex numbers.

Each complex number α_j is a root of some polynomial $f_\chi(t)$, so we have

$$|\alpha_j|^{N-2} \le \sum_{i=0}^{N-3} |\alpha_j|^i;$$

hence $|\alpha_j| < 2$.

It follows that

$$R(f(t), f(1-t)) = \prod_{i,j}(\alpha_i - (1-\alpha_j)) < 5^{k^2} \le 5^{(N-1)^2(N-2)^2/4}.$$

We have proved the next theorem.

3.4. **Theorem.** *Let N be a prime, $11 \le N \le 89$. Then*

$$s(k, N) \equiv 0 \pmod{l}, \qquad 0 \le k \le N-1,$$

for each prime $l > 5^{(N-1)^2(N-2)^2/4}$ for which the first case of Fermat's Last Theorem is false for prime exponent l.

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