

ON A THEOREM OF C. POSSE CONCERNING GAUSSIAN QUADRATURE OF CHEBYSHEV TYPE

KLAUS-JÜRGEN FÖRSTER

ABSTRACT. We consider (n, m) Chebyshev formulae of algebraic degree m using n nodes. The aim of this short note is to show that by a simple algebraic method C. Posse's theorem concerning Gaussian quadrature of Chebyshev type can be improved. Furthermore, we give an application of this method to Gauss-Kronrod quadrature of Chebyshev type.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let L be a (bounded) linear functional on $C[a, b]$. We say that L admits an (n, m) Chebyshev formula if there are n nodes $x_{\nu, n}^C \in \mathbb{R}$ such that

$$(1) \quad L[p_{\mu}] = \frac{L[p_0]}{n} \sum_{\nu=1}^n (x_{\nu, n}^C)^{\mu} \quad \text{for } \mu = 0, 1, \dots, m,$$

where, here and in the following, p_{μ} denotes the monomial $p_{\mu}(x) = x^{\mu}$.

Chebyshev formulae have been investigated for more than a hundred years (see, e.g., [6] and the references cited therein). They were first considered by Chebyshev [3], who showed that the linear functionals $T_{\alpha, \beta, \gamma}$,

$$(2) \quad T_{\alpha, \beta, \gamma}[f] := \alpha \int_{\beta}^{\gamma} \frac{f(x)}{\sqrt{|x - \beta||x - \gamma|}} dx, \quad \alpha, \beta, \gamma \in \mathbb{R},$$

admit $(n, 2n - 1)$ Chebyshev formulae for each $n \in \mathbb{N}$, i.e., that each Gaussian quadrature formula is of Chebyshev type. Let us additionally note that the linear functionals $S_{\eta, \xi}$,

$$(3) \quad S_{\eta, \xi}[f] := \eta f(\xi), \quad \eta, \xi \in \mathbb{R},$$

trivially admit (n, m) Chebyshev formulae for all $n, m \in \mathbb{N}$. By a result of Posse [11], $T_{\alpha, \beta, \gamma}$ and $S_{\eta, \xi}$ are the only linear functionals on $C[a, b]$ admitting $(n, 2n - 1)$ Chebyshev formulae for each $n \in \mathbb{N}$. Recently, using methods of complex analysis and Faber polynomials, Peherstorfer [8] has proved the surprising result, that $T_{\alpha, \beta, \gamma}$ and $S_{\eta, \xi}$ also are the only (positive) linear functionals on $C[a, b]$, admitting $(1, 1)$ and $(n, n + 1)$ Chebyshev formulae for each $n \in \mathbb{N} \setminus \{1\}$. For other improvements of Posse's result see, e.g., [6, 5, 9].

Received by the editor November 6, 1992 and, in revised form, March 24, 1993.

1991 *Mathematics Subject Classification.* Primary 65D30, 41A55.

Key words and phrases. Chebyshev quadrature, Gaussian quadrature, Gauss-Kronrod quadrature.

The aim of this note is to show that by a simple algebraic method, introduced by Radau [12] and extended, e.g., in [2, 1, 4], we can obtain more general results. This method is based on Newton's identities (see, e.g., [10, p. 150 ff]), which, for n arbitrary complex numbers z_ν , yield

$$(4) \quad \begin{cases} s_1 + a_1 = 0, \\ s_2 + a_1 s_1 + 2a_2 = 0, \\ \dots \\ s_{n-1} + a_1 s_{n-2} + \dots + a_{n-2} s_1 + (n-1)a_{n-1} = 0, \\ s_{\lambda+n} + a_1 s_{\lambda+n-1} + \dots + a_{n-1} s_{\lambda+1} + a_n s_\lambda = 0 \quad (\lambda = 0, 1, 2, \dots), \end{cases}$$

where

$$(5) \quad s_\mu = \sum_{\nu=1}^n z_\nu^\mu \quad (\mu = 0, 1, 2, \dots),$$

$$F(t) = (t - z_1)(t - z_2) \cdots (t - z_n) = t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n.$$

Theorem. Let $(n_i)_{i=1}^\infty$ be a strictly increasing sequence of natural numbers. Let L and H be linear functionals on $C[a, b]$, both admitting (n_i, n_{i+1}) Chebyshev formulae for each $i \in \mathbb{N}$. If

$$(6) \quad L[p_\mu] = H[p_\mu] \quad \text{for } \mu = 0, 1, \dots, n_1,$$

then the identity $L = H$ follows.

Proof. First, let $L[p_0] \neq 0$. Applying Newton's identities to (1), we get for $i = 1, 2, \dots$

$$(7) \quad \sum_{\nu=1}^{n_i} (x_\nu^{C, n_i})^\mu = \frac{n_i}{L[p_0]} L[p_\mu] \quad \text{for } \mu = 0, 1, \dots, n_{i+1}.$$

For given $L[p_0], L[p_1], \dots, L[p_{n_i}]$ we directly obtain that the values of $L[p_\mu]$ are uniquely determined for each $\mu \in \{0, 1, 2, \dots, n_{i+1}\}$. Using (6), we have $L[p_\mu] = H[p_\mu]$ for each $\mu \in \mathbb{N}_0$. Since L and H are bounded, by the approximation theorem of Weierstrass the result follows. If $L[p_0] = 0$, then with (1) we have $L = H \equiv 0$. \square

The following corollary extends the results on Chebyshev formulae mentioned in the introduction.

Corollary 1. Let $(n_i)_{i=1}^\infty$ be a strictly increasing sequence of natural numbers. Let L be a linear functional on $C[a, b]$ admitting (n_i, n_{i+1}) Chebyshev formulae for each $i \in \mathbb{N}$. Then there exist $\eta, \xi \in \mathbb{R}$ or $\alpha, \beta, \gamma \in \mathbb{R}$, such that, using the notations in (2) and (3),

$$(i) \quad L = S_{\eta, \xi} \quad \text{if } n_1 = 1,$$

$$(ii) \quad L = S_{\eta, \xi} \quad \text{or } L = T_{\alpha, \beta, \gamma} \quad \text{if } n_1 = 2 \quad \text{and } n_{i+1} < 2n_i \quad \text{for each } i \in \mathbb{N}.$$

Proof. As in the proof of the theorem, we can assume that $L[p_0] \neq 0$. First, let $L[p_2]L[p_0] = L^2[p_1]$. We define

$$(8) \quad H = S_{\eta, \xi}, \quad \eta = L[p_0], \quad \xi = L[p_1]/L[p_0].$$

We have $L[p_\mu] = H[p_\mu]$ for $\mu = 0, 1, 2$; therefore, the result follows from the theorem. Now let $L[p_2]L[p_0] > L^2[p_1]$. We define

$$(9) \quad \bar{H} = T_{\alpha, \beta, \gamma}, \quad \alpha = \frac{L[p_0]}{\pi}, \quad \beta = \frac{L[p_1] - \delta}{L[p_0]}, \quad \gamma = \frac{L[p_1] + \delta}{L[p_0]},$$

$$\delta = [2(L[p_2]L[p_0] - L^2[p_1])]^{1/2}.$$

A short calculation, using

$$(10) \quad \bar{H}[f] = \alpha \int_{-1}^1 f\left(\frac{\gamma - \beta}{2}y + \frac{\gamma + \beta}{2}\right) (1 - y^2)^{-1/2} dy,$$

gives $L[p_\mu] = \bar{H}[p_\mu]$ for $\mu = 0, 1, 2$. Since \bar{H} admits $(n, 2n - 1)$ Chebyshev formulae for each $n \in \mathbb{N}$, the result follows from the theorem. Finally, let $L[p_2]L[p_0] < L^2[p_1]$. This inequality is equivalent to the inequality

$$(11) \quad (x_{1,2}^C)^2 + (x_{2,2}^C)^2 < (x_{1,2}^C + x_{2,2}^C)^2/2,$$

which is impossible for real numbers. \square

Remarks. 1. We say that a linear functional L on $C[a, b]$ admits extended (n, m) Chebyshev formulae if $x_{\nu, n}^C \in \mathbb{C}$, $x_{\nu, n}^C$ real or complex conjugate, and (1) holds. Newton's identities (4) are valid for $z_\nu \in \mathbb{C}$, and therefore the theorem is also valid for extended Chebyshev formulae.

2. In the proof of Corollary 1, we see that (11) is possible if and only if $x_{1,2}^C, x_{2,2}^C \in \mathbb{C}$, $x_{1,2}^C = u + iv$, $x_{2,2}^C = u - iv$, and $v \neq 0$. We obtain $L[p_1] = uL[p_0]$, $L[p_2] = (u^2 - v^2)L[p_0]$. Defining $\tilde{f} := (p_1 - up_0)^2 + v^2p_0/2$, we have

$$(12) \quad L[\tilde{f}] = -\frac{1}{2}v^2L[p_0], \quad \tilde{f} > 0,$$

which is impossible if $L[p_0]L$ is a positive functional. Therefore, for positive functionals L , Corollary 1 also is valid for extended Chebyshev formulae. For $n_i = i + 1$ this has been proved in [8].

3. In Corollary 1(ii) the assumption $n_{i+1} < 2n_i$ cannot be omitted: consider the functional $\bar{S}_{\eta, \xi_1, \xi_2}$ defined by $\bar{S}_{\eta, \xi_1, \xi_2}[f] = \frac{\eta}{2}[f(\xi_1) + f(\xi_2)]$, which admits $(2n, m)$ Chebyshev formulae for all $n, m \in \mathbb{N}$.

4. In Corollary 1, we only have considered $n_1 = 1$ or $n_1 = 2$. For $n_1 > 2$, using the method described, we are also able to investigate linear functionals admitting (extended) Chebyshev formulae. The author intends to state such results and further applications in a forthcoming paper.

2. APPLICATION TO CHEBYSHEV FORMULAE HAVING PREASSIGNED NODES

The above method is also helpful if some of the nodes $x_{\nu, n}^C$ of the Chebyshev formulae are preassigned. As an example, we assume that the linear functional L also admits a Gaussian formula of order k ; i.e., there exist k nodes $x_{\nu, k}^G$ and k weights $a_{\nu, k}^G \in \mathbb{R}$ such that

$$(13) \quad L[p_\mu] = \sum_{\nu=1}^k a_{\nu, k}^G (x_{\nu, k}^G)^\mu \quad \text{for } \mu = 0, 1, \dots, 2k - 1.$$

If $L[p_0] \neq 0$, then a Gaussian formula of order 1 trivially always exists and is uniquely determined by

$$(14) \quad x_{1,1}^G = L[p_1]/L[p_0].$$

If there exists also a (3, 2) Chebyshev formula such that $x_{1,1}^G$ is one of its nodes, then a short calculation using (13) and (1) shows that

$$(15) \quad \begin{aligned} x_{1,3}^C &:= x_{1,1}^G, & x_{2,3}^C &= x_{1,3}^C - \sqrt{3/2}\psi, & x_{3,3}^C &= x_{1,3}^C + \sqrt{3/2}\psi, \\ \psi &= \frac{1}{L[p_0]} \sqrt{L[p_0]L[p_2] - L^2[p_1]}. \end{aligned}$$

Therefore, it is necessary and sufficient for the existence of such a (3, 2) Chebyshev formula that

$$(16) \quad L[p_0]L[p_2] \geq L^2[p_1].$$

Furthermore, if this (3, 2) Chebyshev formula is also a (3, 3) Chebyshev formula, then it follows that

$$(17) \quad L[p_3] = \frac{L[p_1]}{L^2[p_0]} (3L[p_0]L[p_2] - 2L^2[p_1]).$$

Inequality (16) is equivalent to the existence of a (2, 2) Chebyshev formula,

$$(18) \quad x_{1,2}^C = x_{1,1}^G - \psi, \quad x_{2,2}^C = x_{1,1}^G + \psi,$$

where ψ is defined in (15). A further short calculation shows that this (2, 2) Chebyshev formula is also a (2, 3) Chebyshev formula, i.e., a Gaussian formula of order 2, if and only if (17) additionally holds. Therefore, the existence of a (3, 3) Chebyshev formula using the node $x_{1,1}^G$ is equivalent to the existence of a (2, 3) Chebyshev formula.

Now, one may ask, e.g., if there exists an (n, m) Chebyshev formula having some of the nodes $x_{\nu,k}^G$ of a Gaussian formula of order k . In this situation, using methods from the theory of orthogonal polynomials, Notaris [7], for positive L , has recently proved the following interesting result: If, for each $n \in \mathbb{N}$, there exists a $(2n+1, 3n+1)$ Chebyshev formula with $\{x_{\nu,n}^G | \nu = 1, 2, \dots, n\} \subseteq \{x_{\nu,2n+1}^C | \nu = 1, 2, \dots, 2n+1\}$ —i.e., these Chebyshev formulae are so-called Gauss-Kronrod formulae—then L is of type $S_{\eta,\xi}$ being defined in (3). The following Corollary 2 extends this result.

Corollary 2. *Let L be a linear functional admitting a (3, 4) Chebyshev formula and $(2n+1, 2n+3)$ Chebyshev formulae for each $n \in \mathbb{N} \setminus \{1\}$. For $L[p_0] \neq 0$, let the node $x_{1,1}^G$ of the Gaussian formula of order 1 be a node of the (3, 4) Chebyshev formula, and let the two nodes $x_{1,2}^G, x_{2,2}^G$ of the Gaussian formula of order 2 be nodes of the (5, 7) Chebyshev formula. Then, there exist $\eta, \xi \in \mathbb{R}$ such that $L = S_{\eta,\xi}$.*

Proof. Let $L[p_0] \neq 0$. The existence of a (3, 3) Chebyshev formula using the node $x_{1,1}^G$ implies that the Gaussian formula of order 2 is a (2, 3) Chebyshev formula; see (17) and (18) above. If there exists a (5, 3) Chebyshev formula having nodes $x_{1,5}^C := x_{1,2}^G = x_{1,2}^C$ and $x_{2,5}^C := x_{2,2}^G = x_{2,2}^C$, then a short

calculation using

$$\begin{aligned}
 (19) \quad L[p_\mu] &= \frac{L[p_0]}{5} \left\{ \sum_{\nu=1}^2 (x_{\nu,2}^C)^\mu + \sum_{\nu=3}^5 (x_{\nu,5}^C)^\mu \right\} \\
 &= \frac{L[p_0]}{5} \left\{ 2 \frac{L[p_\mu]}{L[p_0]} + \sum_{\nu=3}^5 (x_{\nu,5}^C)^\mu \right\} \quad \text{for } \mu = 0, 1, 2, 3
 \end{aligned}$$

shows that $\{x_{3,5}^C, x_{4,5}^C, x_{5,5}^C\} = \{x_{1,3}^C, x_{2,3}^C, x_{3,3}^C\}$, and therefore, since this (5, 3) Chebyshev formula is also a (5, 4) Chebyshev formula, it follows that

$$(20) \quad L[p_4] = \frac{L[p_0]}{5} \left\{ \sum_{\nu=1}^3 (x_{\nu,3}^C)^4 + \sum_{\nu=1}^2 (x_{\nu,2}^C)^4 \right\}.$$

On the other hand, the existence of a (3, 4) Chebyshev formula having a node $x_{1,3}^C = x_{1,1}^G$ yields the identity

$$(21) \quad L[p_4] = \frac{L[p_0]}{3} \sum_{\nu=1}^3 (x_{\nu,3}^C)^4.$$

By comparing (20) and (21), it follows from (15) and (18) that

$$(22) \quad L^2[p_0]\psi^2 = L[p_0]L[p_2] - L^2[p_1] = 0.$$

Therefore, all nodes $x_{\nu,3}^C$ and $x_{\nu,5}^C$ are equal to $x_{1,1}^G$. Since the (5, 4) Chebyshev formula is also a (5, 7) Chebyshev formula, we have $L[p_\mu] = S_{\eta,\xi}[p_\mu]$ for $\mu = 0, 1, \dots, 7$, where η and ξ are given in (8). The result now follows from the theorem. If $L[p_0] = 0$, then with (1) we have $L \equiv 0$, which is fulfilled for $S_{\eta,\xi}$ with $\eta = 0$. \square

By Corollary 1 it follows that, if there exists a (2, 3) Chebyshev formula and $(2n + 1, 2n + 3)$ Chebyshev formulae for each $n \in \mathbb{N}$, then L is of type $S_{\eta,\xi}$ or $T_{\alpha,\beta,\gamma}$. For $(2n + 1, 2n + 2)$ Chebyshev formulae, using the above method, we have the following result.

Corollary 3. *Let L be a linear functional admitting $(2n + 1, 2n + 2)$ Chebyshev formulae for each $n \in \mathbb{N}$. For $L[p_0] \neq 0$ let $x_{1,1}^G = L[p_1]/L[p_0]$ be a node of each of these $(2n + 1, 2n + 2)$ Chebyshev formulae. Then there exist $\eta, \xi \in \mathbb{R}$ or $\alpha, \beta, \gamma \in \mathbb{R}$ such that $L = S_{\eta,\xi}$ or $L = T_{\alpha,\beta,\gamma}$.*

Proof. Let $L[p_0] \neq 0$. Since L admits a (3, 2) Chebyshev formula having the node $x_{1,3}^C := x_{1,1}^G$, the relations (15) imply that $L[p_0]L[p_2] \geq L^2[p_1]$. Therefore, there exists a functional H of type $S_{\eta,\xi}$ or $T_{\alpha,\beta,\gamma}$ such that $H[p_\mu] = L[p_\mu]$ for $\mu = 0, 1, 2$ —see the proof of Corollary 1. Now assume that, for given $n \in \mathbb{N}$,

$$(23) \quad H[p_\mu] = L[p_\mu] \quad \text{for } \mu = 0, 1, \dots, 2n.$$

In the following, by $x_{\nu,n}^{CH}$ we denote the nodes of the $(n, 2n - 1)$ Chebyshev formula of H . Using (1), we have

$$(24) \quad \sum_{\nu=1}^{2n+1} (x_{\nu,2n+1}^C)^\mu = (2n + 1) \frac{L[p_\mu]}{L[p_0]} \quad \text{for } \mu = 0, 1, \dots, 2n + 2.$$

From (23) it follows that the nodes $x_{1,2n+1}^C, x_{2,2n+1}^C, \dots, x_{2n+1,2n+1}^C$ are also the nodes of a $(2n+1, 2n)$ Chebyshev formula of H . Let F_{2n+1}^C and F_{2n+1}^{CH} be the polynomials

$$(25) \quad F_{2n+1}^C(t) = \prod_{\nu=1}^{2n+1} (t - x_{\nu,2n+1}^C), \quad F_{2n+1}^{CH}(t) = \prod_{\nu=1}^{2n+1} (t - x_{\nu,2n+1}^{CH}).$$

Newton's identities (4) and (5) now show that $F_{2n+1}^C(t) - F_{2n+1}^{CH}(t)$ is equal to a fixed constant c for all $t \in \mathbb{R}$. Since $x_{1,1}^G$ is a zero of F_{2n+1}^{CH} and since $x_{1,1}^G$ is also a zero of F_{2n+1}^C , it follows that $F_{2n+1}^C \equiv F_{2n+1}^{CH}$. Therefore, (24) yields $H[p_\mu] = L[p_\mu]$ for $\mu = 0, 1, 2, \dots, 2n+2$. Now, by induction, we have $H[p_\mu] = L[p_\mu]$ for each $\mu \in \mathbb{N}$, which, using the approximation theorem of Weierstrass, yields $L = H$. \square

BIBLIOGRAPHY

1. L. A. Anderson and W. Gautschi, *Optimal weighted Chebyshev type quadrature formulas*, *Calcolo* **12** (1975), 211–248.
2. R. P. Bailey, *Convergence of sequences of positive linear functional operations*, *Duke Math. J.* **2** (1936), 287–303.
3. P. L. Chebyshev, *Sur les quadratures*, *J. Math. Pures Appl.* (2) **19** (1874), 19–34.
4. K.-J. Förster and G. P. Ostermeyer, *On weighted Chebyshev-type quadrature formulas*, *Math. Comp.* **46** (1986), 591–599.
5. K.-J. Förster, *On the minimal variance of quadrature formulas*, *Numerical Integration III* (H. Braß and G. Hämmerlin, eds.), Birkhäuser, Basel, 1988, pp. 37–48.
6. W. Gautschi, *Advances in Chebyshev quadrature*, *Numerical Analysis* (G. A. Watson, ed.), *Lecture Notes in Math.*, vol. 506, Springer-Verlag, Berlin and New York, 1977, pp. 100–121.
7. S. E. Notaris, *On Gauss-Kronrod quadrature formulae of Chebyshev type*, *Math. Comp.* **58** (1992), 745–753.
8. F. Peherstorfer, *On Tchebycheff quadrature formulas*, *Numerical Integration III* (H. Braß and G. Hämmerlin, eds.), Birkhäuser, Basel, 1988, pp. 172–186.
9. —, *Gauss-Tschebycheff quadratures formulas*, *Numer. Math.* **58** (1990), 273–281.
10. O. Perron, *Algebra*, de Gruyter, Berlin, 1932.
11. C. Posse, *Sur les quadratures*, *Nouv. Ann. Math.* (2) **14** (1875), 49–62.
12. R. Radau, *Sur les formules de quadrature à coefficients égaux*, *C.R. Acad. Sci. Paris* **90** (1880), 520–523.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT HILDESHEIM, 31141 HILDESHEIM, GERMANY