MATHEMATICS OF COMPUTATION
VOLUME 63, NUMBER 207
JULY 1994, PAGES 51-76

ERROR ESTIMATES FOR A FINITE ELEMENT METHOD FOR THE DRIFT-DIFFUSION SEMICONDUCTOR DEVICE EQUATIONS: THE ZERO DIFFUSION CASE

BERNARDO COCKBURN AND IOANA TRIANDAF

ABSTRACT. In this paper new error estimates for an explicit finite element method for numerically solving the so-called zero-diffusion unipolar model (a one-dimensional simplified version of the drift-diffusion semiconductor device equations) are obtained. The method, studied in a previous paper, combines a mixed finite element method using a continuous piecewise-linear approximation of the electric field, with an explicit upwinding finite element method using a piecewise-constant approximation of the electron concentration. By using a suitable extension of Kuznetsov approximation theory for scalar conservation laws, it is proved that, under proper hypotheses on the data, the $L^\infty(L^1)$-error between the approximate and exact electron concentrations of the zero-diffusion unipolar model is of order $\Delta x^{1/2}$. These estimates are sharp.

1. INTRODUCTION

This is the second paper of a series in which we introduce and analyze a new finite element method for numerically solving the equations of the drift-diffusion model for semiconductor devices, [14]. In the first paper of this series [1] we considered the so-called zero-diffusion unipolar model as our model problem:

$$u_\tau + (u\beta)_x = 0, \qquad \tau > 0,\ x \in (0, 1), \tag{1.1a}$$

$$u(\tau, 0) = u_0(\tau), \quad \text{if } \beta(\tau, 0) > 0,\ \tau \geq 0, \tag{1.1b}$$

$$u(\tau, 1) = u_1(\tau), \quad \text{if } \beta(\tau, 1) < 0,\ \tau \geq 0, \tag{1.1c}$$

$$u(0, x) = u_i(x), \qquad x \in (0, 1), \tag{1.1d}$$

where u is the (scaled) electron concentration and β is the (scaled) negative electric field given by

Received by the editor January 20, 1992 and, in revised form, December 14, 1992.
1991 *Mathematics Subject Classification*. Primary 65N30, 65N12, 35L60, 35L65.
Key words and phrases. Semiconductor devices, conservation laws, finite elements, convergence.
The first author was partly supported by the National Science Foundation (Grant DSM-9103997) and by the University of Minnesota Army High Performance Computing Research Center, and the second author was supported by a Fellowship of the University of Minnesota Army High Performance Computing Research Center. Current address: U. S. Naval Research Laboratory, Code 4700.3, Washington, D.C. 20375-5000.

$$-\beta_x = 1 - u, \qquad x \in (0, 1), \tau \geq 0, \tag{1.2a}$$

$$\beta = \phi_x, \qquad x \in (0, 1), \tau \geq 0, \tag{1.2b}$$

$$\phi(\tau, 0) = 0, \quad \text{for } \tau \geq 0, \tag{1.2c}$$

$$\phi(\tau, 1) = \phi_1(\tau), \quad \text{for } \tau \geq 0, \tag{1.2d}$$

where ϕ is the (scaled) electric potential. This model is obtained from the general equations of the drift-diffusion method by a series of simplifying assumptions; see the references given in [1]. In [1], a numerical scheme using a discretization of the equations (1.2) by a mixed finite element method and a discretization of the equations (1.1) by an explicit upwinding scheme was considered. The negative electric field β was approximated by a continuous piecewise linear function and the concentration u by a piecewise constant function. The scheme was proved to be stable and to converge, in a suitable topology, to the unique exact solution; see [1, Theorems 2.1, 2.2, 2.3].

In this paper we show that the $L^\infty(L^1)$-error between the exact electron concentration of (1.1) and (1.2), u, and its numerical approximation under consideration, u_h, is of order $\Delta x^{1/2}$ under proper conditions on the data. The main idea in obtaining the error estimates is to exploit the similarity of equation (1.1a) with classical conservation laws: if β is an evaluation operator, that is, if $\beta = \beta(u)$, then (1.1a) is nothing but a classical scalar conservation law. It is then reasonable to expect that after suitable changes, the Kuznetsov approximation theory for conservation laws [6] could be extended to the case under consideration. In this paper we prove that this is indeed the case. No error estimates seem to be known for bounded domains. Indeed, the error estimates for scalar conservation laws available in the literature consider the domain to be $\mathbb{R}^n$; see [6, 12, 8, 9, 10, 2, 3]. (A single convergence result for the bounded domain case is given in [7].)

Error estimates for the equations of the drift-diffusion model have been obtained in [4] and [11] for a finite element method that uses the modified method of characteristics, and in [13] for a method that extends the Scharfetter-Gummel method to the time-dependent case. In [4] and [11], the authors use the presence of parabolic terms (which we have dropped) to obtain L^2-error estimates that depend on second-order derivatives of the concentration u. Since the second-order derivatives of the concentration depend on a very small "viscosity" parameter λ, the constants for the L^2-error estimates blow up as λ goes to zero. In order to avoid such a situation, we have obtained error estimates depending only on first-order spatial derivatives of the electron concentration u. In [13], the parabolic equations for the concentrations are replaced by parabolic equations for the current densities through a suitable transformation. The resulting equations are then discretized by using a standard finite element method in space. This approach has two advantages: (i) it allows the author to use an L^2-parabolic technique to obtain error estimates, and (ii) it allows the error estimates to be independent of the derivatives of the electron and hole concentrations (although they do depend on second-order derivatives of the currents). Our approach is different in that (i) no previous transformation of the equations is required, (ii) an L^1-hyperbolic error analysis technique is used, and (iii) only first-order derivatives of the electron concentration appear in the error estimates.

The paper is organized as follows. In §2 we display the numerical scheme under consideration. In §3 we state the extension of the Kuznetsov approximation theory [6] to our framework; see Theorem 3.1. We then state and prove our error estimates, Theorems 3.2 and 3.3. In §4 we prove Theorem 3.1.

2. The numerical scheme

For the sake of completeness, we include here the description of the numerical scheme for which we obtain error estimates. For a complete discussion of the ideas of the numerical method we refer the reader to [1].

We first introduce some notation. Let $\{x_{i+1/2}\}_{i=0,\dots,n_x}$ be a partition of $[0, 1]$ with $x_{1/2} = 0$ and $x_{n_x+1/2} = 1$. Similarly, let $\{\tau^n\}_{n=0,\dots,n_T}$ be a partition of $[0, T]$, with $\tau^0 = 0$ and $\tau^{n_T} = T$. We set $I_i = (x_{i-1/2}, x_{i+1/2})$, $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, and $J^n = [\tau^n, \tau^{n+1})$, $\Delta\tau^n = \tau^{n+1} - \tau^n$. Define $\Delta x = \max_{i=1,\dots,n_x}\{\Delta x_i\}$ and $\Delta\tau = \max_{n=0,\dots,n_T-1}\{\Delta\tau^n\}$. We associate with these partitions the following spaces:

$$\begin{aligned}
V_{\Delta x} &= \{v_{\Delta x} \in \mathscr{C}^0(0, 1) \colon v_{\Delta x}|_{I_i} \in P^1(I_i),\ i = 1, \dots, n_x\},\\
W_{\Delta x} &= \{w_{\Delta x} \in L^\infty(0, 1) \colon w_{\Delta x}|_{I_i} \in P^0(I_i),\ i = 1, \dots, n_x\},\\
W_{\Delta\tau} &= \{w_{\Delta\tau} \text{ right-continuous} : w_{\Delta\tau}|_{J^n} \in P^0(J^n),\ n = 0, \dots, n_T - 1\}.
\end{aligned}$$

If $v_{\Delta x} \in V_{\Delta x}$, then $v_{i+1/2}$ denotes $v_{\Delta x}(x_{i+1/2})$ for $i = 0, \dots, n_x$. If $w_{\Delta x} \in W_{\Delta x}$, then w_i denotes the constant value $w_{\Delta x}(x)$, $x \in I_i$, for $i = 1, \dots, n_x$; the values w_0 and w_{n_x+1} denote the exterior trace at $x = 0$, $w_{\Delta x}(0-)$, and at $x = 1$, $w_{\Delta x}(1+)$, respectively. Finally, if $w_{\Delta\tau} \in W_{\Delta\tau}$, then w^n denotes the constant value $w_{\Delta\tau}(\tau)$, $\tau \in J^n$.

To discretize (1.1) and (1.2), we first discretize the data as follows:

$$\phi^n_{1,\Delta\tau} = \frac{1}{\Delta\tau^n}\int_{\tau^n}^{\tau^{n+1}} \phi_1(\tau)\,d\tau, \qquad n = 0, \dots, n_T - 1, \tag{2.1a}$$

$$u^n_{0,\Delta\tau} = \frac{1}{\Delta\tau^n}\int_{\tau^n}^{\tau^{n+1}} u_0(\tau)\,d\tau, \qquad n = 0, \dots, n_T - 1, \tag{2.1b}$$

$$u^n_{1,\Delta\tau} = \frac{1}{\Delta\tau^n}\int_{\tau^n}^{\tau^{n+1}} u_1(\tau)\,d\tau, \qquad n = 0, \dots, n_T - 1, \tag{2.1c}$$

$$(u_{i,\Delta x})_j = \frac{1}{\Delta x_j}\int_{x_{j-1/2}}^{x_{j+1/2}} u_i(x)\,dx, \qquad j = 1, \dots, n_x. \tag{2.1d}$$

The approximate solution u_h is taken to be in the space $W_{\Delta\tau} \otimes W_{\Delta x}$ and is required to satisfy the equation

$$(u_i^{n+1} - u_i^n)/\Delta\tau^n + (f^n_{i+1/2} - f^n_{i-1/2})/\Delta x_i = 0, \tag{2.2a}$$

where the numerical flux $f^n_{i+1/2} = f(u_i^n, u_{i+1}^n; \beta^n_{i+1/2})$ is the upwinding flux given by

$$f^n_{i+1/2} = u_i^n \beta^{n^+}_{i+1/2} + u^n_{i+1}\beta^{n^-}_{i+1/2}. \tag{2.2b}$$

The function $(\beta_h, \phi_h) \in W_{\Delta\tau} \otimes V_{\Delta x} \times W_{\Delta\tau} \otimes W_{\Delta x}$ is defined by the following mixed finite element method:

$$\begin{aligned} -\int_0^1 (\beta_h)_x(\tau^n, x) w_{\Delta x}(x)\, dx \\ = \int_0^1 (1 - u_h(\tau^n, x)) w_{\Delta x}(x)\, dx, \quad \forall w_{\Delta x} \in W_{\Delta x}, \end{aligned} \tag{2.3a}$$

$$\begin{aligned} \int_0^1 \beta_h(\tau^n, x) v_{\Delta x}(x)\, dx \\ = -\int_0^1 \phi_h(\tau^n, x)(v_{\Delta x})_x(x)\, dx + \phi_{1,\Delta\tau}(\tau^n) v_{\Delta x}(1), \quad \forall v_{\Delta x} \in V_{\Delta x}. \end{aligned} \tag{2.3b}$$

Thus the algorithm of our numerical method is:

(1) Compute the functions $u_{0,\Delta\tau}$, $u_{1,\Delta\tau}$, $u_{i,\Delta x}$, and $\phi_{1,\Delta\tau}$ by (2.1);
(2) Set $u_h(0, \cdot) = u_{i,\Delta x}(\cdot)$;
(3) For $n = 0, \dots, n_T - 1$ compute $u_h(\tau^{n+1}, \cdot)$ as follows:
  (i) Compute $(\beta_h(\tau^n, \cdot), \phi_h(\tau^n, \cdot))$ by using the mixed finite element method (2.3);
  (ii) Set $u_h(\tau^n, 0-) = u_{0,\Delta\tau}(\tau^n)$ and $u_h(\tau^n, 1+) = u_{1,\Delta\tau}(\tau^n)$;
  (iii) Compute $u_h(\tau^{n+1}, x)$ for $x \in (0, 1)$ by using the scheme (2.2).

3. The main results

In this section, we state and briefly discuss our main results. We begin with our key result, the extension of Kuznetsov's approximation result [6, Lemma 2] to our framework.

This approximation result gives a measure of the closeness of two arbitrary pairs of functions (u, β) and (v, η) satisfying the following regularity requirements:

$$\begin{aligned} &(u, \beta) \text{ and } (v, \eta) \text{ are right-continuous functions from } [0, T) \\ &\text{to } L^1(0, 1) \times W^{1,\infty}(0, 1) \text{ and have limits from the left on} \\ &(0, T], \end{aligned} \tag{3.1a}$$

$$u, v \in L^\infty(0, T; BV(0, 1)) \cap L^\infty(0, 1; L^1(0, T)), \tag{3.1b}$$

$$\beta(x), \eta(x) \in BV(0, T) \quad \text{for } x \in \{0, 1\}, \tag{3.1c}$$

in terms of the so-called entropy form $E^{\varepsilon_0,\varepsilon}(v, u; \eta)$, which measures how close the function (v, η) is to the exact solution of (1.1) and (1.2), and in terms of the following smoothness-measuring quantities:

$$
\begin{aligned}
\nu^{+}_{x,0}(\varepsilon, v; \eta) &= \sup_{0\le\Delta\le\varepsilon} \int_0^T |v(\tau, \Delta) - v(\tau, 0-)|\eta^{+}(\tau, 0)\, d\tau, \\
\nu^{-}_{x,1}(\varepsilon, v; \eta) &= \sup_{0\le\Delta\le\varepsilon} -\int_0^T |v(\tau, 1-\Delta) - v(\tau, 1+)|\eta^{-}(\tau, 1)\, d\tau, \\
\nu^{+}_{\tau,0}(\varepsilon_0, v) &= \sup_{0\le\Delta\le\varepsilon_0} \|v(\Delta) - v(0)\|_{L^1(0,1)}, \\
\nu^{-}_{\tau,T}(\varepsilon_0, v) &= \sup_{0\le\Delta\le\varepsilon_0} \|v(T-\Delta) - v(T)\|_{L^1(0,1)}, \\
\nu_{\tau}(\varepsilon_0, v) &= \sup_{\substack{|\tau-\tau'|\le\varepsilon_0 \\ \tau,\tau'\in[0,T]}} \|v(\tau) - v(\tau')\|_{L^1(0,1)}, \\
v_{\tau}(\varepsilon_0, \eta) &= \sup_{|\Delta|\le\varepsilon_0} \int_0^T \chi(\tau+\Delta) \left| \int_0^1 (\eta(\tau, x) - \eta(\tau+\Delta, x))\, dx \right| d\tau,
\end{aligned}
\tag{3.2}
$$

where $v(\cdot, 0-)$ denotes the boundary data for v at $x = 0$, $v(\cdot, 1+)$ the boundary data for v at $x = 1$, and χ is the characteristic function of the interval $[0, T]$.

We now define the entropy form $E^{\varepsilon_0,\varepsilon}(v, u; \eta)$, following [6]. Let ε_0 and ε be arbitrary positive real numbers. Let $w: \mathbb{R} \to \mathbb{R}$ be an even nonnegative function in $\mathscr{C}^\infty(\mathbb{R})$ with support contained in $[-1, 1]$ and such that $\int_{-1}^1 w = 1$. We set

$$
\varphi(\tau, x; \tau', x') = w_{\varepsilon_0}(\tau - \tau') w_\varepsilon(x - x'), \tag{3.3a}
$$

where $w_\nu(s) = w(s/\nu)/\nu$, $\forall s \in \mathbb{R}$. Finally, we denote by U an arbitrary even convex function with Lipschitz second derivative, such that $U(0) = 0$. Such a function will be called an "even entropy". Although Kuznetsov [6] used only the Lipschitz entropy $U(u) = |u|$, we need to consider smoother even entropy functions, namely,

$$
U_{1/M}(w) = \begin{cases} |w| - 1/2M & \text{for } |w| \ge 1/M, \\ Mw^2/2 & \text{for } |w| \le 1/M. \end{cases} \tag{3.3b}
$$

Notice that $L = \sup_{u\in\mathbb{R}} U'_{1/M}(u) = 1$, that the support of $U''_{1/M}$ is the interval $[-1/M, 1/M]$, and that $M = \sup_{u\in\mathbb{R}} U''_{1/M}(u)$.

For a general entropy function U, the entropy form $E^{\varepsilon_0,\varepsilon}(v, u; \eta)$ is defined as follows:

$$
E^{\varepsilon_0,\varepsilon}(v, u; \eta) = \int_0^T \int_0^1 \Theta(v, u(\tau, x); \eta; \varphi(\tau, x; \cdot, \cdot))\, dx\, d\tau, \tag{3.4a}
$$

where

$$
\begin{aligned}
&\Theta(v, c; \eta; \varphi(\tau, x; \cdot, \cdot)) \\
&\quad = -\int_0^T \int_0^1 U(v(\tau', x') - c)\varphi_{\tau'}(\tau, x; \tau', x')\, dx'\, d\tau' \\
&\qquad - \int_0^T \int_0^1 U(v(\tau', x') - c)\eta(\tau', x')\varphi_{x'}(\tau, x; \tau', x')\, dx'\, d\tau' \\
&\qquad + \int_0^1 U(v(T, x') - c)\varphi(\tau, x; T, x')\, dx' \\
&\qquad - \int_0^1 U(v(0, x') - c)\varphi(\tau, x; 0, x')\, dx' \\
&\qquad + \int_0^T G(v(\tau', 1-) - c, v(\tau', 1+) - c; \eta(\tau', 1))\varphi(\tau, x; \tau, 1)\, d\tau' \\
&\qquad - \int_0^T G(v(\tau, 0-) - c, v(\tau', 0+) - c; \eta(\tau', 0))\varphi(\tau, x; \tau', 0)\, d\tau' \\
&\qquad - \int_0^T \int_0^1 \{\eta_{x'}(\tau', x')V(v(\tau', x'), c)\varphi(\tau, x; \tau', x')\}\, dx'\, d\tau'
\end{aligned}
\tag{3.4b}
$$

and the "entropy" flux G and the function V are defined by

$$G(v_{\text{left}}, v_{\text{right}}; \eta) = U(v_{\text{left}})\eta^+ + U(v_{\text{right}})\eta^-, \tag{3.4c}$$

$$V(v, c) = U(v - c) - vU'(v - c). \tag{3.4d}$$

We indicate that we are taking U equal to $U_{1/M}$, defined by (3.3b), by writing $E_{1/M}^{\varepsilon_0, \varepsilon}$. We are now ready to state our approximation result.

Theorem 3.1. *Let* (u, β) *and* (v, η) *be functions satisfying the regularity conditions* (3.1). *Then, there exists a constant* C, *which is bounded provided that* T *is bounded and the regularity conditions* (3.1) *are satisfied, such that*

$$
\begin{aligned}
&\|v(T) - u(T)\|_{L^1(0,1)} \\
&\quad \le C \Bigg\{ \|v(0) - u(0)\|_{L^1(0,1)} \\
&\qquad + \|v(1+) - u(1+)\|_{L^1(0,T)} + \|v(0-) - u(0-)\|_{L^1(0,T)} \\
&\qquad + \int_0^T \left| \int_0^1 (\eta(\tau, x) - \beta(\tau, x))\, dx \right| d\tau \\
&\qquad + \int_0^T \|\partial_{x'}\eta(\tau') + (1 - v(\tau'))\|_{L^1(0,1)}\, d\tau' \\
&\qquad + \int_0^T \|\partial_x \beta(\tau) + (1 - u(\tau))\|_{L^1(0,1)}\, d\tau + \varepsilon + \varepsilon_0 + 1/M \\
&\qquad + \{\nu_{x,1}^-(\varepsilon, u; \beta) + \nu_{x,1}^-(\varepsilon, v; \eta) + \nu_{x,0}^+(\varepsilon, u; \beta) + \nu_{x,0}^+(\varepsilon, v; \eta)\} \\
&\qquad + \{\nu_{\tau,T}^-(\varepsilon_0, u) + \nu_{\tau,T}^-(\varepsilon_0, v) + \nu_{\tau,0}^+(\varepsilon_0, u) + \nu_{\tau,0}^+(\varepsilon_0, v)\} \\
&\qquad + \{\nu_\tau(\varepsilon_0, u) + \nu_\tau(\varepsilon_0, v) + v_\tau(\varepsilon_0, \beta)\} \\
&\qquad\qquad + E_{1/M}^{\varepsilon_0, \varepsilon}(v, u; \eta) + E_{1/M}^{\varepsilon_0, \varepsilon}(u, v; \beta) \Bigg\},
\end{aligned}
$$

and, for $\tau \in [0, T]$,

$$\begin{aligned}\|\eta(\tau) - \beta(\tau)\|_{L^\infty(0,1)} &\le \|v(\tau) - u(\tau)\|_{L^1(0,1)} + \|\partial_x \eta(\tau) + (1 - v(\tau))\|_{L^1(0,1)} \\ &\quad + \|\partial_x \beta(\tau) + (1 - u(\tau))\|_{L^1(0,1)} + \left| \int_0^1 (\eta(\tau, x) - \beta(\tau, x))\, dx \right|.\end{aligned}$$

Next, we give estimates for the error between the exact solution (u, β) of (1.1) and (1.2) and the approximate solution (u_h, β_h) given by the numerical scheme described in §2. Such an approximate solution was proven to be stable [1, Theorem 3.1] and to converge to the exact solution [1, Theorem 3.3] under the following regularity conditions on the data:

(3.5a) $\quad u_0(\tau), u_1(\tau), u_i(x) \in [0, u^*], \qquad \tau \in [0, T], x \in [0, 1],$

(3.5b) $\quad u_0, u_1 \in \mathrm{BV}(0, T)$ and $u_i(x) \in \mathrm{BV}(0, 1),$

(3.5c) $\quad \phi_1(\tau) \in [0, \phi_1^*], \qquad \tau \in [0, T],$

(3.5d) $\quad \phi_1 \in \mathrm{BV}(0, T),$

where $u^* \ge 1$, and under the condition that, for $n = 0, \dots, n_T - 1$, the following CFL condition is satisfied:

$$\Delta\tau^n \le \min\left\{ \frac{1}{u^*}, \frac{\Delta x_i}{(2u^* - 1)\Delta x_i + \phi_1^* + \frac{1}{2}\max\{1, u^* - 1\}} \right\}. \tag{3.6}$$

We assume this CFL condition to be satisfied from now on.

Under the general conditions (3.5), it is possible to obtain an upper bound on the entropy forms $E^{\varepsilon_0, \varepsilon}$; see [1, Theorem 3.4]. However, additional hypotheses on the data seem to be necessary, in our technique, to estimate the forms ν defined in (3.2) that measure the smoothness of the flux at the boundaries; see [1, Theorem 3.5]. We consider two different hypotheses that lead to different estimates of the continuity forms ν and hence, by Theorem 3.1, to different error estimates. These hypotheses are the following:

(3.7a) $\quad u^* = 1, \quad u_0 \equiv 0, \quad$ and $\quad u_1|_{K_l}, \phi_1|_{K_l}$ are constant, $l = 1, \dots, N,$

(3.7b) $\quad \phi_1|_{K_l} \in W^{1,\infty}(K_l), \qquad l = 1, \dots, N,$

where $\{K_l\}_{l=1}^N$ is a set of disjoint intervals such that $(0, T) = \bigcup_{l=1}^N K_l$. The first hypothesis allows us to control the signs of β and β_h at the boundary in such a way that the quantities $\nu_x(\varepsilon)$ defined in (3.2) can be bounded by a constant times $(\varepsilon + \Delta x)$. The second hypothesis allows for a possibly highly oscillatory behavior of the signs of β and β_h at the boundary, which is nevertheless suitably controlled by the smoothness of ϕ_1. In this case, smoothness at the boundary deteriorates: the quantities $\nu_x(\varepsilon)$ defined in (3.2) can be bounded only by a constant times $(\varepsilon + \Delta x)^{1/2}$.

Theorem 3.2 (Error estimates: the case (3.7a)). *Suppose that the hypothesis* (3.7a) *is satisfied. Then there exists a constant* C, *depending solely on* T *and the initial and boundary data, such that*

$$\|u - u_h\|_{L^\infty(0,T;L^1(0,1))} \le C\Delta x^{1/2},$$
$$\|\beta - \beta_h\|_{L^1(0,T;L^\infty(0,1))} \le C\Delta x^{1/2}.$$

These error estimates are sharp; see the numerical experiments in [1].

Theorem 3.3 (Error estimates: the case (3.7b)). *Suppose that the hypothesis* (3.7b) *is satisfied. Then there exists a constant* C, *depending solely on* T *and the initial and boundary data, such that*

$$\|u - u_h\|_{L^\infty(0,T;L^1(0,1))} \le C\Delta x^{1/3},$$
$$\|\beta - \beta_h\|_{L^1(0,T;L^\infty(0,1))} \le C\Delta x^{1/3}.$$

Proof of Theorem 3.2. Set $(v, \eta) = (u_h, \beta_h)$ in Theorem 3.1. To prove our result, we first have to estimate each of the terms appearing on the right-hand side of the inequalities of Theorem 3.1. The following estimates follow easily from the fact that the discrete initial and boundary data are local averages of the corresponding continuous functions. Thus,

$$\|v(0) - u(0)\|_{L^1(0,1)} \le |u_i|_{\mathrm{BV}(0,1)}\Delta x \quad \text{by (2.1d) and (3.5b)},$$
$$\|v(1+) - u(1+)\|_{L^1(0,T)} \le |u_1|_{\mathrm{BV}(0,T)}\Delta\tau \quad \text{by (2.1c) and (3.5b)},$$
$$\|v(0-) - u(0-)\|_{L^1(0,T)} \le |u_0|_{\mathrm{BV}(0,T)}\Delta\tau \quad \text{by (2.1b) and (3.5b)},$$

$$\begin{aligned}\int_0^T \left| \int_0^1 (\eta(\tau, x) - \beta(\tau, x))\, dx \right| d\tau \\ &= \int_0^T |\phi_{1,\Delta\tau}(\tau) - \phi_1(\tau)|\, d\tau \quad \text{by (2.3) and (1.2)} \\ &\le |\phi_1|_{\mathrm{BV}(0,T)}\Delta\tau \quad \text{by (2.1a) and (3.5d)},\end{aligned}$$

where $|f|_{\mathrm{BV}(0,1)}$ denotes the total variation of the function f. We also have

$$\int_0^T \|\partial_{x'}\eta(\tau') + (1 - v(\tau'))\|_{L^1(0,1)}\, d\tau' = 0 \quad \text{by (2.3a)},$$
$$\int_0^T \|\partial_x\beta(\tau) + (1 - u(\tau))\|_{L^1(0,1)}\, d\tau = 0 \quad \text{by (1.2a)}.$$

Finally, (most of) the following estimates have been obtained in [1]:

$\{\nu^-_{x,1}(\varepsilon, v; \eta) + \nu^+_{x,0}(\varepsilon, v; \eta)\} \le C(\varepsilon + \Delta x)$ by [1, (2.13)],

$\{\nu^-_{x,1}(\varepsilon, u; \beta) + \nu^+_{x,0}(\varepsilon, u; \beta)\} \le C\varepsilon$ by [1, (2.13)] and [1, Theorem 2.3],

$\{\nu^-_{\tau,T}(\varepsilon_0, v) + \nu^+_{\tau,0}(\varepsilon_0, v)\} \le C(\varepsilon_0 + \Delta\tau)$ by [1, Proposition 3.14],

$\{\nu^-_{\tau,T}(\varepsilon_0, u) + \nu^+_{\tau,0}(\varepsilon_0, u)\} \le C\varepsilon_0$ by [1, Proposition 3.14, Theorem 2.3],

$\nu_\tau(\varepsilon_0, v) \le C(\varepsilon_0 + \Delta\tau)$ by [1, Proposition 3.14],

$\nu_\tau(\varepsilon_0, u) \le C(\varepsilon_0)$ by [1, Proposition 3.14, Theorem 2.3],

$v_\tau(\varepsilon_0, \beta) \le C\varepsilon_0$ by (1.2) and (3.5d),

$E^{\varepsilon_0,\varepsilon}_{1/M}(u, v; \beta) \le 0$ by [1, Theorem 2.4],

$E^{\varepsilon_0,\varepsilon}_{1/M}(v, u; \eta) \le C(\Delta x/\varepsilon + \Delta\tau/\varepsilon_0) + C\Delta\tau M$ by [1, Theorem 2.4].

By Theorem 3.1 and the above estimates, we easily obtain that

$$\|v(T) - u(T)\|_{L^1(0,1)} \le C\{\Delta x + \Delta\tau + \varepsilon + \Delta x/\varepsilon + \varepsilon_0 + \Delta\tau/\varepsilon_0 + \Delta\tau M + 1/M\}.$$

The first error estimate follows by minimizing with respect to M, ε, and ε_0 and by enforcing the CFL condition (3.6), which is of the form $\Delta\tau \le C\Delta x$.

The second error estimate can now easily be obtained from Theorem 3.1, the first error estimate of Theorem 3.2, and from the fifth and sixth inequalities of this proof. □

Proof of Theorem 3.3. The only difference from the preceding proof is that now we have, by [1, (2.14)],

$$\tfrac{1}{2}L\{\nu^-_{x,1}(\varepsilon, u; \beta)+\nu^-_{x,1}(\varepsilon, v; \eta)+\nu^+_{x,0}(\varepsilon, u; \beta)+\nu^+_{x,0}(\varepsilon, \nu; \eta)\} \le C(\varepsilon+\Delta x)^{1/2},$$

and hence, for ε small enough,

$$\|v(T)-u(T)\|_{L^1(0,1)} \le C\{\Delta x^{1/2}+\Delta\tau+\varepsilon^{1/2}+\Delta x/\varepsilon+\varepsilon_0+\Delta\tau/\varepsilon_0+\Delta\tau M+1/M\}.$$

The results follow as in the previous proof. □

4. Proof of Theorem 3.1

In this section we prove Theorem 3.1. We begin with a very simple result linking the distance between η and β and the distance between v and u.

Lemma 4.1. *We have*

$$\begin{aligned}&\|\eta(\tau)-\beta(\tau)\|_{L^\infty(0,1)}\\ &\quad\le \|v(\tau)-u(\tau)\|_{L^1(0,1)}+\|\partial_x\eta(\tau)+(1-v(\tau))\|_{L^1(0,1)}\\ &\qquad+\|\partial_x\beta(\tau)+(1-u(\tau))\|_{L^1(0,1)}+\left|\int_0^1(\eta(\tau,x)-\beta(\tau,x))\,dx\right|.\end{aligned}$$

Proof. It is very simple to see that, for $\tau\in[0,T]$,

$$\begin{aligned}&\|\eta(\tau)-\beta(\tau)\|_{L^\infty(0,1)}\\ &\quad\le \|\partial_x\eta(\tau)-\partial_x\beta(\tau)\|_{L^1(0,1)}+\left|\int_0^1(\eta(\tau,x)-\beta(\tau,x))\,dx\right|.\end{aligned}$$

The results follow from the triangle inequality. □

The above result will allow us to proceed as in [5] and [6] to obtain an estimate on the approximation of the electron concentration. First, we consider the sum $E^{\varepsilon_0,\varepsilon}(v, u; \eta)+E^{\varepsilon_0,\varepsilon}(u, v; \beta)$ and rewrite it as a sum of seven terms.

Lemma 4.2 (The basic equality). *We have*

$$\begin{aligned}&E^{\varepsilon_0,\varepsilon}(v, u; \eta)+E^{\varepsilon_0,\varepsilon}(u, v; \beta)\\ &\quad= T_{\mathrm{err},\tau}(u, v)+T_{\mathrm{err},x}(u, \beta; v, \eta)+T_{u^-}(u, v)\\ &\qquad+T_{\mathrm{vel}}(u, \beta; v, \eta)+T_\beta(u, v)+T_\eta(u, v)+T_{U''}(u, v),\end{aligned}$$

where

$$\begin{aligned}T_{\mathrm{err},\tau}(u, v) = &\int_0^T\int_0^1\int_0^1 U(v(T, x')-u(\tau, x))\varphi(\tau, x; T, x')\,dx'\,dx\,d\tau\\ &+\int_0^T\int_0^1\int_0^1 U(v(\tau', x')-u(T, x))\varphi(T, x; \tau', x')\,dx\,dx'\,d\tau'\\ &-\int_0^T\int_0^1\int_0^1 U(v(0, x')-u(\tau, x))\varphi(\tau, x; 0, x')\,dx'\,dx\,d\tau\\ &-\int_0^T\int_0^1\int_0^1 U(v(\tau', x')-u(0, x))\varphi(0, x; \tau', x')\,dx\,dx'\,d\tau',\end{aligned}$$

$$
\begin{aligned}
&T_{\mathrm{err},x}(u,\beta;v,\eta)\\
&\quad=\int_0^1\int_0^T\int_0^T G(v(\tau',1-)-u(\tau,x),\,v(\tau',1+)-u(\tau,x);\,\eta(\tau',1))\\
&\qquad\qquad\cdot\varphi(\tau,x;\tau',1)\,d\tau'\,dx\,d\tau\\
&\qquad+\int_0^1\int_0^T\int_0^T G(v(\tau',x')-u(\tau,1-),\,v(\tau',x')-u(\tau,1+);\,\beta(\tau',1))\\
&\qquad\qquad\cdot\varphi(\tau,1;\tau',x')\,d\tau\,dx'\,d\tau'\\
&\qquad-\int_0^1\int_0^T\int_0^T G(v(\tau',0-)-u(\tau,x),\,v(\tau',0+)-u(\tau,x);\,\eta(\tau',0))\\
&\qquad\qquad\cdot\varphi(\tau,x;\tau',0)\,d\tau'\,dx\,d\tau\\
&\qquad-\int_0^1\int_0^T\int_0^T G(v(\tau',x')-u(\tau,0-),\,v(\tau',x')-u(\tau,0+);\,\beta(\tau'0))\\
&\qquad\qquad\cdot\varphi(\tau,0;\tau',x')\,d\tau'\,dx'\,d\tau',
\end{aligned}
$$

$$
\begin{aligned}
T_{u^-}(u,v)&=\int_0^1\int_0^T\int_0^1\int_0^T u(\tau,x)U(v(\tau',x')\\
&\qquad\qquad-u(\tau,x))\varphi(\tau,x;\tau',x')\,dx'\,d\tau'\,dx\,d\tau,\\
T_{\mathrm{vel}}(u,\beta;v,\eta)&=-\int_0^T\int_0^1\int_0^T\int_0^1 U(v(\tau',x')-u(\tau,x))\\
&\qquad\cdot\partial_x((\beta(\tau,x)-\eta(\tau',x'))\varphi(\tau,x;\tau',x'))\,dx'\,d\tau'\,dx\,d\tau,\\
T_\beta(u,v)&=-\int_0^T\int_0^1\int_0^T\int_0^1(\partial_x\beta(\tau,x)+(1-u(\tau,x)))u(\tau,x)\\
&\qquad\cdot U'(v(\tau',x')-u(\tau,x))\varphi(\tau,x;\tau',x')\,dx'\,d\tau'\,dx\,d\tau,\\
T_\eta(u,v)&=-\int_0^T\int_0^1\int_0^T\int_0^1(\partial_{x'}\eta(\tau',x')+(1-v(\tau',x')))\\
&\qquad\cdot V(v(\tau',x'),u(\tau,x))\varphi(\tau,x;\tau',x')\,dx'\,d\tau'\,dx\,d\tau,\\
T_{U''}(u,v)&=-\int_0^T\int_0^1\int_0^T\int_0^1(1-v(\tau',x')-u(\tau,x))\\
&\qquad\cdot\left\{\int_0^{v(\tau',x')-u(\tau,x)}sU''(s)\,ds\right\}\varphi(\tau,x;\tau',x')\,dx'\,d\tau'\,dx\,d\tau.
\end{aligned}
$$

We now briefly discuss the meaning of each of the terms appearing in this result. First, notice that the term $T_{\mathrm{err},\tau}$ goes (formally) to

$$\int_0^1 U(v(T,x)-u(T,x))\,dx-\int_0^1 U(v(0,x)-u(0,x))\,dx$$

as ε and ε_0 go to zero. Thus, this term contains the information of "the flow of the errors" of the approximation of the electron concentration across the boundary $\{0,T\}\times(0,1)$. Similarly, the error term $T_{\mathrm{err},x}$ contains the information of "the flow of the errors" of the approximation of the electron concentration across the boundary $(0,T)\times\{0,1\}$. The term $T_{u^-}(u,v)$ can be considered to be a measure of the influence of the negative values of u in the approximation. Notice that only nonnegative values of u have a physical

meaning and that negative values of u might be very difficult to control, since the point $u = 0$ is an unstable equilibrium point of the equation along the characteristics

$$\frac{d}{d\tau}u = u(1-u).$$

The term $T_{\mathrm{vel}}(u, \beta; v, \eta)$ contains information about the smoothness and closeness of the velocities β and η; notice that even if $\beta = \eta$, this term is not equal to zero. The terms $T_\eta(u, v)$ and $T_\beta(u, v)$ contain information about how close $\partial_{x'}\eta$ and $\partial_x\beta$ are from being the negative electric fields associated with the corresponding electron concentrations. Finally, the term $T_{U''}(u, v)$ contains information of how far the entropy $U(\cdot)$ is from $|\cdot|$; notice that this term is equal to zero for $U(\cdot) = |\cdot|$.

Proof of Lemma 4.2. With (3.4) taken into account, it is very easy to see that

$$\begin{aligned} &E^{\varepsilon_0,\varepsilon}(v, u; \eta) + E^{\varepsilon_0,\varepsilon}(u, v; \beta) \\ &\quad = T_{\mathrm{err},\tau}(u, v) + T_{\mathrm{err},x}(u, \beta; v, \eta) + T(u, \beta; v, \eta), \end{aligned}$$

where

$$T(u, \beta; v, \eta) = -\int_0^T \int_0^1 \int_0^T \int_0^1 \Phi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau$$

and (supressing the arguments of the functions for the sake of clarity)

$$\begin{aligned} \Phi &= U(v-u)(\beta-\eta)\partial_x\varphi + \partial_{x'}\eta V(v, u)\varphi + \partial_x\beta V(u, v)\varphi \\ &= U(v-u)(\beta-\eta)\partial_x\varphi \\ &\quad + (\partial_{x'}\eta - (-1+v))V(v, u)\varphi + (\partial_x\beta - (-1+u))V(u, v)\varphi \\ &\quad + \{-U(v-u) - (1-v-u)\{U(u-v) - (u-v)U'(u-v)\}\}\varphi, \end{aligned}$$

by (3.4d). Since $U(w) - wU'(w) = -\int_0^w sU''(s)\, ds$, we obtain

$$\begin{aligned} \Phi &= U(v-u)\{-\varphi + (\beta-\eta)\partial_x\varphi\} \\ &\quad + (\partial_{x'}\eta - (-1+v))V(v, u)\varphi + (\partial_x\beta - (-1+u))V(u, v)\varphi \\ &\quad + \{(1-v-u)\int_0^{v-u} sU''(s)\, ds\}\varphi. \end{aligned}$$

Finally, since, by (3.4d), $V(u, v) = U(u-v) - uU'(u-v)$, we get

$$\begin{aligned} \Phi &= U(v-u)\{-u\varphi + \partial_x((\beta-\eta)\varphi)\} \\ &\quad + (\partial_{x'}\eta - (-1+v))V(v, u)\varphi + (\partial_x\beta - (-1+u))uU'(v-u)\varphi \\ &\quad + \left\{(1-v-u)\int_0^{v-u} sU''(s)\, ds\right\}\varphi. \end{aligned}$$

This completes the proof. □

We now obtain lower bounds for each of the T-terms appearing in the right-hand side of the main equality of Lemma 4.2. The following lemmas contain these bounds. We begin with a couple of lemmas in which we collect all the inequalities involving the auxiliary functions w_ν.

Lemma 4.3. *We have*

$$\int_0^T w_{\varepsilon_0}(\tau - T)\,d\tau = 1/2, \qquad \int_0^1 \left\{1 - \int_0^1 w_\varepsilon(x - x')\,dx\right\} dx' \le \varepsilon,$$

$$\int_0^1 w_\varepsilon(x - x')\,dx \le 1, \quad \forall x' \in [0, 1],$$

$$\int_0^1 \int_0^1 w_\varepsilon(x - x')|f(x) - f(x')|\,dx\,dx' \le \varepsilon |f|_{\mathrm{BV}(0,1)}.$$

Proof. The first three inequalities can be easily obtained by using the definition of w_ν, (3.3a). We now prove the last inequality. Assume that the function f is very smooth. Then,

$$\begin{aligned}
&\int_0^1 \int_0^1 w_\varepsilon(x - x')|f(x) - f(x')|\,dx\,dx' \\
&\quad = \int_{-1}^1 w_\varepsilon(y) \int_{a(y)}^{b(y)} |f(x' + y) - f(x')|\,dx'\,dy \\
&\quad = \int_{-\varepsilon}^{\varepsilon} w_\varepsilon(y) \|f(\cdot + y) - f(\cdot)\|_{L^1(a(y),\,b(y))}\,dy \\
&\quad \le \|\partial_x f\|_{L^1(0,1)} \int_{-\varepsilon}^{\varepsilon} w_\varepsilon(y)|y|\,dy = \|f\|_{\mathrm{BV}(0,1)} \int_{-\varepsilon}^{\varepsilon} w_\varepsilon(y)|y|\,dy \\
&\quad \le \varepsilon \|f\|_{\mathrm{BV}(0,1)} \int_{-\varepsilon}^{\varepsilon} w_\varepsilon(y)\,dy = \varepsilon \|f\|_{\mathrm{BV}(0,1)}.
\end{aligned}$$

This completes the proof for smooth functions f. The general result follows by a standard density argument. □

Lemma 4.4. *Set* $f^+ = \max\{f, 0\}$. *Then we have*

$$\begin{aligned}
&\int_0^T \int_0^1 \int_0^1 \mathbb{F}(x')\varphi(\tau, x; T, x')\,dx'\,dx\,d\tau \\
&\qquad \ge \frac{1}{2}\int_0^1 \mathbb{F}(x')\,dx' - \frac{1}{2}\varepsilon \sup_{0 \le x' \le 1} \mathbb{F}^+(x'),
\end{aligned} \tag{4.1a}$$

$$\begin{aligned}
&-\int_0^T \int_0^1 \int_0^1 |\mathbb{F}(\tau, x')|\varphi(\tau, x; T, x')\,dx'\,dx\,d\tau \\
&\qquad \ge -\frac{1}{2} \sup_{\substack{|\tau - T| \le \varepsilon_0 \\ \tau \in [0, T]}} \left\{\int_0^1 |\mathbb{F}(\tau, x')|\,dx'\right\},
\end{aligned} \tag{4.1b}$$

$$\begin{aligned}
&-\int_0^T \int_0^1 \int_0^1 |f(\tau, x') - f(\tau, x)|\varphi(\tau, x; T, x')\,dx'\,dx\,d\tau \\
&\qquad \ge -\frac{1}{2}\varepsilon \sup_{0 \le \tau \le T} |f(\tau)|_{\mathrm{BV}(0,1)}.
\end{aligned} \tag{4.1c}$$

Proof. We begin with the proof of (4.1a). Let T_1 denote the left-hand side of inequality (4.1a). We have

$$\begin{aligned}
T_1 &= \left\{\int_0^T w_{\varepsilon_0}(\tau - T)\,d\tau\right\}\int_0^1 \left\{\int_0^1 w_\varepsilon(x - x')\,dx\right\}\mathbb{F}(x')\,dx' \quad \text{by (3.3a)}\\
&= \left\{\int_0^T w_{\varepsilon_0}(\tau - T)\,d\tau\right\}\int_0^1 \mathbb{F}(x')\,dx'\\
&\quad - \left\{\int_0^T w_{\varepsilon_0}(\tau - T)\,d\tau\right\}\int_0^1 \left\{1 - \int_0^1 w_\varepsilon(x - x')\,dx\right\}\mathbb{F}(x')\,dx'\\
&\geq \frac{1}{2}\int_0^1 \mathbb{F}(x')\,dx' - \frac{1}{2}\varepsilon \sup_{0\leq x'\leq 1} \mathbb{F}^+(x'),
\end{aligned}$$

by the first and second inequalities of Lemma 4.3.

Now we prove (4.1b). Let T_2 denote the left-hand side of inequality (4.1b). Then,

$$\begin{aligned}
T_2 &= -\int_0^T w_{\varepsilon_0}(\tau - T)\left\{\int_0^1 \left\{\int_0^1 w_\varepsilon(x - x')\,dx\right\}|\mathbb{F}(\tau, x')|\,dx'\right\}d\tau\\
&\geq -\int_0^T w_{\varepsilon_0}(\tau - T)\left\{\int_0^1 |\mathbb{F}(\tau, x')|\,dx'\right\}d\tau\\
&\geq -\frac{1}{2}\sup_{\substack{|\tau - T|\leq\varepsilon_0\\ \tau\in[0,T]}}\left\{\int_0^1 |\mathbb{F}(\tau, x')|\,dx'\right\},
\end{aligned}$$

by the third and first inequalities of Lemma 4.3.

Finally, if T_3 denotes the left-hand side of inequality (4.1c), we have

$$\begin{aligned}
T_3 &= \int_0^T w_{\varepsilon_0}(\tau - T)\left\{\int_0^1\int_0^1 w_\varepsilon(x - x')|f(\tau, x) - f(\tau, x')|\,dx'\,dx\right\}d\tau\\
&\geq -\varepsilon \sup_{0\leq\tau\leq T}|f(\tau)|_{\mathrm{BV}(0,1)}\int_0^T w_{\varepsilon_0}(\tau - T)\,d\tau\\
&= -\frac{1}{2}\varepsilon \sup_{0\leq\tau\leq T}|f(\tau)|_{\mathrm{BV}(0,1)},
\end{aligned}$$

by the fourth and first inequalities of Lemma 4.3. This completes the proof. □

We are now ready to estimate all the T-terms appearing in the right-hand side of the main equality of Lemma 4.2.

Lemma 4.5 (Lower bound for $T_{\mathrm{err},\tau}(u, v)$). *We have*

$$\begin{aligned} T_{\mathrm{err},\tau}(u, v) \geq & \int_0^1 U(v(T, x) - u(T, x))\, dx - \int_0^1 U(v(0, x) - u(0, x))\, dx \\ & - \varepsilon \sup_{x \in \mathbf{R}} U(v(T, x) - u(T, x)) \\ & - \frac{1}{2} L\{\nu^-_{\tau,T}(\varepsilon_0, u) + \nu^+_{\tau,0}(\varepsilon_0, u)\} \\ & - \frac{1}{2} L\{\nu^-_{\tau,T}(\varepsilon_0, v) + \nu^+_{\tau,0}(\varepsilon_0, v)\} \\ & - L\varepsilon\{|u|_{L^\infty(0,T;\mathrm{BV}(0,1))} + |v|_{L^\infty(0,T;\mathrm{BV}(0,1))}\}. \end{aligned}$$

Proof. Consider the first term of $T_{\mathrm{err},\tau}$,

$$\Phi = \int_0^T \int_0^1 \int_0^1 U(v(T, x') - u(\tau, x))\varphi(\tau, x; T, x')\, dx'\, dx\, d\tau.$$

By using the definition of φ, (3.3a), and the following simple inequality,

$$\begin{aligned} U(v(T, x') - u(\tau, x)) = & U(v(T, x') - u(T, x')) \\ & + \{U(v(T, x') - u(\tau, x')) - U(v(T, x') - u(T, x'))\} \\ & + \{U(v(T, x') - u(\tau, x)) - U(v(T, x') - u(\tau, x'))\} \\ \geq & U(v(T, x') - u(T, x')) \\ & - L|u(\tau, x') - u(T, x')| - L|u(\tau, x) - u(\tau, x')|, \end{aligned}$$

we get

$$\Phi \geq T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 &= \int_0^T \int_0^1 \int_0^1 U(v(T, x') - u(T, x'))\varphi(\tau, x; T, x')\, dx'\, dx\, d\tau, \\ T_2 &= -L \int_0^T \int_0^1 \int_0^1 |u(\tau, x') - u(T, x')|\varphi(\tau, x; T, x')\, dx'\, dx\, d\tau, \\ T_3 &= -L \int_0^T \int_0^1 \int_0^1 |u(\tau, x) - u(\tau, x')|\varphi(\tau, x; T, x')\, dx'\, dx\, d\tau. \end{aligned}$$

By (4.1a) with $\mathbb{F}(x') = U(v(T, x') - u(T, x'))$, we get

$$T_1 \geq \frac{1}{2} \int_0^1 U(v(T, x) - u(T, x))\, dx - \frac{1}{2}\varepsilon \sup_{0 \leq x \leq 1} U(v(T, x) - u(T, x)).$$

By (4.1b) with $\mathbb{F}(\tau, x') = u(\tau, x') - u(T, x')$, we get

$$T_2 \geq -\frac{1}{2} L \sup_{\substack{|\tau - T| \leq \varepsilon_0 \\ \tau \in [0,T]}} \left\{ \int_0^1 |u(\tau, x') - u(T, x')|\, dx' \right\} = -\frac{1}{2} L \nu^-_{\tau,T}(\varepsilon_0, u),$$

by the definition of $\nu^-_{\tau,T}(\varepsilon_0, u)$, (3.2).

Finally, by (4.1c) with $f = u$, we get

$$T_3 \geq -\frac{1}{2} L\varepsilon \sup_{0 \leq \tau \leq T} |u(\tau)|_{\mathrm{BV}(0,1)} = -\frac{1}{2} L\varepsilon |u|_{L^\infty(0,T;\mathrm{BV}(0,1))}.$$

As a consequence, we have

$$\Phi \geq \frac{1}{2}\int_0^1 U(v(T,x)-u(T,x))\,dx - \frac{1}{2}\varepsilon \sup_{0\leq x\leq 1} U(v(T,x)-u(T,x)) - \frac{1}{2}L\nu^-_{\tau,T}(\varepsilon_0,u) - \frac{1}{2}L\varepsilon|u|_{L^\infty(0,T;\mathrm{BV}(0,1))}.$$

The remaining three terms are treated in a similar way. □

Lemma 4.6 (Lower bound for $T_{\mathrm{err},x}(u,\beta;v,\eta)$). *We have*

$$\begin{aligned}
&T_{\mathrm{err},x}(u,\beta;v,\eta)\\
&\quad\geq \inf_{\tau\in[0,T]}\{(\eta^-(\tau,1)+\beta^-(\tau,1))/2\}\int_0^1 U(v(\tau,1+)-u(\tau,1+))\,d\tau\\
&\qquad - \sup_{\tau\in[0,T]}\{(\eta^+(\tau,0)+\beta^+(\tau,0))/2\}\int_0^1 U(v(\tau,0-)-u(\tau,0-))\,d\tau\\
&\qquad - \frac{1}{2}L\{\nu^-_{x,1}(\varepsilon,u;\beta)+\nu^-_{x,1}(\varepsilon,v;\eta)\} - \frac{1}{2}L\{\nu^+_{x,0}(\varepsilon,u;\beta)+\nu^+_{x,0}(\varepsilon,v;\eta)\}\\
&\qquad - \frac{1}{2}L\{\|\beta\|_{L^\infty(0,T;L^\infty(0,1))}\{|u(1+)|_{\mathrm{BV}(0,T)}+|u(0-)|_{\mathrm{BV}(0,T)}\}\\
&\qquad\qquad + 2\{|\beta(1)|_{\mathrm{BV}(0,1)}+|\beta(0)|_{\mathrm{BV}(0,1)}\}\|u\|_{L^\infty(0,T;L^\infty(0,1))}\}\varepsilon_0\\
&\qquad - \frac{1}{2}L\{\|\eta\|_{L^\infty(0,T;L^\infty(0,1))}\{|v(1+)|_{\mathrm{BV}(0,T)}+|v(0-)|_{\mathrm{BV}(0,T)}\}\\
&\qquad\qquad + 2\{|\eta(1)|_{\mathrm{BV}(0,1)}+|\eta(0)|_{\mathrm{BV}(0,1)}\}\|v\|_{L^\infty(0,T;L^\infty(0,1))}\}\varepsilon_0\\
&\qquad - 4L\{\|u\|_{L^\infty(0,T;L^\infty(0,1))}+\|v\|_{L^\infty(0,T;L^\infty(0,1))}\}\\
&\qquad \cdot\int_0^T\Bigg\{\|v(\tau)-u(\tau)\|_{L^1(0,1)}+\|\partial_x\eta(\tau)+(1-v(\tau))\|_{L^1(0,1)}\\
&\qquad\qquad + \|\partial_x\beta(\tau)+(1-u(\tau))\|_{L^1(0,1)}+\left|\int_0^1(\eta(\tau,x)-\beta(\tau,x))\,dx\right|\Bigg\}\,d\tau.
\end{aligned}$$

Proof. The proof of this lemma is similar to that of Lemma 4.5. We begin by considering the first term of $T_{\mathrm{err},x}(u,\beta;v,\eta)$,

$$\Phi = \int_0^1\int_0^T\int_0^T G(v(\tau',1-)-u(\tau,x),\,v(\tau',1+)-u(\tau,x);\,\eta(\tau',1)) \cdot\varphi(\tau,x;\tau',1)\,d\tau'\,dx\,d\tau.$$

By (3.4c), we have

$$\begin{aligned}
&G(v(\tau',1-)-u(\tau,x),\,v(\tau',1+)-u(\tau,x);\,\eta(\tau',1))\\
&\quad= U(v(\tau',1+)-u(\tau,x))\eta^-(\tau',1)+U(v(\tau',1-)-u(\tau,x))\eta^+(\tau',1)\\
&\quad\geq U(v(\tau',1+)-u(\tau,x))\eta^-(\tau',1).
\end{aligned}$$

Since

$$
\begin{aligned}
&U(v(\tau', 1+) - u(\tau, x))\eta^-(\tau', 1) \\
&\quad = U(v(\tau', 1+) - u(\tau', 1+))\eta^-(\tau', 1) \\
&\qquad + \{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\eta^-(\tau', 1) \\
&\quad = U(v(\tau', 1+) - u(\tau', 1+))\eta^-(\tau', 1) \\
&\qquad + \{\{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\eta^-(\tau', 1) \\
&\qquad\quad - \{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\beta^-(\tau', 1)\} \\
&\qquad + \{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\beta^-(\tau', 1) \\
&\quad = U(v(\tau', 1+) - u(\tau', 1+))\eta^-(\tau', 1) \\
&\qquad + \{\{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\eta^-(\tau', 1) \\
&\qquad\quad - \{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\beta^-(\tau', 1)\} \\
&\qquad + \{\{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\beta^-(\tau', 1) \\
&\qquad\quad - \{U(v(\tau', 1+) - u(\tau, x)) - U(v(\tau', 1+) - u(\tau', 1+))\}\beta^-(\tau, 1)\} \\
&\qquad + \{U(v(\tau', 1+) - u(\tau, 1+)) - U(v(\tau', 1+) - u(\tau', 1+))\}\beta^-(\tau, 1) \\
&\qquad + \{U(v(\tau', 1+) - u(\tau, x) - U(v(\tau', 1+) - u(\tau, 1+))\}\beta^-(\tau, 1),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&G(v(\tau', 1-) - u(\tau, x), v(\tau', 1+) - u(\tau, x); \eta(\tau', 1)) \\
&\quad \geq U(v(\tau', 1+) - u(\tau', 1+))\eta^-(\tau', 1) \\
&\qquad - L|u(\tau', 1+) - u(\tau, x)|\,|\eta^-(\tau', 1) - \beta^-(\tau', 1)| \\
&\qquad - L|u(\tau', 1+) - u(\tau, x)|\,|\beta^-(\tau', 1) - \beta^-(\tau, 1)| \\
&\qquad + L|u(\tau, 1+) - u(\tau', 1+)|\beta^-(\tau, 1) + L|u(\tau, x) - u(\tau, 1+)|\beta^-(\tau, 1),
\end{aligned}
$$

and so

$$\Phi \geq T_1 + T_2 + T_3 + T_4 + T_5,$$

where

$$
\begin{aligned}
T_1 &= \int_0^1 \int_0^T \int_0^T U(v(\tau', 1+) - u(\tau', 1+))\eta^-(\tau', 1)\varphi(\tau, x; \tau', 1)\, d\tau'\, dx\, d\tau, \\
T_2 &= -L \int_0^1 \int_0^T \int_0^T |u(\tau', 1+) - u(\tau, x)| \\
&\qquad\qquad \cdot |\eta^-(\tau', 1) - \beta^-(\tau', 1)|\varphi(\tau, x; \tau', 1)\, d\tau'\, dx\, d\tau, \\
T_3 &= -L \int_0^1 \int_0^T \int_0^T |u(\tau', 1+) - u(\tau, x)| \\
&\qquad\qquad \cdot |\beta^-(\tau', 1) - \beta^-(\tau, 1)|\varphi(\tau, x; \tau', 1)\, d\tau'\, dx\, d\tau, \\
T_4 &= L \int_0^1 \int_0^T \int_0^T |u(\tau, 1+) - u(\tau', 1+)|\beta^-(\tau, 1)\varphi(\tau, x; \tau', 1)\, d\tau'\, dx\, d\tau, \\
T_5 &= L \int_0^1 \int_0^T \int_0^T |u(\tau, x) - u(\tau, 1+)|\beta^-(\tau, 1)\varphi(\tau, x; \tau', 1)\, d\tau'\, dx\, d\tau.
\end{aligned}
$$

To estimate each of these terms, we use the inequalities of Lemma 4.4 in which the role of (x, x') is now played by (τ, τ') and the role of (τ, T) is played

by $(x, 1)$. Thus, by (4.1a), we get

$$T_1 \geq \frac{1}{2}\int_0^T U(v(\tau', 1+) - u(\tau', 1+))\eta^-(\tau', 1)\, d\tau'.$$

Since

$$T_2 \geq -2L\|u\|_{L^\infty(0,T;L^\infty(0,1))}\int_0^1\int_0^T\int_0^T |\eta^-(\tau', 1) - \beta^-(\tau', 1)| \cdot \varphi(\tau, x; \tau', 1)\, d\tau'\, dx\, d\tau,$$

we have, by (4.1b),

$$\begin{aligned} T_2 &\geq -2L\|u\|_{L^\infty(0,T;L^\infty(0,1))}\int_0^T |\eta^-(\tau', 1) - \beta^-(\tau', 1)|\, d\tau' \\ &\geq -2L\|u\|_{L^\infty(0,T;L^\infty(0,1))}\int_0^T \|\eta(\tau') - \beta(\tau')\|_{L^\infty(0,1)}\, d\tau'. \end{aligned}$$

Since

$$T_3 \geq -2L\|u\|_{L^\infty(0,T;L^\infty(0,1))}\int_0^1\int_0^T\int_0^T |\beta^-(\tau, 1) - \beta^-(\tau', 1)| \cdot \varphi(\tau, x; \tau', 1)\, d\tau'\, dx\, d\tau,$$

we have, by (4.1c),

$$T_3 \geq -L\|u\|_{L^\infty(0,T;L^\infty(0,1))}\varepsilon_0|\beta(1)|_{\mathrm{BV}(0,T)}.$$

Since

$$T_4 \geq -C\int_0^1 w_\varepsilon(x-1)\left\{\int_0^T\int_0^T w_{\varepsilon_0}(\tau-\tau')|u(\tau, 1+) - u(\tau', 1+)|\, d\tau'\, d\tau\right\} dx,$$

where $C = L\|\beta\|_{L^\infty(0,T;L^\infty(0,1))}$, we have, by (4.1c),

$$T_4 \geq -\frac{1}{2}L\|\beta\|_{L^\infty(0,T;L^\infty(0,1))}\varepsilon_0|u(1+)|_{\mathrm{BV}(0,T)}.$$

Finally, by (4.1b),

$$\begin{aligned} T_5 &\geq \frac{1}{2}L\sup_{\substack{|x-1|\leq\varepsilon \\ x\in[0,1]}}\left\{\int_0^T |u(\tau, x) - u(\tau, 1+)|\beta^-(\tau, 1)\, d\tau\right\} \\ &= -\frac{1}{2}L\nu^-_{x,1}(\varepsilon, u; \beta). \end{aligned}$$

We can thus write

$$\begin{aligned} \Phi \geq\ & \frac{1}{2}\int_0^1 U(v(\tau, 1+) - u(\tau, 1+))\eta^-(\tau, 1)\, d\tau \\ & - 2L\|u\|_{L^\infty(0,T;L^\infty(0,1))}\int_0^T \|\beta(\tau) - \eta(\tau)\|_{L^\infty(0,1)}\, d\tau \\ & - \frac{1}{2}L\{\|\beta\|_{L^\infty(0,T;L^\infty(0,1))}|u(1+)|_{\mathrm{BV}(0,T)} \\ & \qquad + 2|\beta(1)|_{\mathrm{BV}(0,1)}\|u\|_{L^\infty(0,T;L^\infty(0,1))}\}\varepsilon_0 \\ & - \frac{1}{2}L\nu^-_{x,1}(\varepsilon, u; \beta). \end{aligned}$$

Finally, by Lemma 4.1,

$$\begin{aligned}\Phi \geq \frac{1}{2}\int_0^1 U(v(\tau,1+)-u(\tau,1+))\eta^-(\tau,1)\,d\tau \\ -2L\|u\|_{L^\infty(0,T;L^\infty(0,1))}\int_0^T\Big\{\|v(\tau)-u(\tau)\|_{L^1(0,1)} \\ +\|\partial_x\eta(\tau)+(1-v(\tau))\|_{L^1(0,1)} \\ +\|\partial_x\beta(\tau)+(1-u(\tau))\|_{L^1(0,1)} \\ +\left|\int_0^1(\eta(\tau,x)-\beta(\tau,x))\,dx\right|\Big\}\,d\tau \\ -\frac{1}{2}L\{\|\beta\|_{L^\infty(0,T;L^\infty(0,1))}|u(1+)|_{\mathrm{BV}(0,T)} \\ +2|\beta(1)|_{\mathrm{BV}(0,1)}\|u\|_{L^\infty(0,T;L^\infty(0,1))}\}\varepsilon_0 \\ -\frac{1}{2}L\nu^-_{x,1}(\varepsilon,u;\beta).\end{aligned}$$

The remaining three terms are treated similarly. □

To estimate the remaining T-terms, we need an auxiliary result similar to Lemma 4.4.

Lemma 4.7. *We have*

$$\begin{aligned}-\int_0^T\int_0^T\int_0^1\int_0^1|\mathbb{F}(\tau',x')|\varphi(\tau,x;\tau',x')\,d\tau\,dx\,d\tau'\,dx' \\ \geq -\int_0^T\int_0^1|\mathbb{F}(\tau',x')|\,d\tau'\,dx',\end{aligned} \tag{4.2a}$$

$$\begin{aligned}-\int_0^T\int_0^T\int_0^1\int_0^1|f(\tau',x')-f(\tau,x')|\varphi(\tau,x;\tau',x')\,d\tau\,dx\,d\tau'\,dx' \\ \geq -T\nu_\tau(\varepsilon_0,f),\end{aligned} \tag{4.2b}$$

$$\begin{aligned}-\int_0^T\int_0^T\int_0^1\int_0^1|f(\tau,x)-f(\tau,x')|\varphi(\tau,x';\tau',x')\,d\tau\,dx\,d\tau'\,dx' \\ \geq -T\varepsilon|f|_{L^\infty(0,T;\mathrm{BV}(0,1))},\end{aligned} \tag{4.2c}$$

$$\begin{aligned}-\int_0^T\int_0^T|T(\tau,\tau')|w_{\varepsilon_0}(\tau-\tau')\,d\tau\,d\tau' \\ \geq -\sup_{|\zeta|\leq\varepsilon_0}\left\{\int_0^T\chi(\tau'+\zeta)|T(\tau'+\zeta,\tau')|\,d\tau'\right\}d\zeta,\end{aligned} \tag{4.2d}$$

where χ is the characteristic function of the interval $[0,T]$.

Proof. Let T_1 denote the left-hand side of inequality (4.2a). Then,

$$\begin{aligned}T_1 &= -\int_0^T\int_0^1|\mathbb{F}(\tau',x')|\left\{\int_0^T\int_0^1\varphi(\tau,x;\tau',x')\,dx\,d\tau\right\}dx'\,d\tau' \\ &\geq -\int_0^T\int_0^1|\mathbb{F}(\tau',x')|\,dx'\,d\tau',\end{aligned}$$

by the definition of φ, (3.3a).

We now prove (4.2b). Let T_2 be the left-hand side of inequality (4.2b). Then,

$$T_2 = -\int_0^1 \left\{ \int_0^T \int_0^T \left\{ \int_0^1 w_\varepsilon(x-x')\,dx \right\} w_{\varepsilon_0}(\tau-\tau') \cdot |f(\tau, x') - f(\tau', x')|\,d\tau\,d\tau' \right\} dx'$$

$$\geq -\int_0^T \int_0^T w_{\varepsilon_0}(\tau-\tau') \|f(\tau) - f(\tau')\|_{L^1(0,1)}\,d\tau\,d\tau',$$

by Lemma 4.3. Finally, by the definition of $\nu_\tau(\varepsilon_0, f)$, (3.2), we get

$$\begin{aligned} T_2 &\geq -\nu_\tau(\varepsilon_0, f) \int_0^T \int_0^T w_{\varepsilon_0}(\tau-\tau')\,d\tau\,d\tau' \\ &\geq -T\nu_\tau(\varepsilon_0, f), \end{aligned}$$

by the third inequality of Lemma 4.3 with τ and τ' playing the role of x and x', respectively. This proves (4.2b).

Next, let T_3 be the left-hand side of inequality (4.2c). Then,

$$T_3 = -\int_0^T \left\{ \left\{ \int_0^T w_{\varepsilon_0}(\tau-\tau')\,d\tau' \right\} \cdot \int_0^1 \int_0^1 w_\varepsilon(x-x') |f(\tau, x) - f(\tau, x')|\,dx'\,dx \right\} d\tau$$

$$\geq -\int_0^T \left\{ \int_0^1 \int_0^1 w_\varepsilon(x-x') |f(\tau, x) - f(\tau, x')|\,dx'\,dx \right\} d\tau,$$

by the third inequality of Lemma 4.3 with τ and τ' playing the role of x and x', respectively. By the last inequality of Lemma 4.3, we get

$$T_3 \geq -\varepsilon \int_0^T |f(\tau)|_{\mathrm{BV}(0,1)}\,d\tau \geq -\varepsilon T |f|_{L^\infty(0,T;\mathrm{BV}(0,1))}.$$

Finally, let T_4 denote the left-hand side of inequality (4.2d). Then,

$$\begin{aligned} T_4 &= -\int_0^T \int_0^T |T(\tau, \tau')| w_{\varepsilon_0}(\tau-\tau')\,d\tau\,d\tau' \\ &= -\int_{-\varepsilon_0}^{\varepsilon_0} w_{\varepsilon_0}(\zeta) \left\{ \int_{a(\zeta)}^{b(\zeta)} |T(\tau'+\zeta, \tau')|\,d\tau \right\} d\zeta, \end{aligned}$$

where, for $\zeta \in [-\varepsilon_0, \varepsilon_0]$,

$$a(\zeta) = \max\{0, -\zeta\}, \qquad b(\zeta) = \min\{T, T-\zeta\}.$$

The above integral can also be rewritten as follows:

$$T_4 = -\int_{-\varepsilon_0}^{\varepsilon_0} w_{\varepsilon_0}(\zeta) \left\{ \int_0^T \chi(\tau'+\zeta) |T(\tau'+\zeta, \tau')|\,d\tau' \right\} d\zeta.$$

The inequality (4.2d) follows easily from the above expression. This completes the proof. □

We are now ready to complete the remaining estimates.

Lemma 4.8 (Lower bound for $T_{u^-}(u, v)$). *We have*

$$T_{u^-}(u, v) \geq u_* \left\{ \int_0^T \int_0^1 U(v(\tau, x) - u(\tau, x))\, dx\, d\tau \right.$$
$$\left. + L\varepsilon T |u|_{L^\infty(0,T;\mathrm{BV}(0,1))} + LT\nu_\tau(\varepsilon_0, u) \right\},$$

where $u_* = \inf_{(\tau,x)\in(0,T)\times(0,1)} u^-(\tau, x)$.

Proof. We have

$$T_{u^-}(u, v) \geq u_* \int_0^T \int_0^1 \int_0^T \int_0^1 U(v(\tau', x') - u(\tau, x))$$
$$\cdot \varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau.$$

Since, by the triangle inequality,

$$\begin{aligned} &U(v(\tau', x') - u(\tau, x)) \\ &\quad = U(v(\tau', x') - u(\tau', x')) \\ &\qquad + \{U(v(\tau', x') - u(\tau, x')) - U(v(\tau', x') - u(\tau', x'))\} \\ &\qquad + \{U(v(\tau', x') - u(\tau, x)) - U(v(\tau', x') - u(\tau, x'))\} \\ &\quad \leq U(v(\tau', x') - u(\tau', x')) + L|u(\tau, x') - u(\tau', x')| + L|u(\tau, x) - u(\tau, x')|, \end{aligned}$$

we have that

$$T_{u^-}(u, v) \geq T_1 + T_2 + T_3,$$

where

$$T_1 = u_* \int_0^T \int_0^1 \int_0^T \int_0^1 U(v(\tau', x') - u(\tau', x'))\varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau,$$
$$T_2 = u_* L \int_0^T \int_0^1 \int_0^T \int_0^1 |u(\tau, x') - u(\tau', x')|\varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau,$$
$$T_3 = u_* L \int_0^T \int_0^1 \int_0^T \int_0^1 |u(\tau, x) - u(\tau, x')|\varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau.$$

Since $u_* \leq 0$, by (4.2a) with $|\mathbb{F}| = U(v - u)$, we have

$$T_1 \geq u_* \int_0^T \int_0^1 U(v(\tau', x') - u(\tau', x'))\, dx'\, d\tau'.$$

By (4.2b) with $f = u$,

$$T_2 \geq u_* LT\nu_\tau(\varepsilon_0, u).$$

Finally, by (4.2c) with $f = u$,

$$T_3 \geq u_* L\varepsilon T |u|_{L^\infty(0,T;\mathrm{BV}(0,1))}.$$

Thus,

$$T_{u^-}(u, v) \geq u_* \left\{ \int_0^T \int_0^1 U(v(\tau, x) - u(\tau, x))\, dx\, d\tau \right.$$
$$\left. + L\varepsilon T |u|_{L^\infty(0,T;\mathrm{BV}(0,1))} + LT\nu_\tau(\varepsilon_0, u) \right\}.$$

This completes the proof. □

Lemma 4.9 (Lower bound for $T_{\mathrm{vel}}(u, \beta; v, \eta)$). *We have*

$$
\begin{aligned}
T_{\mathrm{vel}}(u, \beta; v, \eta) \geq & -\{2U^* + L|u|_{L^\infty(0,T;\mathrm{BV}(0,1))}\} \\
& \cdot \Bigg\{ \int_0^T \|u(\tau) - v(\tau)\|_{L^1(0,1)}\, d\tau \\
& \quad + \int_0^T \|\partial_{x'}\eta(\tau) + (1 - v(\tau))\|_{L^1(0,1)}\, d\tau \\
& \quad + 3\int_0^T \|\partial_x \beta(\tau) + (1 - u(\tau))\|_{L^1(0,1)}\, d\tau \\
& \quad + \int_0^T \left| \int_0^1 (\beta(\tau, x) - \eta(\tau, x))\, dx \right| d\tau \\
& \quad + T\varepsilon \|\partial_x \beta\|_{L^\infty(0,T;L^\infty(0,1))} + T\nu_\tau(\varepsilon_0, u) + v_\tau(\varepsilon_0, \beta) \Bigg\},
\end{aligned}
$$

where $U^* = \sup_{(\tau', x'), (\tau, x) \in (0,T)\times(0,1)} U(v(\tau', x') - u(\tau, x))$.

Proof. To avoid too many technicalities, we assume that u and β are very smooth functions; the result remains true if u and β satisfy the regularity assumptions (3.1). Integrating by parts in $T_{\mathrm{vel}}(u, \beta; v, \eta)$, we obtain

$$T_{\mathrm{vel}}(u, \beta; v, \eta) = T_1 + T_2 + T_3,$$

where

$$
\begin{aligned}
T_1 = & \int_0^T \int_0^1 \int_0^T \int_0^1 \partial_x U(v(\tau', x') - u(\tau, x)) \\
& \qquad \cdot \{(\beta(\tau, x) - \eta(\tau', x'))\varphi(\tau, x; \tau', x')\}\, dx'\, d\tau'\, dx\, d\tau, \\
T_2 = & -\int_0^T \int_0^1 \int_0^T U(v(\tau', x') - u(\tau, 1)) \\
& \qquad \cdot \{(\beta(\tau, 1) - \eta(\tau', x'))\varphi(\tau, 1; \tau', x')\}\, d\tau\, dx'\, d\tau', \\
T_3 = & \int_0^T \int_0^1 \int_0^T U(v(\tau', x') - u(\tau, 0)) \\
& \qquad \cdot \{(\beta(\tau, 0) - \eta(\tau', x'))\varphi(\tau, 0; \tau', x')\}\, d\tau\, dx'\, d\tau'.
\end{aligned}
$$

Consider T_1. By the triangle inequality, we have

$$
\begin{aligned}
& -|\beta(\tau, x) - \eta(\tau', x')| \\
& \quad \geq -|\beta(\tau', x') - \eta(\tau', x')| - |\beta(\tau', x) - \beta(\tau', x')| - |\beta(\tau, x) - \beta(\tau, x')| \\
& \quad \geq -\|\beta(\tau') - \eta(\tau')\|_{L^\infty(0,1)} \\
& \qquad - |x - x'|\, \|\partial_x \beta(\tau')\|_{L^\infty(0,1)} - \|\beta(\tau) - \beta(\tau')\|_{L^\infty(0,1)}.
\end{aligned}
$$

By Lemma 4.1, we have

$$
\begin{aligned}
&-|\beta(\tau, x)-\eta(\tau', x')| \\
&\quad \geq -\|u(\tau')-v(\tau')\|_{L^1(0,1)}-\|\partial_{x'}\eta(\tau')+(1-v(\tau'))\|_{L^1(0,1)} \\
&\qquad -\|\partial_x\beta(\tau')+(1-u(\tau'))\|_{L^1(0,1)}-\left|\int_0^1(\beta(\tau', x)-\eta(\tau', x))\,dx\right| \\
&\qquad -|x-x'|\,\|\partial_x\beta(\tau')\|_{L^\infty(0,1)}-\|u(\tau)-u(\tau')\|_{L^1(0,1)} \\
&\qquad -\|\partial_x\beta(\tau')+(1-u(\tau'))\|_{L'(0,1)}-\|\partial_x\beta(\tau)+(1-u(\tau))\|_{L^1(0,1)} \\
&\qquad -\left|\int_0^1(\beta(\tau, x)-\beta(\tau', x))\,dx\right| \\
&\quad \equiv R_1(\tau, \tau')-|x-x'|R_2(\tau') \equiv R(\tau, \tau'; x-x'),
\end{aligned}
$$

where $R_2(\tau')=\|\partial_x\beta(\tau')\|_{L^\infty(0,1)}$. Hence,

$$
\begin{aligned}
T_1 &\geq \int_0^T\int_0^1\int_0^T\int_0^1 |U'(v(\tau', x')-u(\tau, x))|\,|\partial_x u(\tau, x)|R(\tau, \tau'; x-x') \\
&\qquad\qquad \cdot\varphi(\tau, x; \tau', x')\,dx'\,d\tau'\,dx\,d\tau \\
&\geq L\int_0^T\int_0^T\left\{\int_0^1|\partial_x u(\tau, x)|\left\{\int_0^1 w_\varepsilon(x-x')\,dx'\right\}dx\right\} \\
&\qquad\qquad \cdot R_1(\tau, \tau')w_{\varepsilon_0}(\tau-\tau')\,d\tau'\,d\tau \\
&\quad -L\int_0^T\int_0^T\left\{\int_0^1|\partial_x u(\tau, x)|\left\{\int_0^1|x-x'|w_\varepsilon(x-x')\,dx'\right\}dx\right\} \\
&\qquad\qquad \cdot R_2(\tau')w_{\varepsilon_0}(\tau-\tau')\,d\tau'\,d\tau \\
&\geq L|u|_{L^\infty(0,T;\mathrm{BV}(0,1))}\int_0^T\int_0^T w_{\varepsilon_0}(\tau-\tau')R_1(\tau, \tau'))\,d\tau'\,d\tau \\
&\quad -L|u|_{L^\infty(0,T;\mathrm{BV}(0,1))}\varepsilon\int_0^T R_2(\tau')\,d\tau',
\end{aligned}
$$

by the third inequality of Lemma 4.3. By (4.2d), we get

$$
\begin{aligned}
T_1 &\geq L|u|_{L^\infty(0,T;\mathrm{BV}(0,1))}\sup_{|\zeta|\leq\varepsilon_0}\int_0^T\chi(\tau'+\zeta)R_1(\tau'+\zeta, \tau')\,d\tau' \\
&\quad -L|u|_{L^\infty(0,T;\mathrm{BV}(0,1))}\varepsilon T\|\partial_x\beta\|_{L^\infty(0,T;L^\infty(0,1))},
\end{aligned}
$$

where χ is the characteristic function of the interval $[0, T]$.

Finally, using the definition of the $T(\tau, \tau')$ and the definitions of $\nu_\tau(\varepsilon_0, u)$ and $v_\tau(\varepsilon_0, \beta)$ given in (3.2), we easily get

$$\begin{aligned} T_1 \geq -L|u|_{L^\infty(0,T;\mathrm{BV}(0,1))} \Bigg\{ & \int_0^T \|u(\tau) - v(\tau)\|_{L^1(0,1)}\, d\tau \\ & + \int_0^T \|\partial_{x'}\eta(\tau) + (1 - v(\tau))\|_{L^1(0,1)}\, d\tau \\ & + 3\int_0^T \|\partial_x \beta(\tau) + (1 - u(\tau))\|_{L^1(0,1)}\, d\tau \\ & + \int_0^T \left| \int_0^1 (\beta(\tau, x) - \eta(\tau, x))\, dx \right| d\tau \\ & + T\varepsilon \|\partial_x \beta\|_{L^\infty(0,T;L^\infty(0,1))} + T\nu_\tau(\varepsilon_0, u) + v_\tau(\varepsilon_0, \beta) \Bigg\}. \end{aligned}$$

The other terms are treated in a similar way. □

Lemma 4.10 (Lower bound for $T_\beta(u, v)$). *We have*

$$T_\beta(u, v) \geq -L\|u\|_{L^\infty(0,T;L^\infty(0,1))} \int_0^T \|\partial_x \beta(\tau) + (1 - u(\tau))\|_{L^1(0,1)}\, d\tau.$$

Proof. By the definition of $T_\beta(u, v)$ in Lemma 4.2, we have

$$\begin{aligned} T_\beta(u, v) \geq -C \int_0^T \int_0^1 \int_0^T \int_0^1 & |\partial_x \beta(\tau, x) + (1 - u(\tau, x))| \\ & \cdot \varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau, \end{aligned}$$

where $C = L\|u\|_{L^\infty(0,T;L^\infty(0,1))}$. The result follows from (4.2a). This completes the proof. □

Lemma 4.11 (Lower bound for $T_\eta(u, v)$). *We have*

$$T_\eta(u, v) \geq -\{U^* + L\|v\|_{L^\infty(0,T;L^\infty(0,1))}\} \int_0^T \|\partial_{x'}\eta(\tau') + (1 - v(\tau'))\|_{L^1(0,1)}\, d\tau',$$

where $U^* = \sup_{\tau,\tau' \in (0,T);\, x, x' \in (0,1)} U(v(\tau', x') - u(\tau, x))$.

Proof. By the definition of $T_\eta(u, v)$ in Lemma 4.2, we have

$$\begin{aligned} T_\eta(u, v) \geq -C \int_0^T \int_0^1 \int_0^T \int_0^1 & |\partial_{x'}\eta(\tau', x') + (1 - v(\tau', x'))| \\ & \cdot \varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau, \end{aligned}$$

where $C = \sup_{\tau,\tau' \in (0,T);\, x, x' \in (0,1)} |V(v(\tau', x'), u(\tau, x))|$. By the definition of

V, (3.4d), we get

$$C \leq \{U^* + L\|v\|_{L^\infty(0,T;L^\infty(0,1))}\} \equiv C',$$

and hence,

$$T_\eta(u, v) \geq -C' \int_0^T \int_0^1 \int_0^T \int_0^1 |\partial_{x'}\eta(\tau', x') + (1 - v(\tau', x'))| \cdot \varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau.$$

The result follows from (4.2a). □

Lemma 4.12 (Lower bound for $T_{U''_{1/M}}(u, v)$). *We have*

$$T_{U''_{1/M}}(u, v) \geq -\frac{1}{2}M^{-1}(T + \|v\|_{L^1(0,T;L^1(0,1))} + \|u\|_{L^1(0,T;L^1(0,1))}).$$

Proof. By the definition in Lemma 4.2, we have

$$\begin{aligned} T_{U''_{1/M}}(u, v) &\geq -C \int_0^T \int_0^1 \int_0^T \int_0^1 |1 - v(\tau', x') - u(\tau, x)| \cdot \varphi(\tau, x; \tau', x')\, dx'\, d\tau'\, dx\, d\tau \\ &\geq -C(T + \|v\|_{L^1(0,T;L^1(0,1))} + \|u\|_{L^1(0,T;L^1(0,1))}), \end{aligned}$$

by (4.2a), where

$$C = \sup_{\tau, \tau' \in (0,T); x, x' \in (0,1)} \left| \int_0^{v(\tau', x') - u(\tau, x)} s U''_{1/M}(s)\, ds \right|.$$

Since, by (3.3b),

$$U''_{1/M}(s) = \begin{cases} M & \text{for } |s| \leq 1/M, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\left| \int_0^{v(\tau', x') - u(\tau, x)} s U''_{1/M}(s)\, ds \right| \leq \int_0^{1/M} sM\, ds = 1/2M,$$

and the result follows. □

We now put together the estimates obtained in the previous results.

Corollary 4.13. *Let* (u, β) *and* (v, η) *be functions satisfying the regularity conditions* (3.1). *Then there is a constant* C, *which is bounded provided that* T *is bounded and the regularity conditions* (3.1) *are satisfied, such that*

$$
\begin{aligned}
&\int_0^1 U_{1/M}(v(T,x)-u(T,x))\,dx \\
&\quad \le C\Bigg\{\int_0^1 U_{1/M}(v(0,x)-u(0,x))\,dx \\
&\qquad + \int_0^1 U_{1/M}(v(\tau,1+)-u(\tau,1+))\,d\tau \\
&\qquad + \int_0^1 U_{1/M}(v(\tau,0-)-u(\tau,0-))\,d\tau \\
&\qquad + \int_0^T\int_0^1 U_{1/M}(v(\tau,x)-u(\tau,x))\,dx\,d\tau \\
&\qquad + \int_0^T \|v(\tau)-u(\tau)\|_{L^1(0,1)}\,d\tau \\
&\qquad + \int_0^T \left|\int_0^1 (\eta(\tau,x)-\beta(\tau,x))\,dx\right| d\tau \\
&\qquad + \int_0^T \|\partial_{x'}\eta(\tau')+(1-v(\tau'))\|_{L^1(0,1)}\,d\tau' \\
&\qquad + \int_0^T \|\partial_x\beta(\tau)+(1-u(\tau))\|_{L^1(0,1)}\,d\tau \\
&\qquad + \varepsilon+\varepsilon_0+1/M \\
&\qquad + \{\nu^-_{x,1}(\varepsilon,u;\beta)+\nu^-_{x,1}(\varepsilon,v;\eta)+\nu^+_{x,0}(\varepsilon,u;\beta)+\nu^+_{x,0}(\varepsilon,v;\eta)\} \\
&\qquad + \{\nu^-_{\tau,T}(\varepsilon_0,u)+\nu^-_{\tau,T}(\varepsilon_0,v)+\nu^+_{\tau,0}(\varepsilon_0,u)+\nu^+_{\tau,0}(\varepsilon_0,v)\} \\
&\qquad + \{\nu_\tau(\varepsilon_0,u)+\nu_\tau(\varepsilon_0,v)+v_\tau(\varepsilon_0,\beta)\} \\
&\qquad\qquad + E^{\varepsilon_0,\varepsilon}_{1/M}(v,u;\eta)+E^{\varepsilon_0,\varepsilon}_{1/M}(u,v;\beta)\Bigg\}.
\end{aligned}
$$

We can now prove Theorem 3.1.

Proof of Theorem 3.1. The second inequality of Theorem 3.1 is nothing but Lemma 4.1. To obtain the first inequality of Theorem 3.1, we notice that Corollary 4.13 holds for (u,β) equal to the weak solution of the problem (1.1), (1.2), and (v,η) equal to (u_h,β_h), the approximate solution defined by the numerical method described in §2, if these functions satisfy the regularity conditions (3.1). The regularity properties (3.1a) and (3.1b) follow from [1, Theorem 2.1] and [1, Theorem 2.3]. To prove the regularity property (3.1c), we can proceed as in Lemma 4.1 to get, for $x\in\{0,1\}$,

$$
\begin{aligned}
|\beta(x)|_{\mathrm{BV}(0,T)} &\le \left|\int_0^1 \beta(\cdot,x)\right|_{\mathrm{BV}(0,T)} + |\partial_x\beta|_{\mathrm{BV}(0,T;L^1(0,1))} \\
&= |\phi_1|_{\mathrm{BV}(0,T)} + |u|_{\mathrm{BV}(0,T;L^1(0,1))} < \infty,
\end{aligned}
$$

by hypothesis (3.5d) and [1, Theorem 2.3]. A similar result can be easily proven for $|\beta_h(x)|_{\mathrm{BV}(0,T)}$ for $x\in\{0,1\}$.

Thus, the inequality of Corollary 4.13 holds for the exact and approximate solutions of (1.1) and (1.2). In such an inequality, we can replace the terms

involving the function $U_{1/M}$ by using the following inequalities:

$$\|v\|_{L^1(0,1)} - 1/2M \leq \int_0^1 U_{1/M}(v(x))\,dx \leq \|v\|_{L^1(0,1)},$$

which follow from the definition of $U_{1/M}$, (3.3b). After doing that, we obtain an inequality of the form

$$\|u_h(T) - u(T)\|_{L^1(0,1)} \leq A + C\int_0^T \|u_h(\tau) - u(\tau)\|_{L^1(0,1)}\,d\tau,$$

which after a simple application of Gronwall's Lemma becomes

$$\|u_h(T) - u(T)\|_{L^1(0,1)} \leq Ae^{CT}.$$

This completes the proof of Theorem 3.1. □

Acknowledgments

The author wants to thank the reviewer for helpful comments that led to a better presentation of this paper.

Bibliography

1. B. Cockburn and I. Triandaf, *Convergence of a finite element method for the drift-diffusion semiconductor equations: the zero-diffusion case*, Math. Comp. **59** (1992), 383–401.
2. B. Cockburn, *The quasi-monotone schemes for scalar conservation laws.* I, SIAM J. Numer. Anal. **26** (1989), 1325–1341.
3. B. Cockburn, F. Coquel, and P. LeFloch, *An error estimate for finite volume methods for multidimensional conservation laws*, Math. Comp. **63** (1994), 77–103.
4. J. Douglas, Jr., I. Martínez-Gamba, and M. C. J. Squeff, *Simulation of the transient behavior of a one-dimensional semiconductor device*, Mat. Apl. Comput. **5** (1986), 103–122.
5. S. N. Kružkov, *First order quasilinear equations in several independent variables*, Math. USSR-Sb. **10** (1970), 217–243.
6. N. N. Kuznetsov, *Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation*, U.S.S.R. Comput. Math. and Math. Phys. **16** (1976), 105–119.
7. A. Y. LeRoux, *Etude du problème mixte pour une équation quasilinéaire du premier ordre*, C. R. Acad. Sci. Paris Sér. A **285** (1977), 351–354.
8. B. J. Lucier, *Error bounds for the methods of Glimm, Godunov and LeVeque*, SIAM J. Numer. Anal. **22** (1985), 1074–1081.
9. ——, *A moving mesh numerical method for hyperbolic conservation laws*, Math. Comp. **46** (1986), 59–69.
10. ——, *On nonlocal monotone difference schemes for scalar conservation laws*, Math. Comp. **47** (1986), 19–36.
11. I. Martínez-Gamba and M. C. J. Squeff, *Simulation of the transient behavior of a one-dimensional semiconductor device.* II, SIAM J. Numer. Math. **26** (1989), 539–552.
12. R. Sanders, *On convergence of monotone finite difference schemes with variable spatial differencing*, Math. Comp. **40** (1983), 91–106.
13. M. Sever, *Analysis of a discretization algorithm for time-dependent semiconductor models*, COMPEL **6** (1987), 171–189.
14. V. W. Van Roosbroeck, *Theory of flow of electrons and holes in germanium and other semiconductors*, Bell Syst. Tech. J. **29** (1950), 560–607.

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
E-mail address: cockburn@math.umn.edu
E-mail address: triandaf@ppdpi2.nrl.navy.mil