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COMPUTING DIVISION POLYNOMIALS

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ABSTRACT. Recurrence relations for the coefficients of the nth division polynomial for elliptic curves are presented. These provide an algorithm for computing the general division polynomial without using polynomial multiplications; also a bound is given for the coefficients, and their general shape is revealed, with a means for computing the coefficients as explicit functions of n.

1. INTRODUCTION

Let k be a field with characteristic $\neq 2$ or 3. Given $a, b \in k$ with $4a^3 + 27b^2 \neq 0$, let E be the elliptic curve over k defined (as a projective plane curve over k) by the affine equation

$$y^2 = x^3 + ax + b,$$

with the special point being the point at infinity.

With the usual abelian group law on E, we have the notion of a multiplication-by-n map, for any integer n, denoted $[n]$. For positive integers n, we define *division polynomials* $f_n \in \mathbb{Z}[a, b][x]$ by the recursion formulae (cf. [4, p. 200])

$$\begin{aligned}
f_1 &= 1, \\
f_2 &= 2, \\
f_3 &= 3x^4 + 6ax^2 + 12bx - a^2, \\
f_4 &= 4x^6 + 20ax^4 + 80bx^3 - 20a^2x^2 - 16abx - 32b^2 - 4a^3, \\
f_{2m} &= f_m(f_{m+2}f_{m-1}^2 - f_{m-2}f_{m+1}^2)/2, \qquad m \geq 3, \\
f_{4l+1} &= (x^3 + ax + b)^2 f_{2l+2}f_{2l}^3 - f_{2l-1}f_{2l+1}^3, \qquad l \geq 1, \\
f_{4l+3} &= f_{2l+3}f_{2l+1}^3 - (x^3 + ax + b)^2 f_{2l}f_{2l+2}^3, \qquad l \geq 1.
\end{aligned} \tag{1}$$

The vanishing of $f_n(x)$ for n odd, or of $yf_n(x)$ for n even, characterizes the kernel of $[n]$. As a polynomial in x, f_n has degree $\chi(n)$, where $\chi(n) = (n^2-1)/2$ if n is odd, and $\chi(n) = (n^2-4)/2$ if n is even. The relation between f_n and Weber's ψ_n [3, p. 105] is that $f_n = \psi_n$ for n odd, and $f_n = \psi_n/y$ for n even.

If x is given weight 1, a is given weight 2, and b is given weight 3, then all the terms in $f_n(a, b, x)$ have weight $\chi(n)$. Thus, the coefficient of $x^{\chi(n)-1}$

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must be 0, and we have

$$f_n(a, b, x) = \alpha_{0,0}(n)x^{\chi(n)} + \alpha_{1,0}(n)ax^{\chi(n)-2} + \alpha_{0,1}(n)bx^{\chi(n)-3} + \cdots + \alpha_{r,s}(n)a^r b^s x^{\chi(n)-2r-3s} + \cdots,$$

where $\alpha_{r,s}(n) \in \mathbb{Z}$.

In this paper we give recurrence relations for the coefficients of a *fixed* division polynomial; these can be used to compute the coefficients $\alpha_{r,s}(n)$ *as functions of* n and to compute the general nth division polynomial $f_n(a, b, x)$ using $O(n^6)$ integer operations. The recurrence relations also provide bounds for the coefficients and reveal their general shape.

2. Statement of Main Lemma and deduction of results

Define $\alpha_{r,s}(n) = 0$ if either r or s is negative, or if $2r + 3s > \chi(n)$. Then $f_n(a, b, x) = \sum_t \beta_t(n)x^t$, where

$$\beta_t(n) = \sum_{2r+3s=\chi(n)-t} \alpha_{r,s}(n)a^r b^s \in \mathbb{Z}[a, b].$$

Main Lemma. *For n odd, and any $i \in \mathbb{Z}$,*

$$(i+3)(i+2)b\beta_{i+3}(n) - (i+2)(2n^2/3 - 3/2 - i)a\beta_{i+2}(n) + ((n^2-2i)(n^2-2i-1)/4)\beta_i(n) - 3n^2 b\frac{\partial \beta_{i+1}(n)}{\partial a} + (2n^2a^2/3)\frac{\partial \beta_{i+1}(n)}{\partial b} = 0, \tag{2}$$

and, with $d = 2r + 3s$, for any $r, s \in \mathbb{Z}$,

$$\begin{aligned} d(d+1/2)\alpha_{r,s}(n) = {} & ((n^2+3)/2 - d)(n^2/6 - 1 + d)\alpha_{r-1,s}(n) \\ & - ((n^2+5)/2 - d)((n^2+3)/2 - d)\alpha_{r,s-1}(n) \\ & + 3(r+1)n^2\alpha_{r+1,s-1}(n) \\ & - (2(s+1)n^2/3)\alpha_{r-2,s+1}(n). \end{aligned} \tag{3}$$

For n even, we have similarly

$$(i+3)(i+2)b\beta_{i+3}(n) - (i+2)(2n^2/3 - 5/2 - i)a\beta_{i+2}(n) + ((n^2-2i-3)(n^2-2i-4)/4)\beta_i(n) - 3n^2 b\frac{\partial \beta_{i+1}(n)}{\partial a} + (2n^2a^2/3)\frac{\partial \beta_{i+1}(n)}{\partial b} = 0, \tag{4}$$

and

$$\begin{aligned} d(d+1/2)\alpha_{r,s}(n) = {} & (n^2/2 - d)(n^2/6 - 1/2 + d)\alpha_{r-1,s}(n) \\ & - ((n^2+2)/2 - d)(n^2/2 - d)\alpha_{r,s-1}(n) \\ & + 3(r+1)n^2\alpha_{r+1,s-1}(n) \\ & - (2(s+1)n^2/3)\alpha_{r-2,s+1}(n). \end{aligned} \tag{5}$$

TABLE 1

n	Computed maximum number of decimal digits in $\alpha_{r,s}(n)$	Bound on number of digits implied by (6)
6	5	22
12	22	93
24	90	381

Corollary 1. *There holds*

$$\log(1+|\alpha_{r,s}(n)|) = O(n^2),$$

where the implied constant is independent of r and s.

Proof. Let B_d be a bound for $|\alpha_{r,s}(n)|$ over $2r+3s \le d$. We have $B_0 = B_1 = n$, and from (3) and (5) we deduce that

$$B_d \le \frac{n^2(d+n^2/2)}{d^2} B_{d-1},$$

for $d \ge 2$ and $n \ge 5$, and the cases $n < 5$ can be checked directly. Hence,

$$\begin{aligned} |\alpha_{r,s}(n)| \le B_{\chi(n)} &\le \frac{n^{n^2}(n^2-1/2)!}{[((n^2-1)/2)!]^2(n^2/2+1)!} \\ &\sim 2^{(3n^2+1)/2} e^{n^2/2}/\pi n^3. \end{aligned} \tag{6}$$

Taking logarithms gives the desired bound. □

Remark. This corollary suggests that the maximum number of digits in the coefficients of f_n should grow like n^2. This is reflected in Table 1.

Corollary 2. *There holds*

$$\alpha_{r,s}(n) = P_{r,s}(n) + (-1)^n Q_{r,s}(n),$$

where $P_{r,s}$ and $Q_{r,s}$ are both odd polynomials in $\mathbb{Q}[n]$ (i.e., only odd powers of n occur), $P_{r,s}$ has degree at most $4r+6s+1$, and $Q_{r,s}$ has degree at most $4r+6s-3$. The denominators of $P_{r,s}$ and $Q_{r,s}$ are $(4r+6s+1)$-smooth (i.e., they have no prime divisors greater than $4r+6s+1$).

Proof. Induction on $2r+3s$, using (3) and (5). □

Remark. Using (3) and (5), one can compute explicit formulae for any desired $\alpha_{r,s}(n)$, e.g.,

$$\alpha_{1,0}(n) = \begin{cases} \frac{1}{60}n(n^2-1)(n^2+6), & n \text{ odd}, \\ \frac{1}{60}n(n^2-4)(n^2+9), & n \text{ even}. \end{cases}$$

Corollary 3. *The general division polynomial $f_n(a, b, x)$ can be computed using $O(n^6)$ multiplications and divisions (of integers with $O(n^2)$ digits by integers with $O(\log n)$ digits) and $O(n^6)$ additions (of integers with $O(n^2)$ digits).*

Proof. Set $x = 1$. Starting with $\beta_{\chi(n)}(n) = n$, and $\beta_t(n) = 0$ for $t > \chi(n)$, one can use (2) or (4) as appropriate to compute $\beta_t(n)$ for $t = \chi(n)-1$, $\chi(n)-2, \ldots, 0$. Each application of (2) or (4) requires $O(n^4)$ integer operations of the type given in the statement of the corollary (using Corollary 1 to bound the coefficients), and $O(n^2)$ applications are needed. □

3. A comparison with the traditional means for computing f_n

For specific values of a and b, using the recursion formulae (1) seems to be the best (i.e., quickest) method for computing $f_n(a, b, x)$. For computing the general division polynomial $f_n(a, b, x) \in \mathbb{Z}[a, b][x]$, however, this approach is very slow. By homogeneity, it suffices to compute $f_n(a, b, 1)$. The most time-consuming step is the final use of (1), which involves multiplying together polynomials in two variables, of degree $O(n^2)$ in each, so having $O(n^4)$ terms. Thus $O(n^8)$ multiplications of integer coefficients are needed, if one uses "ordinary" polynomial multiplication. By using divide and conquer [1, pp. 62–64] this can be reduced to $O(n^{4\log_2 3}) = O(n^{6.34})$ multiplications of integer coefficients (with $O(n^2)$ digits). Using FFT techniques [1, pp. 252 ff.] we can further reduce this to $O(n^4(\log n)^2)$ multiplications of integer coefficients. Thus, using (1) with FFT would be *ultimately* faster than (2)/(4), but, for reasonable values of n, using (2)/(4) is better.

Using PARI-GP on a Sun 3/60 workstation, we timed the last step in using (1) to compute f_n for a few values of n ($t_1(n)$ in Table 2—this is an underestimate for the time to compute $f_n(a, b, 1)$). By comparison, $t_2(n)$ in Table 2 gives the time taken to compute $f_n(a, b, 1)$ from scratch, using (2) or (4) as appropriate. The polynomial $f_{25}(a, b, 1)$ has 8269 terms with coefficients up to 97 decimal digits long. For small n, using (1) beats using (2)/(4), but the latter method soon becomes better.

Table 2. Comparing $t_1(n)$, an underestimate of the time taken to compute $f_n(a, b, 1)$ using (1), with $t_2(n)$, the time taken using (2) or (4) as appropriate

n	$t_1(n)$	$t_2(n)$
10	1s	6s
15	29s	47s
20	2 min 44s	3 min 5s
23	13 min 31s	9 min 29s
25	27 min 23s	15 min 29s

4. Proof of lemma

First suppose n is odd. Fricke, in [2, p. 191], derives a partial differential equation for ψ_n, which for n odd translates directly into a partial differential equation for f_n:

$$\begin{aligned}(x^3+ax+b)\frac{\partial^2 f_n}{\partial x^2} &- ((n^2-3/2)x^2+(2n^2/3-1/2)a)\frac{\partial f_n}{\partial x} \\ &- 3n^2 b\frac{\partial f_n}{\partial a} + (2n^2a^2/3)\frac{\partial f_n}{\partial b} + n^2(n^2-1)xf_n/4 = 0.\end{aligned} \tag{7}$$

He comments that this provides linear relations between the coefficients of f_n, which together with $\alpha_{0,0}(n) = n$ suffice to determine f_n, but he complains that this "freilich schon bei $n = 5$ einen erheblichen Aufwand von Rechnung erfordert", implying that this is not a profitable approach. Here we disagree. Our aim is to make the solution more explicit. Note that although (7) is derived over $\mathbb{C}$ using complex-variable methods, it is just a formal identity in

$\mathbb{Z}[1/6, a, b][x]$ and as such holds over any field with characteristic not dividing 6.

Equating coefficients of x^{i+1} in (7) gives (2), at least for $i \geq 0$, but since $\beta_t = 0$ for $t < 0$ one soon checks that (2) holds for negative i too.

Set $i = (n^2-1)/2 - 2r - 3s$ in (2); then equating coefficients of $a^r b^s$ gives (3).

For n even, replace f_n by yf_n in (7), giving

$$(x^3+ax+b)\frac{\partial^2 f_n}{\partial x^2} - ((n^2-9/2)x^2 + (2n^2/3 - 3/2)a)\frac{\partial f_n}{\partial x} + ((n^2-3)(n^2-4)x/4)f_n - 3n^2 b\frac{\partial f_n}{\partial a} + (2n^2a^2/3)\frac{\partial f_n}{\partial b} = 0.$$

Equating coefficients of x^{i+1} gives (4) for $i \geq 0$, but again this extends to all i.

Set $i = (n^2-4)/2 - 2r - 3s$ in (4); then equating coefficients of $a^r b^s$ gives (5). $\square$

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