

COMPUTING DIVISION POLYNOMIALS

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ABSTRACT. Recurrence relations for the coefficients of the n th division polynomial for elliptic curves are presented. These provide an algorithm for computing the general division polynomial without using polynomial multiplications; also a bound is given for the coefficients, and their general shape is revealed, with a means for computing the coefficients as explicit functions of n .

1. INTRODUCTION

Let k be a field with characteristic $\neq 2$ or 3 . Given $a, b \in k$ with $4a^3 + 27b^2 \neq 0$, let E be the elliptic curve over k defined (as a projective plane curve over k) by the affine equation

$$y^2 = x^3 + ax + b,$$

with the special point being the point at infinity.

With the usual abelian group law on E , we have the notion of a multiplication-by- n map, for any integer n , denoted $[n]$. For positive integers n , we define *division polynomials* $f_n \in \mathbb{Z}[a, b][x]$ by the recursion formulae (cf. [4, p. 200])

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 2, \\ f_3 &= 3x^4 + 6ax^2 + 12bx - a^2, \\ (1) \quad f_4 &= 4x^6 + 20ax^4 + 80bx^3 - 20a^2x^2 - 16abx - 32b^2 - 4a^3, \\ f_{2m} &= f_m(f_{m+2}f_{m-1}^2 - f_{m-2}f_{m+1}^2)/2, \quad m \geq 3, \\ f_{4l+1} &= (x^3 + ax + b)^2 f_{2l+2}f_{2l}^3 - f_{2l-1}f_{2l+1}^3, \quad l \geq 1, \\ f_{4l+3} &= f_{2l+3}f_{2l+1}^3 - (x^3 + ax + b)^2 f_{2l}f_{2l+2}^3, \quad l \geq 1. \end{aligned}$$

The vanishing of $f_n(x)$ for n odd, or of $yf_n(x)$ for n even, characterizes the kernel of $[n]$. As a polynomial in x , f_n has degree $\chi(n)$, where $\chi(n) = (n^2 - 1)/2$ if n is odd, and $\chi(n) = (n^2 - 4)/2$ if n is even. The relation between f_n and Weber's ψ_n [3, p. 105] is that $f_n = \psi_n$ for n odd, and $f_n = \psi_n/y$ for n even.

If x is given weight 1, a is given weight 2, and b is given weight 3, then all the terms in $f_n(a, b, x)$ have weight $\chi(n)$. Thus, the coefficient of $x^{\chi(n)-1}$

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must be 0, and we have

$$f_n(a, b, x) = \alpha_{0,0}(n)x^{\chi(n)} + \alpha_{1,0}(n)ax^{\chi(n)-2} + \alpha_{0,1}(n)bx^{\chi(n)-3} + \dots + \alpha_{r,s}(n)a^r b^s x^{\chi(n)-2r-3s} + \dots,$$

where $\alpha_{r,s}(n) \in \mathbb{Z}$.

In this paper we give recurrence relations for the coefficients of a *fixed* division polynomial; these can be used to compute the coefficients $\alpha_{r,s}(n)$ as *functions of n* and to compute the general n th division polynomial $f_n(a, b, x)$ using $O(n^6)$ integer operations. The recurrence relations also provide bounds for the coefficients and reveal their general shape.

2. STATEMENT OF MAIN LEMMA AND DEDUCTION OF RESULTS

Define $\alpha_{r,s}(n) = 0$ if either r or s is negative, or if $2r + 3s > \chi(n)$. Then $f_n(a, b, x) = \sum_t \beta_t(n)x^t$, where

$$\beta_t(n) = \sum_{2r+3s=\chi(n)-t} \alpha_{r,s}(n)a^r b^s \in \mathbb{Z}[a, b].$$

Main Lemma. For n odd, and any $i \in \mathbb{Z}$,

$$(2) \quad \begin{aligned} &(i + 3)(i + 2)b\beta_{i+3}(n) - (i + 2)(2n^2/3 - 3/2 - i)a\beta_{i+2}(n) \\ &+ ((n^2 - 2i)(n^2 - 2i - 1)/4)\beta_i(n) - 3n^2b \frac{\partial \beta_{i+1}(n)}{\partial a} \\ &+ (2n^2a^2/3) \frac{\partial \beta_{i+1}(n)}{\partial b} = 0, \end{aligned}$$

and, with $d = 2r + 3s$, for any $r, s \in \mathbb{Z}$,

$$(3) \quad \begin{aligned} d(d + 1/2)\alpha_{r,s}(n) &= ((n^2 + 3)/2 - d)(n^2/6 - 1 + d)\alpha_{r-1,s}(n) \\ &- ((n^2 + 5)/2 - d)((n^2 + 3)/2 - d)\alpha_{r,s-1}(n) \\ &+ 3(r + 1)n^2\alpha_{r+1,s-1}(n) \\ &- (2(s + 1)n^2/3)\alpha_{r-2,s+1}(n). \end{aligned}$$

For n even, we have similarly

$$(4) \quad \begin{aligned} &(i + 3)(i + 2)b\beta_{i+3}(n) - (i + 2)(2n^2/3 - 5/2 - i)a\beta_{i+2}(n) \\ &+ ((n^2 - 2i - 3)(n^2 - 2i - 4)/4)\beta_i(n) - 3n^2b \frac{\partial \beta_{i+1}(n)}{\partial a} \\ &+ (2n^2a^2/3) \frac{\partial \beta_{i+1}(n)}{\partial b} = 0, \end{aligned}$$

and

$$(5) \quad \begin{aligned} d(d + 1/2)\alpha_{r,s}(n) &= (n^2/2 - d)(n^2/6 - 1/2 + d)\alpha_{r-1,s}(n) \\ &- ((n^2 + 2)/2 - d)(n^2/2 - d)\alpha_{r,s-1}(n) \\ &+ 3(r + 1)n^2\alpha_{r+1,s-1}(n) \\ &- (2(s + 1)n^2/3)\alpha_{r-2,s+1}(n). \end{aligned}$$

TABLE 1

| n | Computed maximum number of decimal digits in $\alpha_{r,s}(n)$ | Bound on number of digits implied by (6) |
|-----|--|--|
| 6 | 5 | 22 |
| 12 | 22 | 93 |
| 24 | 90 | 381 |

Corollary 1. *There holds*

$$\log(1 + |\alpha_{r,s}(n)|) = O(n^2),$$

where the implied constant is independent of r and s .

Proof. Let B_d be a bound for $|\alpha_{r,s}(n)|$ over $2r + 3s \leq d$. We have $B_0 = B_1 = n$, and from (3) and (5) we deduce that

$$B_d \leq \frac{n^2(d + n^2/2)}{d^2} B_{d-1},$$

for $d \geq 2$ and $n \geq 5$, and the cases $n < 5$ can be checked directly. Hence,

$$(6) \quad |\alpha_{r,s}(n)| \leq B_{\chi(n)} \leq \frac{n^n (n^2 - 1/2)!}{[(n^2 - 1)/2!]^2 (n^2/2 + 1)!} \\ \sim 2^{(3n^2+1)/2} e^{n^2/2} / \pi n^3.$$

Taking logarithms gives the desired bound. \square

Remark. This corollary suggests that the maximum number of digits in the coefficients of f_n should grow like n^2 . This is reflected in Table 1.

Corollary 2. *There holds*

$$\alpha_{r,s}(n) = P_{r,s}(n) + (-1)^n Q_{r,s}(n),$$

where $P_{r,s}$ and $Q_{r,s}$ are both odd polynomials in $\mathbb{Q}[n]$ (i.e., only odd powers of n occur), $P_{r,s}$ has degree at most $4r + 6s + 1$, and $Q_{r,s}$ has degree at most $4r + 6s - 3$. The denominators of $P_{r,s}$ and $Q_{r,s}$ are $(4r + 6s + 1)$ -smooth (i.e., they have no prime divisors greater than $4r + 6s + 1$).

Proof. Induction on $2r + 3s$, using (3) and (5). \square

Remark. Using (3) and (5), one can compute explicit formulae for any desired $\alpha_{r,s}(n)$, e.g.,

$$\alpha_{1,0}(n) = \begin{cases} \frac{1}{60} n(n^2 - 1)(n^2 + 6), & n \text{ odd,} \\ \frac{1}{60} n(n^2 - 4)(n^2 + 9), & n \text{ even.} \end{cases}$$

Corollary 3. *The general division polynomial $f_n(a, b, x)$ can be computed using $O(n^6)$ multiplications and divisions (of integers with $O(n^2)$ digits by integers with $O(\log n)$ digits) and $O(n^6)$ additions (of integers with $O(n^2)$ digits).*

Proof. Set $x = 1$. Starting with $\beta_{\chi(n)}(n) = n$, and $\beta_t(n) = 0$ for $t > \chi(n)$, one can use (2) or (4) as appropriate to compute $\beta_t(n)$ for $t = \chi(n) - 1, \chi(n) - 2, \dots, 0$. Each application of (2) or (4) requires $O(n^4)$ integer operations of the type given in the statement of the corollary (using Corollary 1 to bound the coefficients), and $O(n^2)$ applications are needed. \square

3. A COMPARISON WITH THE TRADITIONAL MEANS FOR COMPUTING f_n

For specific values of a and b , using the recursion formulae (1) seems to be the best (i.e., quickest) method for computing $f_n(a, b, x)$. For computing the general division polynomial $f_n(a, b, x) \in \mathbb{Z}[a, b][x]$, however, this approach is very slow. By homogeneity, it suffices to compute $f_n(a, b, 1)$. The most time-consuming step is the final use of (1), which involves multiplying together polynomials in two variables, of degree $O(n^2)$ in each, so having $O(n^4)$ terms. Thus $O(n^8)$ multiplications of integer coefficients are needed, if one uses "ordinary" polynomial multiplication. By using divide and conquer [1, pp. 62–64] this can be reduced to $O(n^{4 \log_2 3}) = O(n^{6.34})$ multiplications of integer coefficients (with $O(n^2)$ digits). Using FFT techniques [1, pp. 252 ff.] we can further reduce this to $O(n^4(\log n)^2)$ multiplications of integer coefficients. Thus, using (1) with FFT would be *ultimately* faster than (2)/(4), but, for reasonable values of n , using (2)/(4) is better.

Using PARI-GP on a Sun 3/60 workstation, we timed the last step in using (1) to compute f_n for a few values of n ($t_1(n)$ in Table 2—this is an underestimate for the time to compute $f_n(a, b, 1)$). By comparison, $t_2(n)$ in Table 2 gives the time taken to compute $f_n(a, b, 1)$ from scratch, using (2) or (4) as appropriate. The polynomial $f_{25}(a, b, 1)$ has 8269 terms with coefficients up to 97 decimal digits long. For small n , using (1) beats using (2)/(4), but the latter method soon becomes better.

TABLE 2. Comparing $t_1(n)$, an underestimate of the time taken to compute $f_n(a, b, 1)$ using (1), with $t_2(n)$, the time taken using (2) or (4) as appropriate

| n | $t_1(n)$ | $t_2(n)$ |
|-----|------------|------------|
| 10 | 1s | 6s |
| 15 | 29s | 47s |
| 20 | 2 min 44s | 3 min 5s |
| 23 | 13 min 31s | 9 min 29s |
| 25 | 27 min 23s | 15 min 29s |

4. PROOF OF LEMMA

First suppose n is odd. Fricke, in [2, p. 191], derives a partial differential equation for ψ_n , which for n odd translates directly into a partial differential equation for f_n :

$$(7) \quad (x^3 + ax + b) \frac{\partial^2 f_n}{\partial x^2} - ((n^2 - 3/2)x^2 + (2n^2/3 - 1/2)a) \frac{\partial f_n}{\partial x} - 3n^2 b \frac{\partial f_n}{\partial a} + (2n^2 a^2/3) \frac{\partial f_n}{\partial b} + n^2(n^2 - 1)x f_n/4 = 0.$$

He comments that this provides linear relations between the coefficients of f_n , which together with $\alpha_{0,0}(n) = n$ suffice to determine f_n , but he complains that this "freilich schon bei $n = 5$ einen erheblichen Aufwand von Rechnung erfordert", implying that this is not a profitable approach. Here we disagree. Our aim is to make the solution more explicit. Note that although (7) is derived over \mathbb{C} using complex-variable methods, it is just a formal identity in

$\mathbb{Z}[1/6, a, b][x]$ and as such holds over any field with characteristic not dividing 6.

Equating coefficients of x^{i+1} in (7) gives (2), at least for $i \geq 0$, but since $\beta_t = 0$ for $t < 0$ one soon checks that (2) holds for negative i too.

Set $i = (n^2 - 1)/2 - 2r - 3s$ in (2); then equating coefficients of $a^r b^s$ gives (3).

For n even, replace f_n by $y f_n$ in (7), giving

$$(x^3 + ax + b) \frac{\partial^2 f_n}{\partial x^2} - ((n^2 - 9/2)x^2 + (2n^2/3 - 3/2)a) \frac{\partial f_n}{\partial x} + ((n^2 - 3)(n^2 - 4)x/4)f_n - 3n^2 b \frac{\partial f_n}{\partial a} + (2n^2 a^2/3) \frac{\partial f_n}{\partial b} = 0.$$

Equating coefficients of x^{i+1} gives (4) for $i \geq 0$, but again this extends to all i .

Set $i = (n^2 - 4)/2 - 2r - 3s$ in (4); then equating coefficients of $a^r b^s$ gives (5). \square

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