MATHEMATICS OF COMPUTATION
VOLUME 63, NUMBER 208
OCTOBER 1994, PAGES 747–748

SOME ZEROS OF THE TITCHMARSH COUNTEREXAMPLE

ROBERT SPIRA

ABSTRACT. Zeros on and off the critical line are found for Titchmarsh's function $f(s)$.

Let $s = \sigma + it$. E. C. Titchmarsh [1, pp. 240–244] introduced the function

$$\begin{aligned} f(s) &= \tfrac{1}{2}\sec\theta\{e^{-i\theta}L_1(s) + e^{i\theta}L_2(s)\} \\ &= \frac{1}{1^s} + \frac{\tan\theta}{2^s} - \frac{\tan\theta}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \cdots \\ &= 5^{-s}\{\zeta(s, 1/5) + \tan\theta\zeta(s, 2/5) - \tan\theta\zeta(s, 3/5) - \zeta(s, 4/5)\}, \end{aligned}$$

where

$$\tan\theta = \frac{\sqrt{10 - 2\sqrt{5}} - 2}{\sqrt{5} - 1} = .28407\ 90438\ 40412\ 296\ldots$$

and $L_1(s) = \sum_{n=1}^{\infty} \chi_1(n)n^{-s}$, $L_2(s) = \sum_{n=1}^{\infty} \chi_2(n)n^{-s}$ are Dirichlet L-functions mod 5 with χ_1 and χ_2 the Dirichlet characters determined by $\chi_1(2) = i$ and $\chi_2(2) = -i$.

Titchmarsh showed that though $f(s)$ satisfies a functional equation identical to that of a Dirichlet L-function:

$$f(s) = 5^{1/2-s}2(2\pi)^{s-1}\Gamma(1-s)\cos(\tfrac{1}{2}s\pi)f(1-s),$$

it has zeros with $\sigma > 1$ (together with infinitely many zeros on the line $\sigma = \frac{1}{2}$). According to a theorem of Voronin [2], $f(s)$ has zeros in the critical strip off the critical line. Titchmarsh gave the equation $\sin 2\theta = 2\cos(2\pi/5)$, but $\sin 2\theta$ should be $\tan 2\theta$. This minor error was carried over to Voronin [2] and the review MR 86g:11048 in *Mathematical Reviews.*

With the help of programs for computing L and L' (Spira [3]), an exploratory computation of $f(s)$ in the critical strip for $0 \le t \le 200$ revealed the following zeros off the critical line:

$$\begin{aligned} &.808517 + 85.699348i \\ &.650830 + 114.163343i \\ &.574356 + 166.479306i \\ &.724258 + 176.702461i \end{aligned}$$

Received by the editor January 13, 1993 and, in revised form, December 7, 1993.
1991 *Mathematics Subject Classification.* Primary 11M26.
Key words and phrases. Riemann zeta function.

The first few zeros on the line have t-coordinates: 5.094160, 8.939914, 12.133545, 14.404003, 17.130239, 19.308800, 22.159708, 23.345370, 26.094967, 27.923799, 30.159418, 31.964500, 33.699862, 35.890855, 37.455462, 40.162578, 40.682953, 43.081265, 44.947134, 46.456355, 48.477787, 50.240086.

It was also found that $f(0) = .6568158$, $f(\frac{1}{2}) = .8253830$. The program reproduced check values of an L-function mod 5 and its derivative, and then new values for the character were inserted simply and easily to calculate $f(s)$. A further check was to calculate the zeros off the line reflected in $\sigma = \frac{1}{2}$.

By Rolle's Theorem for a zero $\sigma_0 + it_0$ off the line, there is a σ_1 between σ_0 and $1 - \sigma_0$ such that $|f(\sigma_1 + it_0)|$ is a maximum, or

$$(\operatorname{Re} f \cdot \operatorname{Re} f' + \operatorname{Im} f \cdot \operatorname{Im} f')(\sigma_1 + it_0) = 0.$$

For the first zero at least, $\sigma_1 < \frac{1}{2}$ and $f'(\sigma_1 + it_0) \neq 0$, so the vectors $(\operatorname{Re} f, \operatorname{Im} f)$ and $(\operatorname{Re} f', \operatorname{Im} f')$ are orthogonal. In Spira [4] it was conjectured that for $|t| > 6.3$, $(\operatorname{Re}\zeta \cdot \operatorname{Re}\zeta' + \operatorname{Im}\zeta \cdot \operatorname{Im}\zeta') < 0$ in the left half of the critical strip, which is stronger than the Riemann hypothesis.

No zeros of $f'(s)$ were found with $\sigma < \frac{1}{2}$ for $t \leq 200$, nor any zeros of $f(s)$ with $\sigma > 1$ for $t \leq 200$, though this last is not unusual since Titchmarsh's proof relies on methods which ordinarily require a very large t. If one multiplies $\zeta(s)$ by the four linear factors $(s - \frac{1}{2} \pm \frac{1}{4} \pm i)$ one obtains a function with a functional equation which is not zero for $\sigma \geq 1$, but vanishes off $\sigma = \frac{1}{2}$.

Karatsuba and Voronin [5, Chapter VI, §5, pp. 212–240] is devoted to a study of zeros of $f(s)$ in the critical strip and on $\sigma = \frac{1}{2}$.

Bibliography

1. E. C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford Univ. Press, London, 1951.
2. S. M. Voronin, *On the distribution of zeros of some Dirichlet series*, Proc. Steklov Inst. Math. (1985), no. 4, 89–92.
3. R. Spira, *Calculation of Dirichlet L-functions*, Math. Comp. **23** (1969), 489–498.
4. ——, *On the Riemann zeta function*, J. London Math. Soc. **44** (1969), 325–328.
5. A. A. Karatsuba and S. M. Voronin, *The Riemann zeta-function*, de Gruyter, Berlin, 1992.

392 Taylor, Ashland, Oregon 97520