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SOME ZEROS OF THE TITCHMARSH COUNTEREXAMPLE

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ABSTRACT. Zeros on and off the critical line are found for Titchmarsh's function $f(s)$.

Let $s = \sigma + it$. E. C. Titchmarsh [1, pp. 240–244] introduced the function

$$\begin{aligned} f(s) &= \tfrac{1}{2}\sec\theta\{e^{-i\theta}L_1(s) + e^{i\theta}L_2(s)\} \\ &= \frac{1}{1^s} + \frac{\tan\theta}{2^s} - \frac{\tan\theta}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \cdots \\ &= 5^{-s}\{\zeta(s, 1/5) + \tan\theta\,\zeta(s, 2/5) - \tan\theta\,\zeta(s, 3/5) - \zeta(s, 4/5)\}, \end{aligned}$$

where

$$\tan\theta = \frac{\sqrt{10 - 2\sqrt{5}} - 2}{\sqrt{5} - 1} = .28407\ 90438\ 40412\ 296\ldots$$

and $L_1(s) = \sum_{n=1}^{\infty} \chi_1(n)n^{-s}$, $L_2(s) = \sum_{n=1}^{\infty} \chi_2(n)n^{-s}$ are Dirichlet L-functions mod 5 with χ_1 and χ_2 the Dirichlet characters determined by $\chi_1(2) = i$ and $\chi_2(2) = -i$.

Titchmarsh showed that though $f(s)$ satisfies a functional equation identical to that of a Dirichlet L-function:

$$f(s) = 5^{1/2-s}2(2\pi)^{s-1}\Gamma(1-s)\cos(\tfrac{1}{2}s\pi)f(1-s),$$

it has zeros with $\sigma > 1$ (together with infinitely many zeros on the line $\sigma = \frac{1}{2}$). According to a theorem of Voronin [2], $f(s)$ has zeros in the critical strip off the critical line. Titchmarsh gave the equation $\sin 2\theta = 2\cos(2\pi/5)$, but $\sin 2\theta$ should be $\tan 2\theta$. This minor error was carried over to Voronin [2] and the review MR 86g:11048 in *Mathematical Reviews.*

With the help of programs for computing L and L' (Spira [3]), an exploratory computation of $f(s)$ in the critical strip for $0 \le t \le 200$ revealed the following zeros off the critical line:

$$\begin{aligned} &.808517 + 85.699348i \\ &.650830 + 114.163343i \\ &.574356 + 166.479306i \\ &.724258 + 176.702461i \end{aligned}$$

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The first few zeros on the line have t-coordinates: 5.094160, 8.939914, 12.133545, 14.404003, 17.130239, 19.308800, 22.159708, 23.345370, 26.094967, 27.923799, 30.159418, 31.964500, 33.699862, 35.890855, 37.455462, 40.162578, 40.682953, 43.081265, 44.947134, 46.456355, 48.477787, 50.240086.

It was also found that $f(0) = .6568158$, $f(\frac{1}{2}) = .8253830$. The program reproduced check values of an L-function mod 5 and its derivative, and then new values for the character were inserted simply and easily to calculate $f(s)$. A further check was to calculate the zeros off the line reflected in $\sigma = \frac{1}{2}$.

By Rolle's Theorem for a zero $\sigma_0 + it_0$ off the line, there is a σ_1 between σ_0 and $1 - \sigma_0$ such that $|f(\sigma_1 + it_0)|$ is a maximum, or

$$(\operatorname{Re} f \cdot \operatorname{Re} f' + \operatorname{Im} f \cdot \operatorname{Im} f')(\sigma_1 + it_0) = 0.$$

For the first zero at least, $\sigma_1 < \frac{1}{2}$ and $f'(\sigma_1 + it_0) \neq 0$, so the vectors $(\operatorname{Re} f, \operatorname{Im} f)$ and $(\operatorname{Re} f', \operatorname{Im} f')$ are orthogonal. In Spira [4] it was conjectured that for $|t| > 6.3$, $(\operatorname{Re}\zeta \cdot \operatorname{Re}\zeta' + \operatorname{Im}\zeta \cdot \operatorname{Im}\zeta') < 0$ in the left half of the critical strip, which is stronger than the Riemann hypothesis.

No zeros of $f'(s)$ were found with $\sigma < \frac{1}{2}$ for $t \leq 200$, nor any zeros of $f(s)$ with $\sigma > 1$ for $t \leq 200$, though this last is not unusual since Titchmarsh's proof relies on methods which ordinarily require a very large t. If one multiplies $\zeta(s)$ by the four linear factors $(s - \frac{1}{2} \pm \frac{1}{4} \pm i)$ one obtains a function with a functional equation which is not zero for $\sigma \geq 1$, but vanishes off $\sigma = \frac{1}{2}$.

Karatsuba and Voronin [5, Chapter VI, §5, pp. 212–240] is devoted to a study of zeros of $f(s)$ in the critical strip and on $\sigma = \frac{1}{2}$.

Bibliography

1. E. C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford Univ. Press, London, 1951.
2. S. M. Voronin, *On the distribution of zeros of some Dirichlet series*, Proc. Steklov Inst. Math. (1985), no. 4, 89–92.
3. R. Spira, *Calculation of Dirichlet L-functions*, Math. Comp. **23** (1969), 489–498.
4. ——, *On the Riemann zeta function*, J. London Math. Soc. **44** (1969), 325–328.
5. A. A. Karatsuba and S. M. Voronin, *The Riemann zeta-function*, de Gruyter, Berlin, 1992.

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