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A NONCONFORMING MIXED MULTIGRID METHOD FOR THE PURE TRACTION PROBLEM IN PLANAR LINEAR ELASTICITY

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ABSTRACT. A robust optimal-order multigrid method for the pure traction problem in two-dimensional linear elasticity is studied. The finite element discretization is a variant of a mixed method proposed by Falk where the displacement is approximated by nonconforming piecewise linear functions and the "pressure" is approximated by piecewise constant functions on a coarser grid. Full multigrid convergence is obtained. The performance of this multigrid algorithm does not deteriorate as the material becomes nearly incompressible.

1. INTRODUCTION

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$. The pure traction boundary value problem for planar linear elasticity is given by

$$
(1.1)\qquad
\begin{aligned}
-\underset{\sim}{\mathrm{div}}\{2\,\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u})+\lambda\,\mathrm{tr}\,(\underset{\approx}{\epsilon}(\underset{\sim}{u}))\,\underset{\approx}{\delta}\} &= \underset{\sim}{f} \quad \text{in } \Omega,\\
(2\,\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u})+\lambda\,\mathrm{tr}\,(\underset{\approx}{\epsilon}(\underset{\sim}{u}))\,\underset{\approx}{\delta})\,\underset{\sim}{\nu} &= \underset{\sim}{g} \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $\underset{\sim}{u}$ is the displacement, $\underset{\sim}{f}$ is the body force, $\underset{\sim}{g}$ is the boundary traction, $\mu>0$, $\lambda>0$ are the Lamé constants, and $\underset{\sim}{\nu}$ is the unit outer normal. Throughout this paper, an undertilde is used to denote vector-valued functions, operators and their associated spaces. Double undertildes are used for matrix-valued functions and operators. We define

$$
\underset{\sim}{\mathrm{grad}}\,p=\begin{pmatrix}\partial p/\partial x_1\\ \partial p/\partial x_2\end{pmatrix},\quad \underset{\sim}{\mathrm{div}}\,\underset{\approx}{\tau}=\begin{pmatrix}\partial\tau_{11}/\partial x_1+\partial\tau_{12}/\partial x_2\\ \partial\tau_{21}/\partial x_1+\partial\tau_{22}/\partial x_2\end{pmatrix},
$$
$$
\underset{\sim}{\mathrm{curl}}\,p=\begin{pmatrix}\partial p/\partial x_2\\ -\partial p/\partial x_1\end{pmatrix},
$$

$$
\mathrm{div}\,\underset{\sim}{v}=\partial v_1/\partial x_1+\partial v_2/\partial x_2,\qquad \mathrm{rot}\,\underset{\sim}{v}=-\partial v_1/\partial x_2+\partial v_2/\partial x_1,
$$

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and

$$\underset{\approx}{\text{grad}}\ \underset{\sim}{v} = \begin{pmatrix} \partial v_1/\partial x_1 & \partial v_1/\partial x_2 \\ \partial v_2/\partial x_1 & \partial v_2/\partial x_2 \end{pmatrix}.$$

We also define

$$\underset{\approx}{\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underset{\approx}{\chi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$\text{tr}(\underset{\approx}{\tau}) = \underset{\approx}{\tau} : \underset{\approx}{\delta},$$

where

$$\underset{\approx}{\sigma} : \underset{\approx}{\tau} = \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}\tau_{ij}.$$

Finally,

$$\underset{\approx}{\epsilon}(\underset{\sim}{v}) = \frac{1}{2}[\underset{\approx}{\text{grad}}\ \underset{\sim}{v} + (\underset{\approx}{\text{grad}}\ \underset{\sim}{v})^t].$$

We assume that the Lamé constants (μ, λ) belong to the range $[\mu_1, \mu_2] \times [\lambda_1, \infty)$, where μ_1, μ_2, λ_1 are three fixed positive constants.

We shall denote by $\underset{\sim}{H}^k(\Omega)$ the standard L^2 Sobolev spaces of vector-valued functions, and we use the following conventions for Sobolev norms and semi-norms:

$$\|\underset{\sim}{v}\|_{\underset{\sim}{H}^m(\Omega)} := \Big(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha \underset{\sim}{v}|^2 dx\Big)^{1/2} \quad \text{and} \tag{1.2}$$

$$|\underset{\sim}{v}|_{\underset{\sim}{H}^m(\Omega)} := \Big(\int_\Omega \sum_{|\alpha| = m} |\partial^\alpha \underset{\sim}{v}|^2 dx\Big)^{1/2}. \tag{1.3}$$

Let $\underset{\sim}{RM} = \{\underset{\sim}{v} : \underset{\sim}{v}^t = (a + bx_2, c - bx_1), a, b, c \in \mathbb{R}\}$ be the space of infinitesimal rigid motions. Since $\underset{\approx}{\epsilon}(\underset{\sim}{v}) = \underset{\approx}{0}$ for all $\underset{\sim}{v} \in \underset{\sim}{RM}$, one should look for a unique solution of (1.1) in a subspace of $\underset{\sim}{H}^k(\Omega)$ which is transversal to $\underset{\sim}{RM}$. The unique solvability and regularity theorems are usually stated on the space

$$\underset{\sim}{\hat{H}}^k(\Omega) = \Big\{\underset{\sim}{v} \in \underset{\sim}{H}^k(\Omega) : \int_\Omega \underset{\sim}{v}\, dx = \underset{\sim}{0}, \int_\Omega \text{rot}\ \underset{\sim}{v}\, dx = 0\Big\}. \tag{1.4}$$

However, for the design and analysis of a multigrid method, it is better to use the space

$$\underset{\sim}{H}^k_\perp(\Omega) = \Big\{\underset{\sim}{v} \in \underset{\sim}{H}^k(\Omega) : \int_\Omega \underset{\sim}{v} \cdot \underset{\sim}{w}\, dx = 0 \quad \forall \underset{\sim}{w} \in \underset{\sim}{RM}\Big\}. \tag{1.5}$$

We also use the notation $\underset{\sim}{L}^2_\perp$ for $\underset{\sim}{H}^0_\perp$. We assume that the origin of our coordinate system is chosen to be the centroid of Ω, so that $\underset{\sim}{RM}$ has the following orthonormal basis:

$$\underset{\sim}{\psi}_1 = \frac{1}{|\Omega|^{1/2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underset{\sim}{\psi}_2 = \frac{1}{|\Omega|^{1/2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \underset{\sim}{\psi}_3 = \frac{1}{\omega}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{1.6}$$

where $\omega = \sqrt{\int_\Omega (x_1^2 + x_2^2)\, dx}$.

Let the operators $T_{\perp}^{\wedge}$ and $T_{\wedge}^{\perp}$ be defined by

$$T_{\perp}^{\wedge}(\underset{\sim}{v}) = \underset{\sim}{v} - \frac{\omega}{2|\Omega|}\left(\int_{\Omega} \operatorname{rot} \underset{\sim}{v}\, dx\right) \underset{\sim}{\psi}_3 \tag{1.7}$$

and

$$T_{\wedge}^{\perp}(\underset{\sim}{v}) = \underset{\sim}{v} - \left(\int_{\Omega} \underset{\sim}{v} \cdot \underset{\sim}{\psi}_3 dx\right) \underset{\sim}{\psi}_3. \tag{1.8}$$

Then clearly for $k \geq 1$,

$$T_{\perp}^{\wedge} : \underset{\sim}{H}_{\perp}^{k}(\Omega) \longrightarrow \hat{\underset{\sim}{H}}^{k}(\Omega) \quad \text{and} \quad T_{\wedge}^{\perp} : \hat{\underset{\sim}{H}}^{k}(\Omega) \longrightarrow \underset{\sim}{H}_{\perp}^{k}(\Omega), \tag{1.9}$$

$$\begin{aligned} &T_{\wedge}^{\perp}(T_{\perp}^{\wedge}\underset{\sim}{v}) = \underset{\sim}{v} \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_{\perp}^{k}(\Omega) \quad \text{and} \\ &T_{\perp}^{\wedge}(T_{\wedge}^{\perp}\underset{\sim}{v}) = \underset{\sim}{v} \quad \forall \underset{\sim}{v} \in \hat{\underset{\sim}{H}}^{k}(\Omega), \end{aligned} \tag{1.10}$$

$$\underset{\approx}{\epsilon}(\underset{\sim}{v}) = \underset{\approx}{\epsilon}\left(T_{\perp}^{\wedge}(\underset{\sim}{v})\right) \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_{\perp}^{k}(\Omega), \tag{1.11}$$

and

$$\operatorname{div}(\underset{\sim}{v}) = \operatorname{div}\left(T_{\perp}^{\wedge}(\underset{\sim}{v})\right) \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_{\perp}^{k}(\Omega). \tag{1.12}$$

There exist positive constants C_1 and C_2 such that ($k \geq 1$)

$$C_1\|T_{\perp}^{\wedge}(\underset{\sim}{v})\|_{\underset{\sim}{H}^k(\Omega)} \leq \|\underset{\sim}{v}\|_{\underset{\sim}{H}^k(\Omega)} \leq C_2\|T_{\perp}^{\wedge}(\underset{\sim}{v})\|_{\underset{\sim}{H}^k(\Omega)} \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_{\perp}^{k}(\Omega). \tag{1.13}$$

By Friedrichs' inequality (cf. [9, Theorem 1.1.5]), (1.13) also holds for $|\cdot|_{\underset{\sim}{H}^{\ell}(\Omega)}$ ($1 \leq \ell \leq k$) norms:

$$C_1|T_{\perp}^{\wedge}(\underset{\sim}{v})|_{\underset{\sim}{H}^{\ell}(\Omega)} \leq |\underset{\sim}{v}|_{\underset{\sim}{H}^{\ell}(\Omega)} \leq C_2|T_{\perp}^{\wedge}(\underset{\sim}{v})|_{\underset{\sim}{H}^{\ell}(\Omega)} \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_{\perp}^{k}(\Omega). \tag{1.14}$$

Using these operators, one can translate results for the $\hat{\underset{\sim}{H}}^{k}(\Omega)$-space to the $\underset{\sim}{H}_{\perp}^{k}(\Omega)$-space and vice versa.

Since the boundary of a polygon has corners, the boundary conditions in (1.1) must be carefully interpreted. We shall denote by S_i, $1 \leq i \leq n$, the vertices of Ω; by Γ_i, $1 \leq i \leq n$, the open line segments joining S_i to S_{i+1}; by $\underset{\sim}{\tau}_i$ the positively oriented unit tangent along Γ_i; and by $\underset{\sim}{\nu}_i$ the unit outer normal along Γ_i. Henceforth, S_{n+1} and Γ_{n+1} should be interpreted as S_1 and Γ_1, respectively.

Let $p \in H^{1/2}(\Gamma_i)$ and $q \in H^{1/2}(\Gamma_{i+1})$. We say that $p \equiv q$ at S_{i+1} if $\int_0^{\delta} |q(s) - p(-s)|^2 \frac{ds}{s} < \infty$, where s is the oriented arc length measured from S_{i+1}, and δ is a positive number less than $\min\{|\Gamma_i| : 1 \leq i \leq n\}$.

Equation (1.1) can be written more precisely as

$$\begin{aligned} &-\underset{\sim}{\operatorname{div}}\{2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) + \lambda \operatorname{tr}(\underset{\approx}{\epsilon}(\underset{\sim}{u}))\underset{\approx}{\delta}\} = \underset{\sim}{f} \quad \text{in } \Omega, \\ &(2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) + \lambda \operatorname{tr}(\underset{\approx}{\epsilon}\underset{\sim}{u}))\underset{\approx}{\delta})\underset{\sim}{\nu}_i|_{\Gamma_i} = \underset{\sim}{g}_i, \quad 1 \leq i \leq n, \end{aligned} \tag{1.15}$$

where $\underset{\sim}{f} \in \underset{\sim}{L}^2(\Omega)$, and $\underset{\sim}{g}_i \in \underset{\sim}{H}^{1/2}(\Gamma_i)$ satisfy

$$\underset{\sim}{g}_i \cdot \underset{\sim}{\nu}_{i+1} \equiv \underset{\sim}{g}_{i+1} \cdot \underset{\sim}{\nu}_i \quad \text{at } S_{i+1} \text{ for } 1 \leq i \leq n. \tag{1.16}$$

If we assume that the following compatibility condition holds,

$$\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \sum_{i=1}^{n} \int_{\Gamma_i} \underset{\sim}{g}_i \cdot \underset{\sim}{v}|_{\Gamma_i}\, ds = 0 \quad \forall \underset{\sim}{v} \in \underset{\sim}{RM}\,, \tag{1.17}$$

then the pure traction boundary value problem (1.15) has a unique solution $\hat{\underset{\sim}{u}} \in \hat{\underset{\sim}{H}}^2(\Omega)$. Moreover, there exists a positive constant C_Ω such that

$$\|\hat{\underset{\sim}{u}}\|_{\underset{\sim}{H}^2(\Omega)} + \lambda \|\operatorname{div} \hat{\underset{\sim}{u}}\|_{H^1(\Omega)} \le C_\Omega \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\} \tag{1.18}$$

(cf. [3, 7, 8]).

By using (1.10) and (1.11), we see that (1.15) also has a unique solution $\underset{\sim}{u}_\perp \in \underset{\sim}{H}^2_\perp(\Omega)$ and

$$\hat{\underset{\sim}{u}} = T_\perp^\wedge \underset{\sim}{u}_\perp. \tag{1.19}$$

Therefore, (1.12), (1.13), and (1.18) imply that there exists a positive constant C'_Ω such that

$$\|\underset{\sim}{u}_\perp\|_{\underset{\sim}{H}^2(\Omega)} + \lambda \|\operatorname{div} \underset{\sim}{u}_\perp\|_{H^1(\Omega)} \le C'_\Omega \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\}. \tag{1.20}$$

From now on, we will simply denote $\underset{\sim}{u}_\perp$ by $\underset{\sim}{u}$.

Henceforth, (1.16) and (1.17) are assumed to be true. Let $\gamma = \lambda/(2\mu)$ and $p = \gamma \operatorname{div} \underset{\sim}{u}$; then a mixed formulation for (1.15) is:

Find $(\underset{\sim}{u}, p) \in \underset{\sim}{H}^1_\perp(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\epsilon}(\underset{\sim}{v})\, dx + \int_\Omega p(\operatorname{div} \underset{\sim}{v})\, dx \\ &\qquad = \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \sum_{i=1}^{n} \int_{\Gamma_i} \underset{\sim}{g}_i \cdot \underset{\sim}{v}|_{\Gamma_i}\, ds\right], \\ &\int_\Omega (\operatorname{div} \underset{\sim}{u})\, q\, dx - \frac{1}{\gamma}\int_\Omega pq\, dx = 0 \end{aligned} \tag{1.21}$$

for all $(\underset{\sim}{v}, q) \in \underset{\sim}{H}^1_\perp(\Omega) \times L^2(\Omega)$.

Equation (1.21) can be written concisely as

$$\mathscr{B}\Big((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\Big) = \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \sum_{i=1}^{n} \int_{\Gamma_i} \underset{\sim}{g}_i \cdot \underset{\sim}{v}|_{\Gamma_i}\, ds\right] \tag{1.22}$$

for all $(\underset{\sim}{v}, q) \in \underset{\sim}{H}^1_\perp(\Omega) \times L^2(\Omega)$, where the symmetric bilinear form $\mathscr{B}(\cdot, \cdot)$ on $\underset{\sim}{H}^1(\Omega) \times L^2(\Omega)$ is defined by

$$\begin{aligned} &\mathscr{B}((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)) \\ &\quad = \int_\Omega \left\{ \underset{\approx}{\epsilon}(\underset{\sim}{v}_1) : \underset{\approx}{\epsilon}(\underset{\sim}{v}_2) + q_1(\operatorname{div} \underset{\sim}{v}_2) + (\operatorname{div} \underset{\sim}{v}_1)\, q_2 - \frac{1}{\gamma} q_1 q_2 \right\} dx. \end{aligned}$$

It is clear from the definition of $\mathscr{B}$ that

$$\left|\mathscr{B}\left((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\right)\right| \le \max\left(\sqrt{2}, \frac{2\mu_2}{\lambda_1}\right)\left(|\underset{\sim}{v}_1|_{\underset{\sim}{H}^1(\Omega)} + \|q_1\|_{L^2(\Omega)}\right) \cdot \left(|\underset{\sim}{v}_2|_{\underset{\sim}{H}^1(\Omega)} + \|q_2\|_{L^2(\Omega)}\right). \tag{1.23}$$

Let $\mathscr{T}_k$ ($k \ge 0$) be a sequence of triangulations of Ω, where $\mathscr{T}_{k+1}$ is obtained by connecting the midpoints of each of the triangles in $\mathscr{T}_k$. We will denote $\max\{\operatorname{diam} T: T \in \mathscr{T}_k\}$ by h_k. For $k \ge 0$, let $Q_k = \{q: q \in L^2(\Omega)$ and $q|_T$ is a constant for all $T \in \mathscr{T}_k\}$. The nonconforming finite element spaces $\underset{\sim}{V}_k$ ($k \ge 1$) are defined as follows:

$$\underset{\sim}{V}_k = \{\underset{\sim}{v} : \underset{\sim}{v} \in \underset{\sim}{L}^2(\Omega), \underset{\sim}{v}|_T \text{ is linear for all } T \in \mathscr{T}_k, \underset{\sim}{v} \text{ is continuous at the midpoints of interelement boundaries}\}. \tag{1.24}$$

Since $\underset{\sim}{V}_k$ is nonconforming, any differentiation of members of $\underset{\sim}{V}_k$ must be done piecewise. We define the operators div_k, rot_k, and $\underset{\approx}{\operatorname{grad}}_k$ as follows:

$$\left(\operatorname{div}_k \underset{\sim}{v}\right)\Big|_T = \operatorname{div}\left(\underset{\sim}{v}|_T\right),$$
$$\left(\operatorname{rot}_k \underset{\sim}{v}\right)\Big|_T = \operatorname{rot}\left(\underset{\sim}{v}|_T\right),$$
$$\left(\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{v}\right)\Big|_T = \underset{\approx}{\operatorname{grad}}\left(\underset{\sim}{v}|_T\right),$$

for all $T \in \mathscr{T}_k$. We also denote by $\underset{\sim}{V}_k^{\perp}$ the subspaces of $\underset{\sim}{V}_k$ whose members satisfy

$$\int_\Omega \underset{\sim}{v} \cdot \underset{\sim}{w}\, dx = 0 \quad \forall\, \underset{\sim}{w} \in \underset{\sim}{RM}. \tag{1.25}$$

The following discretization of (1.22) is a modification of one introduced by Falk in [6].

Find $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$ ($k \ge 1$) such that

$$\mathscr{B}_k\left((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}, q)\right) = \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \sum_{i=1}^n \int_{\Gamma_i} \underset{\sim}{g}_i \cdot \underset{\sim}{v}|_{\Gamma_i} ds\right] \tag{1.26}$$

for all $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$. Here the symmetric bilinear form $\mathscr{B}_k$ on $\left(\underset{\sim}{H}^1(\Omega) + \underset{\sim}{V}_k\right) \times L^2(\Omega)$ is defined by

$$\mathscr{B}_k\left((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\right) = \int_\Omega \left\{\underset{\approx}{\epsilon}_k^*(\underset{\sim}{v}_1) : \underset{\approx}{\epsilon}_k^*(\underset{\sim}{v}_2) + q_1(\operatorname{div}_k \underset{\sim}{v}_2) + (\operatorname{div}_k \underset{\sim}{v}_1) q_2 - \frac{1}{\gamma} q_1 q_2\right\} dx, \tag{1.27}$$

where

$$\underset{\approx}{\epsilon}_k^*(\underset{\sim}{v}) = \underset{\approx}{\operatorname{grad}}_k \underset{\sim}{v} - \frac{1}{2}\left(P_{k-1} \operatorname{rot}_k \underset{\sim}{v}\right) \underset{\approx}{\chi}, \tag{1.28}$$

and P_k ($k \geq 0$) is the L^2 orthogonal projection onto Q_k. Note that there exists a positive constant C such that

$$(1.29)\qquad \begin{aligned} &\left|\mathscr{B}_k\left((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\right)\right| \\ &\quad \leq C\left(\|\underset{\sim}{v}_1\|_k + \|q_1\|_{L^2(\Omega)}\right)\left(\|\underset{\sim}{v}_2\|_k + \|q_2\|_{L^2(\Omega)}\right), \end{aligned}$$

where the nonconforming energy norm $\|\cdot\|_k$ on $\underset{\sim}{H}^1_\perp(\Omega) + \underset{\sim}{V}^\perp_k$ is defined by

$$(1.30)\qquad \|\underset{\sim}{v}\|_k := \|\underset{\approx}{\operatorname{grad}}_k\, \underset{\sim}{v}\|_{\underset{\approx}{L}^2(\Omega)}.$$

Henceforth, C (with or without subscripts) denotes a generic positive constant independent of the Lamé constants and the mesh parameter k.

We shall show in §2 that (1.26) is uniquely solvable and derive the following discretization error estimate:

$$(1.31)\qquad \begin{aligned} &\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_{\underset{\sim}{L}^2(\Omega)} + h_k\left(\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_k + \|p - p_k\|_{L^2(\Omega)}\right) \\ &\quad \leq C h_k^2\left\{\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right\}. \end{aligned}$$

In this paper we will develop an optimal-order multigrid method for solving (1.26). Let n_k be the dimension of $\underset{\sim}{V}^\perp_k \times Q_{k-1}$. Our full multigrid algorithm will yield an approximate solution $(\underset{\sim}{u}^*_k, p^*_k)$ to (1.26) in $\mathscr{O}(n_k)$ steps such that

$$(1.32)\qquad \begin{aligned} &\|\underset{\sim}{u}_k - \underset{\sim}{u}^*_k\|_{\underset{\sim}{L}^2(\Omega)} + h_k\left(\|\underset{\sim}{u}_k - \underset{\sim}{u}^*_k\|_k + \|p_k - p^*_k\|_{L^2(\Omega)}\right) \\ &\quad \leq C h_k^2\left\{\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right\}. \end{aligned}$$

Since the constant C in (1.32) does not depend on the Lamé constants, the performance of our multigrid method will not deteriorate as the material becomes nearly incompressible. As documented in [11], for nearly incompressible linear elasticity problems, the standard multigrid method using conforming bilinear finite elements requires an extremely large number of smoothing steps in order to achieve convergence. Our algorithm converges with a small number of smoothing steps, independent of the Poisson ratio.

The rest of this paper is organized as follows. We establish the discretization error estimate (1.31) in §2. Since the finite element space $\underset{\sim}{V}_k$ is nonconforming, appropriate intergrid transfer operators must be chosen. This is done in §3, where the mesh-dependent norms are also defined. The estimates for the intergrid transfer operators established in this section are crucial to the convergence analysis of the multigrid method. In §4 we define the multigrid algorithm. The convergence results are stated in §5. The details of the proofs are found in the supplement to this paper. Results of some numerical experiments are reported in §6.

2. The mixed method

The first ingredient in establishing directly the unique solvability of (1.22) is the following well-known Korn's second inequality (cf. [10]):

There exists a positive constant C such that

$$\|\underset{\approx}{\epsilon}(\underset{\sim}{v})\|_{\underset{\approx}{L}^2(\Omega)} \geq C|\underset{\sim}{v}|_{\underset{\sim}{H}^1(\Omega)} \quad \forall \underset{\sim}{v} \in \underset{\sim}{\hat{H}}^1(\Omega). \tag{2.1}$$

Using (1.11) and (1.14), we deduce that there exists a positive constant C such that

$$\|\underset{\approx}{\epsilon}(\underset{\sim}{v})\|_{\underset{\approx}{L}^2(\Omega)} \geq C|\underset{\sim}{v}|_{\underset{\sim}{H}^1(\Omega)} \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}^1_{\perp}(\Omega). \tag{2.2}$$

The second ingredient is the following property of the divergence operator.

Lemma 2.1. *Given any $q \in L^2(\Omega)$, there exists $\underset{\sim}{v} \in \underset{\sim}{H}^1_{\perp}$ such that*

$$\operatorname{div} \underset{\sim}{v} = q \tag{2.3}$$

and

$$\|\underset{\sim}{v}\|_{\underset{\sim}{H}^1(\Omega)} \leq C_\Omega \|q\|_{L^2(\Omega)}. \tag{2.4}$$

Proof. Let D be an open disc that contains $\overline{\Omega}$. Extend q to be zero on $D\backslash\Omega$. Let $\zeta \in H^2(\Omega)$ be the solution of

$$\Delta\zeta = q \quad \text{in } D, \qquad \zeta = 0 \quad \text{on } \partial D. \tag{2.5}$$

Then from elliptic regularity (cf. [9, Theorem 4.2.1]) we have

$$\|\zeta\|_{H^2(D)} \leq C_\Omega \|q\|_{L^2(D)}. \tag{2.6}$$

Let (cf. (1.6))

$$\underset{\sim}{v} = \underset{\sim}{\operatorname{grad}} \zeta|_\Omega - \sum_{i=1}^{3} \left(\int_\Omega \underset{\sim}{\operatorname{grad}} \zeta \cdot \underset{\sim}{\psi}_i \, dx \right) \underset{\sim}{\psi}_i. \tag{2.7}$$

It is clear that $\underset{\sim}{v} \in \underset{\sim}{H}^1_{\perp}(\Omega)$. The lemma now follows from (2.5) and (2.6). □

Proposition 2.2. *There exists a positive constant C, which depends only on Ω, such that for any $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{H}^1_{\perp} \times L^2(\Omega)$,*

$$\sup_{(\underset{\sim}{v}_2, q_2) \in \underset{\sim}{H}^1_{\perp} \times L^2(\Omega)\backslash\{(\underset{\sim}{0}, 0)\}} \frac{\left|\mathscr{B}((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2))\right|}{|\underset{\sim}{v}_2|_{\underset{\sim}{H}^1(\Omega)} + \|q_2\|_{L^2(\Omega)}} \geq C\left(|\underset{\sim}{v}_1|_{\underset{\sim}{H}^1(\Omega)} + \|q_1\|_{L^2(\Omega)}\right). \tag{2.8}$$

Proof. Given $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{H}^1_{\perp} \times L^2(\Omega)$, let

$$s = \sup_{(\underset{\sim}{v}_2, q_2) \in \underset{\sim}{H}^1_{\perp} \times L^2(\Omega)\backslash\{(\underset{\sim}{0}, 0)\}} \frac{\left|\mathscr{B}((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2))\right|}{|\underset{\sim}{v}_2|_{\underset{\sim}{H}^1(\Omega)} + \|q_2\|_{L^2(\Omega)}}.$$

First we consider the case where $q_1 = \gamma \operatorname{div} \underset{\sim}{v}_1$. By Korn's second inequality,

$$\begin{aligned}\mathscr{B}\left((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_1, 0)\right) &= \int_\Omega \left\{\underset{\approx}{\epsilon}(\underset{\sim}{v}_1) : \underset{\approx}{\epsilon}(\underset{\sim}{v}_1) + \gamma(\operatorname{div}\underset{\sim}{v}_1)^2\right\} dx \\ &\geq \int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{v}_1) : \underset{\approx}{\epsilon}(\underset{\sim}{v}_1)\, dx \geq C|\underset{\sim}{v}_1|^2_{\underset{\sim}{H}^1(\Omega)}.\end{aligned}$$

Therefore,

$$s \geq C \,|\underset{\sim}{v}_1|_{\underset{\sim}{H}^1(\Omega)}. \tag{2.9}$$

By Lemma 2.1, there exists $\underset{\sim}{v} \in \underset{\sim}{H}^1_\perp$ such that

$$\operatorname{div} \underset{\sim}{v} = q_1 \quad \text{and} \quad \|\underset{\sim}{v}\|_{\underset{\sim}{H}^1(\Omega)} \leq C\,\|q_1\|_{L^2(\Omega)}. \tag{2.10}$$

Using the definition of s, (2.9), and (2.10), we have

$$\begin{aligned}\|q_1\|^2_{L^2(\Omega)} &\leq \left|\mathscr{B}((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}, 0))\right| + \left|\int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{v}_1) : \underset{\approx}{\epsilon}(\underset{\sim}{v})\,dx\right| \\ &\leq s|\underset{\sim}{v}|_{\underset{\sim}{H}^1(\Omega)} + |\underset{\sim}{v}_1|_{\underset{\sim}{H}^1(\Omega)}\,|\underset{\sim}{v}|_{\underset{\sim}{H}^1(\Omega)} \leq Cs\,\|q_1\|_{L^2(\Omega)}.\end{aligned}$$

Therefore,

$$\|q_1\|_{L^2(\Omega)} \leq Cs. \tag{2.11}$$

Combining (2.9) and (2.11), we obtain (2.8) in the special case where $q_1 = \gamma \operatorname{div} \underset{\sim}{v}_1$.

We now turn to the general case. Given $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{H}^1_\perp \times L^2(\Omega)$, by Lemma 2.1 there exists $\underset{\sim}{w} \in \underset{\sim}{H}^1_\perp$ such that

$$\operatorname{div} \underset{\sim}{w} = \frac{1}{\gamma} q_1 - \operatorname{div} \underset{\sim}{v}_1 \quad \text{and} \quad \|\underset{\sim}{w}\|_{\underset{\sim}{H}^1(\Omega)} \leq C \left\|\frac{1}{\gamma} q_1 - \operatorname{div} \underset{\sim}{v}_1\right\|_{L^2(\Omega)}. \tag{2.12}$$

Then $\gamma \operatorname{div}(\underset{\sim}{v}_1 + \underset{\sim}{w}) = q_1$. From the special case, we know that

$$\begin{aligned}\sup_{(\underset{\sim}{v}_2, q_2) \in \underset{\sim}{H}^1_\perp \times L^2(\Omega) \setminus \{(\underset{\sim}{0}, 0)\}} &\frac{\left|\mathscr{B}((\underset{\sim}{v}_1 + \underset{\sim}{w}, q_1), (\underset{\sim}{v}_2, q_2))\right|}{|\underset{\sim}{v}_2|_{\underset{\sim}{H}^1(\Omega)} + \|q_2\|_{L^2(\Omega)}} \\ &\geq C\left(|\underset{\sim}{v}_1 + \underset{\sim}{w}|_{\underset{\sim}{H}^1(\Omega)} + \|q_1\|_{L^2(\Omega)}\right).\end{aligned} \tag{2.13}$$

Therefore, by (2.13), the bilinearity of $\mathscr{B}$ and (1.23), we have

$$\begin{aligned}|\underset{\sim}{v}_1|_{\underset{\sim}{H}^1(\Omega)} + \|q_1\|_{L^2(\Omega)} &\leq \left[|\underset{\sim}{v}_1 + \underset{\sim}{w}|_{\underset{\sim}{H}^1(\Omega)} + \|q_1\|_{L^2(\Omega)}\right] + |\underset{\sim}{w}|_{\underset{\sim}{H}^1(\Omega)} \\ &\leq s + C\,|\underset{\sim}{w}|_{\underset{\sim}{H}^1(\Omega)}.\end{aligned} \tag{2.14}$$

On the other hand, by the definition of $\mathscr{B}$,

$$\begin{aligned}\left\|\frac{1}{\gamma} q_1 - \operatorname{div} \underset{\sim}{v}_1\right\|^2_{L^2(\Omega)} &= \mathscr{B}\left((\underset{\sim}{v}_1, q_1), \left(\underset{\sim}{0}, \operatorname{div} \underset{\sim}{v}_1 - \frac{1}{\gamma} q_1\right)\right) \\ &\leq s \left\|\frac{1}{\gamma} q_1 - \operatorname{div} \underset{\sim}{v}_1\right\|_{L^2(\Omega)}.\end{aligned} \tag{2.15}$$

Combining (2.12) and (2.15), we see that

$$\|\underset{\sim}{w}\|_{\underset{\sim}{H}^1(\Omega)} \leq Cs. \tag{2.16}$$

The proposition now follows from (2.14) and (2.16). □

The following corollary is a consequence of Proposition 2.2 and Friedrichs' inequality.

Corollary 2.3. *Given any bounded linear functional* F *on* $\underset{\sim}{H}^1_\perp \times L^2(\Omega)$, *there is a unique* $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{H}^1_\perp \times L^2(\Omega)$ *such that*

$$(2.17)\quad \mathscr{B}\Big((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\Big) = F\Big((\underset{\sim}{v}_2, q_2)\Big) \quad \forall (\underset{\sim}{v}_2, q_2) \in \underset{\sim}{H}^1_\perp \times L^2(\Omega).$$

In particular, equation (1.22) *is uniquely solvable.*

Similarly, there are two ingredients in establishing the unique solvability of the discretized equation (1.26), namely a discrete Korn's second inequality and an analog of Lemma 2.1.

We begin with the discrete Korn's second inequality. Let $\hat{\underset{\sim}{V}}_k = \{\underset{\sim}{v} \in \underset{\sim}{V}_k : \int_\Omega \underset{\sim}{v}\, dx = \underset{\sim}{0}$ and $\int_\Omega \mathrm{rot}_k \underset{\sim}{v}\, dx = 0\}$. Analogous to (1.7) and (1.8), define $T^\wedge_\perp : \underset{\sim}{V}^\perp_k \longrightarrow \hat{\underset{\sim}{V}}_k$ and $T^\perp_\wedge : \hat{\underset{\sim}{V}}_k \longrightarrow \underset{\sim}{V}^\perp_k$ by

$$(2.18)\qquad T^\wedge_\perp(\underset{\sim}{v}) = \underset{\sim}{v} - \frac{\omega}{2|\Omega|}\left(\int_\Omega \mathrm{rot}_k \underset{\sim}{v}\, dx\right) \underset{\sim}{\psi}_3$$

and

$$(2.19)\qquad T^\perp_\wedge(\underset{\sim}{v}) = \underset{\sim}{v} - \left(\int_\Omega \underset{\sim}{v} \cdot \underset{\sim}{\psi}_3\, dx\right) \underset{\sim}{\psi}_3.$$

Since $\underset{\approx}{\epsilon}^*_k(\underset{\sim}{\psi}_3) = \underset{\approx}{\epsilon}(\underset{\sim}{\psi_3}) = \underset{\approx}{0}$, we have

$$(2.20)\qquad \underset{\approx}{\epsilon}^*_k(\underset{\sim}{v}) = \underset{\approx}{\epsilon}^*_k\left(T^\wedge_\perp(\underset{\sim}{v})\right) \quad \forall\, \underset{\sim}{v} \in \underset{\sim}{V}^\perp_k.$$

By the discrete Friedrichs' inequality (cf. [14]) for $\underset{\sim}{V}_k$, there exist positive constants C_1 and C_2 such that

$$(2.21)\qquad C_1\|T^\wedge_\perp(\underset{\sim}{v})\|_k \le \|\underset{\sim}{v}\|_k \le C_2\|T^\wedge_\perp(\underset{\sim}{v})\|_k \quad \forall\, \underset{\sim}{v} \in \underset{\sim}{V}^\perp_k.$$

The proof of the following discrete Korn's second inequality can be found in [6].

Lemma 2.4. *There exists a positive constant* C *such that*

$$\|\underset{\approx}{\epsilon}^*_k(\underset{\sim}{v})\|_{\underset{\approx}{L}^2(\Omega)} \ge C\|\underset{\sim}{v}\|_k \quad \forall\, \underset{\sim}{v} \in \hat{\underset{\sim}{V}}_k.$$

With the aid of the operator $T^\wedge_\perp$ we can translate Lemma 2.4 into a result on $\underset{\sim}{V}^\perp_k$.

Corollary 2.5. *There exists a positive constant* C *such that*

$$\|\underset{\approx}{\epsilon}^*_k(\underset{\sim}{v})\|_{\underset{\approx}{L}^2(\Omega)} \ge C\|\underset{\sim}{v}\|_k \quad \forall\, \underset{\sim}{v} \in \underset{\sim}{V}^\perp_k.$$

Proof. Let $\underset{\sim}{v} \in \underset{\sim}{V}^\perp_k$. Then, using (2.20), Lemma 2.4, and (2.21), we get

$$\|\underset{\approx}{\epsilon}^*_k(\underset{\sim}{v})\|_{\underset{\approx}{L}^2(\Omega)} = \|\underset{\approx}{\epsilon}^*_k(T^\wedge_\perp \underset{\sim}{v})\|_{\underset{\approx}{L}^2(\Omega)} \ge C\|T^\wedge_\perp \underset{\sim}{v}\|_k \ge C\|\underset{\sim}{v}\|_k. \quad \square$$

There is an interpolation operator $\Pi_k : \underset{\sim}{H}^1(\Omega) \longrightarrow \underset{\sim}{V}_k$ defined by (cf. [5])

$$(2.22)\qquad \left[\Pi_k(\underset{\sim}{\phi})\right](m_e) := \frac{1}{|e|}\int_e \underset{\sim}{\phi}\, ds$$

at the midpoints m_e of the edge e in $\mathscr{T}_k$.

A simple homogeneity argument shows that

$$\|\Pi_k \underset{\sim}{\phi}\|_k \leq C\,|\underset{\sim}{\phi}|_{\underset{\sim}{H}^1(\Omega)} \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^1(\Omega). \tag{2.23}$$

It follows from a simple calculation (cf. [5]) that

$$\operatorname{div}_k \left[\Pi_k \underset{\sim}{\phi}\right] = P_k(\operatorname{div} \underset{\sim}{\phi}) \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^1(\Omega), \tag{2.24}$$

and

$$\operatorname{rot}_k \left[\Pi_k \underset{\sim}{\phi}\right] = P_k(\operatorname{rot} \underset{\sim}{\phi}) \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^1(\Omega). \tag{2.25}$$

The following interpolation error estimate also holds (cf. [5]):

$$\|\underset{\sim}{\phi} - \Pi_k \underset{\sim}{\phi}\|_{\underset{\sim}{L}^2(\Omega)} + h_k \|\underset{\sim}{\phi} - \Pi_k \underset{\sim}{\phi}\|_k \leq C h_k^2 |\underset{\sim}{\phi}|_{\underset{\sim}{H}^2(\Omega)} \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^2(\Omega). \tag{2.26}$$

Let $\Pi_k^\perp$ be defined by

$$\Pi_k^\perp(\underset{\sim}{\phi}) := \Pi_k(\underset{\sim}{\phi}) - \sum_{i=1}^{3} \left(\int_\Omega \Pi_k(\underset{\sim}{\phi}) \cdot \underset{\sim}{\psi}_i \, dx\right) \underset{\sim}{\psi}_i \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^1_\perp. \tag{2.27}$$

It follows that $\Pi_k^\perp$ maps $\underset{\sim}{H}^1_\perp$ into $\underset{\sim}{V}^\perp_k$. Moreover, we still have

$$\|\Pi_k^\perp \underset{\sim}{\phi}\|_k \leq C\,|\underset{\sim}{\phi}|_{\underset{\sim}{H}^1(\Omega)} \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^1_\perp, \tag{2.28}$$

$$\operatorname{div}_k \left[\Pi_k^\perp \underset{\sim}{\phi}\right] = P_k(\operatorname{div} \underset{\sim}{\phi}) \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^1_\perp, \tag{2.29}$$

and

$$\|\underset{\sim}{\phi} - \Pi_k^\perp \underset{\sim}{\phi}\|_{\underset{\sim}{L}^2(\Omega)} + h_k \|\underset{\sim}{\phi} - \Pi_k^\perp \underset{\sim}{\phi}\|_k \leq C h_k^2 |\underset{\sim}{\phi}|_{\underset{\sim}{H}^2(\Omega)} \quad \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^2_\perp(\Omega). \tag{2.30}$$

Also, a standard interpolation error estimate (cf. [4, Theorem 3.1.6]) shows that

$$\|\psi - P_{k-1}\psi\|_{L^2(\Omega)} \leq C h_k |\psi|_{H^1(\Omega)} \quad \forall \psi \in H^1(\Omega). \tag{2.31}$$

The following lemma is the discrete analog of Lemma 2.1.

Lemma 2.6. *There exists a positive constant C such that, given any $q \in Q_k$, there exists $\underset{\sim}{v} \in \underset{\sim}{V}^\perp_k$ which satisfies*

$$\operatorname{div}_k \underset{\sim}{v} = q \tag{2.32}$$

and

$$\|\underset{\sim}{v}\|_k \leq C\,\|q\|_{L^2(\Omega)}. \tag{2.33}$$

Proof. From Lemma 2.1, there exists $\underset{\sim}{w} \in \underset{\sim}{H}^1_\perp$ such that

$$\operatorname{div} \underset{\sim}{w} = q \tag{2.34}$$

and

$$\|\underset{\sim}{w}\|_{\underset{\sim}{H}^1(\Omega)} \leq C\|q\|_{L^2(\Omega)}. \tag{2.35}$$

Let $\underset{\sim}{v} = \Pi_k^\perp \underset{\sim}{w}$. Then by (2.29) and (2.34) we have

$$\operatorname{div}_k \underset{\sim}{v} = q,$$

and by (2.28) and (2.35),

$$\|\underset{\sim}{v}\|_k \leq C\,\|q\|_{L^2(\Omega)}. \quad \square$$

With Corollary 2.5 and Lemma 2.6 in place, the following discrete version of Proposition 2.2 follows verbatim.

Proposition 2.7. *There exists a positive constant C such that for any $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$,*

$$\sup_{(\underset{\sim}{v}_2, q_2) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1} \setminus \{(\underset{\sim}{0}, 0)\}} \frac{\left|\mathscr{B}_k\left((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\right)\right|}{\|\underset{\sim}{v}_2\|_k + \|q_2\|_{L^2(\Omega)}} \geq C\left(\|\underset{\sim}{v}_1\|_k + \|q_1\|_{L^2(\Omega)}\right). \tag{2.36}$$

Corollary 2.8. *Given any linear functional F on $\underset{\sim}{V}_k^{\perp} \times Q_{k-1}$, there exists a unique $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$ such that*

$$\mathscr{B}\left((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\right) = F\left((\underset{\sim}{v}_2, q_2)\right) \quad \forall (\underset{\sim}{v}_2, q_2) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}. \tag{2.37}$$

In particular, the discretized equation (1.26) *is uniquely solvable.*

We can now establish the discretization error estimate.

Theorem 2.9. *If $(\underset{\sim}{u}, p) \in \underset{\sim}{H}_{\perp}^1 \times L^2(\Omega)$ solves* (1.22) *and $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$ solves* (1.26), *then*

$$\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_k + \|p - p_k\|_{L^2(\Omega)} \leq C h_k \left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right). \tag{2.38}$$

Proof. We follow the ideas of Scott (cf. [12] and [13, pp. 178–179]) for estimating the errors caused by variational crimes. Given any $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$, using (2.36), we have

$$\begin{aligned} &\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_k + \|p - p_k\|_{L^2(\Omega)} \\ &\quad \leq \|\underset{\sim}{u} - \underset{\sim}{v}\|_k + \|p - q\|_{L^2(\Omega)} + \|\underset{\sim}{v} - \underset{\sim}{u}_k\|_k + \|q - p_k\|_{L^2(\Omega)} \\ &\quad \leq \|\underset{\sim}{u} - \underset{\sim}{v}\|_k + \|p - q\|_{L^2(\Omega)} \\ &\qquad + C \sup_{(\underset{\sim}{w}, r) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1} \setminus \{(\underset{\sim}{0}, 0)\}} \frac{\left|\mathscr{B}_k\left((\underset{\sim}{v} - \underset{\sim}{u}_k, q - p_k), (\underset{\sim}{w}, r)\right)\right|}{\|\underset{\sim}{w}\|_k + \|r\|_{L^2(\Omega)}}. \end{aligned} \tag{2.39}$$

By (1.29) and (1.26) we have

(2.40)

$$\begin{aligned} &\frac{\left|\mathscr{B}_k\left((\underset{\sim}{v} - \underset{\sim}{u}_k, q - p_k), (\underset{\sim}{w}, r)\right)\right|}{\|\underset{\sim}{w}\|_k + \|r\|_{L^2(\Omega)}} \\ &\quad \leq C\left(\|\underset{\sim}{u} - \underset{\sim}{v}\|_k + \|p - q\|_{L^2(\Omega)}\right) \\ &\qquad + C \frac{\left|\mathscr{B}_k\left((\underset{\sim}{u}, p), (\underset{\sim}{w}, r)\right) - \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{w}\, dx + \sum_{i=1}^n \int_{\Gamma_i} \underset{\sim}{g}_i \cdot \underset{\sim}{w}|_{\Gamma_i}\, ds\right]\right|}{\|\underset{\sim}{w}\|_k + \|r\|_{L^2(\Omega)}}. \end{aligned}$$

Combining (2.39) and (2.40), we have

(2.41)
$$\begin{aligned}&\|\underset{\sim}{u}-\underset{\sim}{u}_k\|_k+\|p-p_k\|_{L^2(\Omega)}\\&\quad\le C\left(\|\underset{\sim}{u}-\underset{\sim}{v}\|_k+\|p-q\|_{L^2(\Omega)}\right)\\&\qquad+C\sup\frac{\left|\mathscr{B}_k\left((\underset{\sim}{u},p),(\underset{\sim}{w},r)\right)-\frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{w}\,dx+\sum_{i=1}^n\int_{\Gamma_i}\underset{\sim}{g}_i\cdot\underset{\sim}{w}|_{\Gamma_i}\,ds\right]\right|}{\|\underset{\sim}{w}\|_k+\|r\|_{L^2(\Omega)}},\end{aligned}$$

where the supremum term is taken over all $(\underset{\sim}{w}, r)\in \underset{\sim}{V}_k^{\perp}\times Q_{k-1}\backslash\{(\underset{\sim}{0},0)\}$. The supremum term on the right-hand side of (2.41) measures the effect of the nonconformity of $\underset{\sim}{V}_k^{\perp}$ and the effect of reduced integration built into the definition of $\underset{\approx}{\epsilon}_k^*$ (cf. (1.28)). It is bounded by $Ch_k|\underset{\sim}{u}|_{\underset{\sim}{H}^2(\Omega)}$ (cf. [3, 5]). If we take $(\underset{\sim}{v}, q)$ to be $(\Pi_k^{\perp}\underset{\sim}{u}, P_{k-1}p)$, then (2.30), (2.31) and (2.41) imply that

$$\begin{aligned}&\|\underset{\sim}{u}-\underset{\sim}{u}_k\|_k+\|p-p_k\|_{L^2(\Omega)}\\&\quad\le C\left(\|\underset{\sim}{u}-\Pi_k^{\perp}\underset{\sim}{u}\|_k+\|p-P_{k-1}p\|_{L^2(\Omega)}+h_k|\underset{\sim}{u}|_{\underset{\sim}{H}^2(\Omega)}\right)\\&\quad\le Ch_k\left(|\underset{\sim}{u}|_{\underset{\sim}{H}^2(\Omega)}+|p|_{H^1(\Omega)}\right).\end{aligned}\tag{2.42}$$

Since $p=\gamma\,\mathrm{div}\,\underset{\sim}{u}$, the theorem follows from (2.42) and the elliptic regularity estimate (1.20). □

Theorem 2.10. *There exists a positive constant* C *such that*

$$\|\underset{\sim}{u}-\underset{\sim}{u}_k\|_{\underset{\sim}{L}^2(\Omega)}\le Ch_k^2\left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}+\sum_{i=1}^n\|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right).\tag{2.43}$$

Proof. We use a duality argument. Since $\underset{\sim}{u}-\underset{\sim}{u}_k\in\underset{\sim}{L}_{\perp}^2=\{\underset{\sim}{v}\in\underset{\sim}{L}^2(\Omega):\int_\Omega\underset{\sim}{v}\cdot\underset{\sim}{w}\,dx=0\quad\forall\,\underset{\sim}{w}\in\underset{\sim}{RM}\}$, we can write

$$\|\underset{\sim}{u}-\underset{\sim}{u}_k\|_{\underset{\sim}{L}^2(\Omega)}=\sup_{\underset{\sim}{w}\in\underset{\sim}{L}_{\perp}^2(\Omega)\backslash\{\underset{\sim}{0}\}}\frac{\left|\int_\Omega(\underset{\sim}{u}-\underset{\sim}{u}_k)\cdot\underset{\sim}{w}\,dx\right|}{\|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}}.\tag{2.44}$$

Let $\underset{\sim}{w}\in\underset{\sim}{L}_{\perp}^2(\Omega)$. Then

$$\begin{aligned}&-\underset{\sim}{\mathrm{div}}\{2\mu\underset{\approx}{\epsilon}(\underset{\sim}{\zeta})+\lambda\,\mathrm{tr}\left(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta})\right)\underset{\approx}{\delta}\}=\underset{\sim}{w}\quad\text{in }\Omega,\\&\left(2\mu\underset{\approx}{\epsilon}(\underset{\sim}{\zeta})+\lambda\,\mathrm{tr}\left(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta})\right)\underset{\approx}{\delta}\right)\underset{\sim}{\nu}_i|_{\Gamma_i}=\underset{\sim}{0},\qquad 1\le i\le n,\end{aligned}\tag{2.45}$$

has a unique solution $\underset{\sim}{\zeta}\in\underset{\sim}{H}_{\perp}^2(\Omega)$ because conditions (1.16) and (1.17) are satisfied for $(\underset{\sim}{w},0)$.

The boundary value problem (2.45) is equivalent to

$$\mathscr{B}\left((\underset{\sim}{\zeta},\xi),(\underset{\sim}{v},q)\right)=\frac{1}{2\mu}\int_\Omega\underset{\sim}{w}\cdot\underset{\sim}{v}\,dx,\tag{2.46}$$

for all $(\underset{\sim}{v},q)\in\underset{\sim}{H}_{\perp}^1\times L^2(\Omega)$, where $\xi=(\lambda/(2\mu))\,\mathrm{div}\,\underset{\sim}{\zeta}$. The elliptic regularity estimate (1.20) implies that

$$\|\underset{\sim}{\zeta}\|_{\underset{\sim}{H}^2(\Omega)}+\|\xi\|_{H^1(\Omega)}\le C\|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}.\tag{2.47}$$

Let $(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$ satisfy

$$\mathscr{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)\Big) = \frac{1}{2\mu}\int_\Omega \underset{\sim}{w} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}. \tag{2.48}$$

Theorem 2.9 implies that

$$\|\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k\|_k + \|\xi - \xi_k\|_{L^2(\Omega)} \le C h_k \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}. \tag{2.49}$$

The interpolation error estimates (2.30)–(2.31) and (2.47) imply that

$$\begin{aligned} &\|\underset{\sim}{\zeta} - \Pi_k^{\perp}\underset{\sim}{\zeta}\|_{\underset{\sim}{L}^2(\Omega)} + h_k\|\underset{\sim}{\zeta} - \Pi_k^{\perp}\underset{\sim}{\zeta}\|_k + h_k\|\xi - P_{k-1}\xi\|_{L^2(\Omega)} \\ &\quad \le C h_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}. \end{aligned} \tag{2.50}$$

Using (1.29), (2.46) and (2.48), we have

(2.51)

$$\begin{aligned}
&\left|\frac{1}{2\mu}\int_\Omega (\underset{\sim}{u} - \underset{\sim}{u}_k)\cdot \underset{\sim}{w}\, dx\right| \\
&\quad = \Big|\mathscr{B}_k\Big((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\Big) - \mathscr{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{u}_k, p_k)\Big) \\
&\qquad + \mathscr{B}\Big((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\Big) - \mathscr{B}_k\Big((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\Big)\Big| \\
&\quad = \Big|\mathscr{B}_k\Big((\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k, \xi - \xi_k), (\underset{\sim}{u} - \Pi_k^{\perp}\underset{\sim}{u}, p - P_{k-1}p)\Big) \\
&\qquad + \mathscr{B}_k\Big((\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k, \xi - \xi_k), (\Pi_k^{\perp}\underset{\sim}{u}, P_{k-1}p)\Big) \\
&\qquad + \mathscr{B}_k\Big((\underset{\sim}{\zeta}_k - \Pi_k^{\perp}\underset{\sim}{\zeta}_k, \xi_k - P_{k-1}\xi), (\underset{\sim}{u} - \underset{\sim}{u}_k, p - p_k)\Big) \\
&\qquad + \mathscr{B}_k\Big((\Pi_k^{\perp}\underset{\sim}{\zeta}_k, P_{k-1}\xi), (\underset{\sim}{u} - \underset{\sim}{u}_k, p - p_k)\Big) \\
&\qquad + \mathscr{B}\Big((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\Big) - \mathscr{B}_k\Big((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\Big)\Big| \\
&\quad \le C\Big\{\Big(\|\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k\|_k + \|\xi - \xi_k\|_{L^2(\Omega)}\Big)\Big(\|\underset{\sim}{u} - \Pi_k^{\perp}\underset{\sim}{u}\|_k + \|p - P_{k-1}p\|_{L^2(\Omega)}\Big) \\
&\qquad\quad + \Big(\|\underset{\sim}{\zeta}_k - \Pi_k^{\perp}\underset{\sim}{\zeta}_k\|_k + \|\xi_k - P_{k-1}\xi\|_{L^2(\Omega)}\Big)\Big(\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_k + \|p - p_k\|_{L^2(\Omega)}\Big)\Big\} \\
&\qquad + \Big|\mathscr{B}_k\Big((\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k, \xi - \xi_k), (\Pi_k^{\perp}\underset{\sim}{u}, P_{k-1}p)\Big)\Big| \\
&\qquad + \Big|\mathscr{B}_k\Big((\Pi_k^{\perp}\underset{\sim}{\zeta}_k, P_{k-1}\xi), (\underset{\sim}{u} - \underset{\sim}{u}_k, p - p_k)\Big)\Big| \\
&\qquad + \Big|\mathscr{B}\Big((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\Big) - \mathscr{B}_k\Big((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\Big)\Big|.
\end{aligned}$$

From (2.30)–(2.31), (1.20), Theorem 2.9 and (2.49)–(2.50), we know that

(2.52)

$$\begin{aligned}
&\Big(\|\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k\|_k + \|\xi - \xi_k\|_{L^2(\Omega)}\Big)\Big(\|\underset{\sim}{u} - \Pi_k^{\perp}\underset{\sim}{u}\|_k + \|p - P_{k-1}p\|_{L^2(\Omega)}\Big) \\
&\qquad + \Big(\|\underset{\sim}{\zeta}_k - \Pi_k^{\perp}\underset{\sim}{\zeta}_k\|_k + \|\xi_k - P_{k-1}\xi\|_{L^2(\Omega)}\Big)\Big(\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_k + \|p - p_k\|_{L^2(\Omega)}\Big) \\
&\quad \le C h_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}\left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n}\|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right).
\end{aligned}$$

It therefore remains to estimate

$$\left|\mathscr{B}_k\left((\underset{\sim}{\zeta} - \underset{\sim k}{\zeta}, \xi - \xi_k), (\Pi_k^\perp \underset{\sim}{u}, P_{k-1}p)\right)\right|,$$
$$\left|\mathscr{B}_k\left((\Pi_k^\perp \underset{\sim k}{\zeta}, P_{k-1}\xi), (\underset{\sim}{u} - \underset{\sim k}{u}, p - p_k)\right)\right|$$

and

$$\left|\mathscr{B}\left((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\right) - \mathscr{B}_k\left((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{u}, p)\right)\right|.$$

In order to obtain a bound for $\left|\mathscr{B}_k\left((\underset{\sim}{\zeta} - \underset{\sim k}{\zeta}, \xi - \xi_k), (\Pi_k^\perp \underset{\sim}{u}, P_{k-1}p)\right)\right|$, first note that by (1.27), the fact that $\xi = \gamma \operatorname{div} \underset{\sim}{\zeta}$, (2.45) and (2.48), we have
(2.53)

$$\begin{aligned}
&\mathscr{B}_k\left((\underset{\sim}{\zeta} - \underset{\sim k}{\zeta}, \xi - \xi_k), (\Pi_k^\perp \underset{\sim}{u}, P_{k-1}p)\right) \\
&\quad = \mathscr{B}_k\left((\underset{\sim}{\zeta}, \xi), (\Pi_k^\perp \underset{\sim}{u}, P_{k-1}p)\right) - \frac{1}{2\mu}\int_\Omega \underset{\sim}{w} \cdot (\Pi_k^\perp \underset{\sim}{u})\, dx \\
&\quad = \int_\Omega \left[\underset{\approx k}{\epsilon}^*(\underset{\sim}{\zeta}) : \underset{\approx k}{\epsilon}^*(\Pi_k^\perp \underset{\sim}{u}) + \xi \operatorname{div}_k(\Pi_k^\perp \underset{\sim}{u})\right] dx - \frac{1}{2\mu}\int_\Omega \underset{\sim}{w} \cdot (\Pi_k^\perp \underset{\sim}{u})\, dx \\
&\quad = \int_\Omega \left[\underset{\approx k}{\epsilon}^*(\underset{\sim}{\zeta}) : \underset{\approx k}{\epsilon}^*(\Pi_k^\perp \underset{\sim}{u}) - \underset{\approx k}{\epsilon}(\underset{\sim}{\zeta}) : \underset{\approx k}{\epsilon}(\Pi_k^\perp \underset{\sim}{u})\right] dx \\
&\qquad + \int_\Omega \left[\underset{\approx k}{\epsilon}(\underset{\sim}{\zeta}) : \underset{\approx k}{\epsilon}(\Pi_k^\perp \underset{\sim}{u}) + \underset{\sim}{\operatorname{div}}\left(\underset{\approx k}{\epsilon}(\underset{\sim}{\zeta})\right) \cdot \Pi_k^\perp \underset{\sim}{u}\right] dx \\
&\qquad + \int_\Omega \left[\frac{\lambda}{2\mu}\operatorname{tr}(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta}))\underset{\approx}{\delta} : \underset{\approx}{\operatorname{grad}}_k(\Pi_k^\perp \underset{\sim}{u}) + \underset{\sim}{\operatorname{div}}\left(\frac{\lambda}{2\mu}\operatorname{tr}(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta}))\underset{\approx}{\delta}\right) \cdot \left(\Pi_k^\perp \underset{\sim}{u}\right)\right] dx,
\end{aligned}$$

where $\underset{\approx k}{\epsilon}(\underset{\sim}{\phi}) := \frac{1}{2}(\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\phi} + (\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\phi})^t) = \underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\phi} - \frac{1}{2}(\operatorname{rot}_k \underset{\sim}{\phi})\underset{\approx}{\chi}$. Standard nonconforming estimates (cf. [5]) and elliptic regularity (1.20) and (2.47) give estimates on the last two integrals, i.e.,

$$\begin{aligned}
&\left|\int_\Omega \left[\underset{\approx k}{\epsilon}(\underset{\sim}{\zeta}) : \underset{\approx k}{\epsilon}(\Pi_k^\perp \underset{\sim}{u}) + \underset{\sim}{\operatorname{div}}(\underset{\approx k}{\epsilon}(\underset{\sim}{\zeta})) \cdot \Pi_k^\perp \underset{\sim}{u}\right] dx\right| \\
&\quad \le C h_k^2 |\underset{\sim}{\zeta}|_{\underset{\sim}{H}^2(\Omega)} |\underset{\sim}{u}|_{\underset{\sim}{H}^2(\Omega)} \qquad (2.54)\\
&\quad \le C h_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim i}{g}\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right)
\end{aligned}$$

and
(2.55)

$$\begin{aligned}
&\left|\int_\Omega \left[\frac{\lambda}{2\mu}\operatorname{tr}(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta}))\underset{\approx}{\delta} : \underset{\approx}{\operatorname{grad}}_k(\Pi_k^\perp \underset{\sim}{u}) + \underset{\sim}{\operatorname{div}}\left(\frac{\lambda}{2\mu}\operatorname{tr}(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta}))\underset{\approx}{\delta}\right) \cdot \Pi_k^\perp \underset{\sim}{u}\right] dx\right| \\
&\quad \le C h_k^2 |\underset{\sim}{\zeta}|_{\underset{\sim}{H}^2(\Omega)} |\underset{\sim}{u}|_{\underset{\sim}{H}^2(\Omega)} \\
&\quad \le C h_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim i}{g}\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right).
\end{aligned}$$

We must still show that

$$(2.56)\quad \begin{aligned}&\left|\int_\Omega \left[\underset{\approx}{\epsilon}^*_k(\underset{\sim}{\zeta}) : \underset{\approx}{\epsilon}^*_k(\Pi_k^\perp \underset{\sim}{u}) - \underset{\approx}{\epsilon}_k(\underset{\sim}{\zeta}) : \underset{\approx}{\epsilon}_k(\Pi_k^\perp \underset{\sim}{u})\right] dx\right| \\ &\quad \le Ch_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right).\end{aligned}$$

The left-hand side of (2.56) is bounded by

(2.57)

$$\begin{aligned}&\left|\int_\Omega \underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\zeta} : \left[-\frac{1}{2} P_{k-1}(\operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u}))\underset{\approx}{\chi}\right] - \underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\zeta} : \left[-\frac{1}{2}(\operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u}))\underset{\approx}{\chi}\right] dx\right| \\ &+ \left|\int_\Omega \left[-\frac{1}{2} P_{k-1}(\operatorname{rot}_k \underset{\sim}{\zeta})\underset{\approx}{\chi}\right] : \underset{\approx}{\operatorname{grad}}_k(\Pi_k^\perp \underset{\sim}{u}) - \left[-\frac{1}{2}(\operatorname{rot}_k \underset{\sim}{\zeta})\underset{\approx}{\chi}\right] : \underset{\approx}{\operatorname{grad}}_k(\Pi_k^\perp \underset{\sim}{u})\, dx\right| \\ &+ \left|\int_\Omega \frac{1}{4}\left[P_{k-1}(\operatorname{rot}_k \underset{\sim}{\zeta})\underset{\approx}{\chi} : \operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u})\underset{\approx}{\chi}\right] - \frac{1}{4}\left[(\operatorname{rot}_k \underset{\sim}{\zeta})\underset{\approx}{\chi} : \operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u})\underset{\approx}{\chi}\right] dx\right|.\end{aligned}$$

For the first term of (2.57) we have by (2.30) and (2.31)

$$\begin{aligned}&\left|\int_\Omega \frac{1}{2} \underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\zeta} : \left(\operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u}) - P_{k-1}\operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u})\right)\underset{\approx}{\chi}\, dx\right| \\ &\quad = \left|\frac{1}{2}\int_\Omega \left(\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\zeta} - P_{k-1}\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\zeta}\right) : \left(\operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u}) - P_{k-1}\operatorname{rot}_k(\Pi_k^\perp \underset{\sim}{u})\right)\underset{\approx}{\chi}\, dx\right| \\ &\quad = \left|\frac{1}{2}\int_\Omega \left(\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\zeta} - P_{k-1}\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{\zeta}\right) : \left[\operatorname{rot}_k\left((\Pi_k^\perp \underset{\sim}{u}) - \underset{\sim}{u}\right) + (\operatorname{rot}_k \underset{\sim}{u} - P_{k-1}\operatorname{rot}_k \underset{\sim}{u})\right.\right. \\ &\qquad\qquad \left.\left. + P_{k-1}\left(\operatorname{rot}_k(\underset{\sim}{u} - \Pi_k^\perp \underset{\sim}{u})\right)\underset{\approx}{\chi}\right] dx\right| \\ &\quad \le Ch_k^2 |\underset{\sim}{\zeta}|_{\underset{\sim}{H}^2(\Omega)} |\underset{\sim}{u}|_{\underset{\sim}{H}^2(\Omega)} \\ &\quad \le Ch_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right).\end{aligned}$$

Similar arguments yield the same bound for the second and third terms of (2.57).

Putting these bounds into (2.53), we have

$$(2.58)\quad \begin{aligned}&\left|\mathscr{B}_k\left((\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k, \xi - \xi_k), (\Pi_k^\perp \underset{\sim}{u}, P_{k-1}p)\right)\right| \\ &\quad \le Ch_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right).\end{aligned}$$

We can analogously (by interchanging $(\underset{\sim}{u}, p)$ with $(\underset{\sim}{\zeta}, \xi)$, and $(\underset{\sim}{u}_k, p_k)$ with $(\underset{\sim}{\zeta}_k, \xi_k)$) derive the following estimate:

$$(2.59)\quad \begin{aligned}&\left|\mathscr{B}_k\left((\Pi_k^\perp \underset{\sim}{\zeta}_k, P_{k-1}\xi), (\underset{\sim}{u} - \underset{\sim}{u}_k, p - p_k)\right)\right| \\ &\quad \le Ch_k^2 |\underset{\sim}{\zeta}|_{\underset{\sim}{H}^2(\Omega)} |\underset{\sim}{u}|_{\underset{\sim}{H}^2(\Omega)} \\ &\quad \le Ch_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^n \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}&\left|\mathscr{B}\left((\underset{\sim}{\zeta},\xi),(\underset{\sim}{u},p)\right)-\mathscr{B}_k\left((\underset{\sim}{\zeta},\xi),(\underset{\sim}{u},p)\right)\right|\\&=\left|\int_\Omega \underset{\approx}{\epsilon}_k(\underset{\sim}{\zeta}):\underset{\approx}{\epsilon}(\underset{\sim}{u})-\underset{\approx}{\epsilon}_k^*(\underset{\sim}{\zeta}):\underset{\approx}{\epsilon}_k^*(\underset{\sim}{u})\right|\\&\leq Ch_k^2\,\|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}\left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}+\sum_{i=1}^n\|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right),\end{aligned}\tag{2.60}$$

which is obtained by arguments similar to those in the proof of (2.56). Combining (2.51), (2.52), (2.58), (2.59), and (2.60), we therefore have

$$\begin{aligned}&\left|\frac{1}{2\mu}\int_\Omega(\underset{\sim}{u}-\underset{\sim}{u}_k)\cdot\underset{\sim}{w}\,dx\right|\\&\leq Ch_k^2\|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}\left(\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}+\sum_{i=1}^n\|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right)\end{aligned}\tag{2.61}$$

for all $\underset{\sim}{w}\in\underset{\sim}{L}^2_\perp(\Omega)$. Combining (2.61) and (2.44) completes the proof. □

Remark. The L^2 estimate in Theorem 2.10 is crucial in our proof of convergence of the multigrid method. This is one reason why $\underset{\sim}{V}_k^\perp$ is preferable over $\hat{\underset{\sim}{V}}_k$. The duality argument in the proof of Theorem 2.10 would not work for $\hat{\underset{\sim}{V}}_k$ because condition (1.17) is only satisfied by $(\underset{\sim}{w},0)$ where $\underset{\sim}{w}\in\underset{\sim}{L}^2_\perp$.

3. Intergrid transfer operators and mesh-dependent norms

In this section we define the intergrid transfer operators and the mesh-dependent norms. The estimates involving the intergrid transfer operators will play an important role in the convergence analysis of the multigrid method.

The coarse-to-fine operator $I_{k-1}^k:\underset{\sim}{V}_{k-1}\times Q_{k-2}\longrightarrow\underset{\sim}{V}_k\times Q_{k-1}$ is defined by

$$I_{k-1}^k(\underset{\sim}{v},q)=(\underset{\sim}{P}_k\underset{\sim}{v},q),\tag{3.1}$$

where $\underset{\sim}{P}_k:\underset{\sim}{L}^2(\Omega)\longrightarrow\underset{\sim}{V}_k$ is the L^2 orthogonal projection operator. Note that $\underset{\sim}{P}_k\underset{\sim}{v}$ can be explicitly defined by

$$\begin{aligned}&\underset{\sim}{P}_k(\underset{\sim}{v})(m_e)\\&=\begin{cases}\underset{\sim}{v}(m_e) & \text{if } m_e\in\operatorname{int}T\text{ for some }T\in\mathscr{T}_{k-1}\text{ or if }m_e\in\partial\Omega,\\ \frac{1}{|T_1|+|T_2|}\left(|T_1|[\underset{\sim}{v}|_{T_1}(m_e)]+|T_2|[\underset{\sim}{v}|_{T_2}(m_e)]\right) & \text{if } e=T_1\cap T_2\text{ for some}\\ & \qquad T_1,\,T_2\in\mathscr{T}_{k-1}.\end{cases}\end{aligned}\tag{3.2}$$

In order to define the fine-to-coarse operator I_k^{k-1}, we introduce the following mesh-dependent inner product:

$$\left((\underset{\sim}{v}_1,q_1),(\underset{\sim}{v}_2,q_2)\right)_k:=(\underset{\sim}{v}_1,\underset{\sim}{v}_2)_{\underset{\sim}{L}^2(\Omega)}+h_k^2(q_1,q_2)_{L^2(\Omega)}.\tag{3.3}$$

Then $I_k^{k-1}:\underset{\sim}{V}_k\times Q_{k-1}\longrightarrow\underset{\sim}{V}_{k-1}\times Q_{k-2}$ is defined by

$$\left(I_k^{k-1}(\underset{\sim}{v}_1,q_1),(\underset{\sim}{v}_2,q_2)\right)_{k-1}=\left((\underset{\sim}{v}_1,q_1),I_{k-1}^k(\underset{\sim}{v}_2,q_2)\right)_k\tag{3.4}$$

for all $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{V}_k \times Q_{k-1}$ and $(\underset{\sim}{v}_2, q_2) \in \underset{\sim}{V}_{k-1} \times Q_{k-2}$.

Lemma 3.1. *The following properties of I^k_{k-1} and I^{k-1}_k hold*:

(i) $\underset{\sim}{\mathrm{RM}} \subseteq \underset{\sim}{V}_k$ *for* $k = 1, 2, \ldots$.

(ii) *Given any* $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times Q_{k-1}$, *then* $(\underset{\sim}{v}, q) \in \underset{\sim}{V}^{\perp}_k \times Q_{k-1}$ *if and only if* $\Big((\underset{\sim}{v}, q), (\underset{\sim}{w}, 0)\Big)_k = 0$ *for all* $\underset{\sim}{w} \in \underset{\sim}{\mathrm{RM}}$.

(iii) $I^k_{k-1} : \underset{\sim}{V}^{\perp}_{k-1} \times Q_{k-2} \longrightarrow \underset{\sim}{V}^{\perp}_k \times Q_{k-1}$.

(iv) $I^{k-1}_k : \underset{\sim}{V}^{\perp}_k \times Q_{k-1} \longrightarrow \underset{\sim}{V}^{\perp}_{k-1} \times Q_{k-2}$.

Proof. Property (i) is trivial. Property (ii) follows from (3.3), (iii) follows from (3.1), (i) and (ii), and (iv) follows from (3.1), (3.4), (i) and (ii). □

Remark. The fact that the constraints on $\underset{\sim}{V}^{\perp}_k$ can be enforced by the mesh-dependent inner product as described in Lemma 3.1 (ii) is another reason why $\underset{\sim}{V}^{\perp}_k$ is preferable to $\hat{\underset{\sim}{V}}_k$. As a consequence, these constraints are preserved by the intergrid transfer operators I^k_{k-1} and I^{k-1}_k (Lemmas 3.1 (iii) and (iv)). We will take advantage of this in the construction of the multigrid algorithm in §4.

Let $B_k : \underset{\sim}{V}_k \times Q_{k-1} \longrightarrow \underset{\sim}{V}_k \times Q_{k-1}$ be defined by

$$\begin{aligned} &\Big(B_k(\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\Big)_k \\ &\quad = \mathscr{B}_k\Big((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\Big) \quad \forall (\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2) \in \underset{\sim}{V}_k \times Q_{k-1}. \end{aligned} \tag{3.5}$$

Lemma 3.2. *The operator B_k maps $\underset{\sim}{V}_k \times Q_{k-1}$ into $\underset{\sim}{V}^{\perp}_k \times Q_{k-1}$.*

Proof. Let $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times Q_{k-1}$. Then

$$\Big(B_k(\underset{\sim}{v}, q), (\underset{\sim}{w}, 0)\Big)_k = \mathscr{B}_k\Big((\underset{\sim}{v}, q), (\underset{\sim}{w}, 0)\Big) = 0 \qquad \forall\, \underset{\sim}{w} \in \underset{\sim}{\mathrm{RM}}.$$

Therefore, $B_k(\underset{\sim}{v}, q) \in \underset{\sim}{V}^{\perp}_k \times Q_{k-1}$ by Lemma 3.1 (ii). □

Let $B^{\perp}_k : \underset{\sim}{V}^{\perp}_k \times Q_{k-1} \longrightarrow \underset{\sim}{V}^{\perp}_k \times Q_{k-1}$ be the restriction of B_k to $\underset{\sim}{V}^{\perp}_k \times Q_{k-1}$.

Lemma 3.3. *There exists a positive constant C such that the spectral radius of $B^{\perp}_k$ is less than or equal to Ch_k^{-2} for $k = 1, 2, \ldots$.*

Proof. This is an immediate consequence of (3.3), (3.5), (1.29) and the following inverse estimate (cf. [4, Theorem 3.2.6]):

$$\|\underset{\sim}{w}\|_k \leq Ch_k^{-1} \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \quad \forall\, \underset{\sim}{w} \in \underset{\sim}{V}_k. \tag{3.6}$$ □

The mesh-dependent norms on $\underset{\sim}{V}^{\perp}_k \times Q_{k-1}$ are defined as follows:

$$|\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{s,k} := \sqrt{\Big(({B^{\perp}_k}^2)^{s/2}(\underset{\sim}{v}, q), (\underset{\sim}{v}, q)\Big)_k} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}^{\perp}_k \times Q_{k-1}. \tag{3.7}$$

Since $B^{\perp}_k$ is nonsingular (cf. Proposition 2.7) and symmetric, $(B^{\perp}_k)^2$ is positive definite and (3.7) defines a norm on $\underset{\sim}{V}^{\perp}_k \times Q_{k-1}$ for each $s \in \mathbb{R}$.

The following properties of the mesh-dependent norms are trivial:

$$\text{(3.8)} \qquad |||(\underset{\sim}{v}, q)|||_{0,k} = \sqrt{\|\underset{\sim}{v}\|^2_{\underset{\sim}{L}^2(\Omega)} + h_k^2\|q\|^2_{L^2(\Omega)}} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1},$$

$$\text{(3.9)} \qquad \begin{aligned} &\left|\mathscr{B}_k\left((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\right)\right| \\ &\le |||(\underset{\sim}{v}_1, q_1)|||_{2,k}\, |||(\underset{\sim}{v}_2, q_2)|||_{0,k} \quad \forall (\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}, \end{aligned}$$

and

$$\text{(3.10)} \qquad |||(\underset{\sim}{v}, q)|||_{2,k} = \sup_{(\underset{\sim}{v}', q') \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1} \setminus \{(\underset{\sim}{0}, 0)\}} \frac{\left|\mathscr{B}_k((\underset{\sim}{v}, q), (\underset{\sim}{v}', q'))\right|}{|||(\underset{\sim}{v}', q')|||_{0,k}}$$

for all $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$.

Lemma 3.4. *There exists a positive constant* C *such that*

(i) $|||I_{k-1}^k(\underset{\sim}{v}, q)|||_{0,k} \le C\, |||(\underset{\sim}{v}, q)|||_{0,k-1}$ *for all* $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$.

(ii) $\|\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)} \le C h_k \|\underset{\sim}{v}\|_{k-1}$ *for all* $\underset{\sim}{v} \in \underset{\sim}{V}_{k-1}^{\perp}$.

(iii) $|||I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\phi}, P_{k-2}\psi) - (\Pi_k^{\perp}\underset{\sim}{\phi}, P_{k-1}\psi)|||_{0,k} \le C h_k^2 \left\{|\underset{\sim}{\phi}|_{\underset{\sim}{H}^2(\Omega)} + |\psi|_{H^1(\Omega)}\right\}$ *for all* $(\underset{\sim}{\phi}, \psi) \in \underset{\sim}{H}_{\perp}^2(\Omega) \times H^1(\Omega)$.

Proof. Since P_k is the L^2 orthogonal projection onto $\underset{\sim}{V}_k$, the estimate (i) follows from (3.1) and (3.8). The proof of inequality (ii) is based on the averaging formula (3.2), and a straightforward computation. For more details we refer the reader to the proof of Theorem 2.3 in [2]. Estimate (iii) is a consequence of (3.8) and the interpolation error estimates (2.30) and (2.31). □

Let $P_k^{k-1} : \underset{\sim}{V}_k^{\perp} \times Q_{k-1} \longrightarrow \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$ be defined by

$$\text{(3.11)} \qquad \mathscr{B}_{k-1}\left(P_k^{k-1}(\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2)\right) = \mathscr{B}_k\left((\underset{\sim}{v}_1, q_1), I_{k-1}^k(\underset{\sim}{v}_2, q_2)\right)$$

for all $(\underset{\sim}{v}_1, q_1) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$ and $(\underset{\sim}{v}_2, q_2) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$. The operator P_k^{k-1} will appear in the convergence analysis but not in the multigrid algorithm.

It follows from Lemma 3.4 (i) and (3.9)–(3.11) that

$$\text{(3.12)} \qquad |||P_k^{k-1}(\underset{\sim}{v}, q)|||_{2,k-1} \le C\, |||(\underset{\sim}{v}, q)|||_{2,k} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}.$$

The next lemma is the basis of Lemmas 3.6 and 3.7, which are crucial for the proof of the approximation property (Lemma S.3) in the Supplement.

Lemma 3.5. *Let* $k \ge 2$ *and* $\underset{\sim}{w} \in \underset{\sim}{L}^2(\Omega)$. *Assume that* $(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$ *satisfies*

$$\text{(3.13)} \qquad \mathscr{B}_k\left((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)\right) = \int_\Omega \underset{\sim}{w} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1},$$

and $(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$ *satisfies*

$$\text{(3.14)} \qquad \mathscr{B}_{k-1}\left((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}, q)\right) = \int_\Omega \underset{\sim}{w} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}.$$

Then there exists a positive constant C *such that*

$$\||(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)\||_{0,k-1} \leq Ch_k^2 \| \underset{\sim}{w} \|_{\underset{\sim}{L}^2(\Omega)}. \tag{3.15}$$

Proof. Let $(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k) = (\underset{\sim}{\eta}, \tau) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$. Recall that by (3.8), $\||(\underset{\sim}{\eta}, \tau)\||_{0,k-1}^2 = \| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)}^2 + h_{k-1}^2 \|\tau\|_{L^2(\Omega)}^2$. We will estimate $\| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)}$ and $h_{k-1}\|\tau\|_{L^2(\Omega)}$ by two duality arguments.

Since $\underset{\sim}{\eta} \in \underset{\sim}{L}_{\perp}^2(\Omega)$, the boundary value problem

$$\begin{aligned} -\underset{\sim}{\operatorname{div}} \left\{2\mu \underset{\approx}{\epsilon}(\underset{\sim}{\phi}) + \lambda \operatorname{tr}\left(\underset{\approx}{\epsilon}(\underset{\sim}{\phi})\right) \underset{\approx}{\delta}\right\} &= 2\mu \underset{\sim}{\eta} \quad \text{in}\, \Omega, \\ \left(2\mu \underset{\approx}{\epsilon}(\underset{\sim}{\phi}) + \lambda \operatorname{tr}\left(\underset{\approx}{\epsilon}(\underset{\sim}{\phi})\right) \underset{\approx}{\delta}\right) \underset{\sim}{\nu}_i|_{\Gamma_i} &= 0, \qquad 1 \leq i \leq n, \end{aligned} \tag{3.16}$$

has a unique solution $\underset{\sim}{\phi} \in \underset{\sim}{H}_{\perp}^2(\Omega)$ because (1.16) and (1.17) are satisfied. The elliptic regularity estimate (1.20) implies that

$$\| \underset{\sim}{\phi} \|_{\underset{\sim}{H}^2(\Omega)} \leq C \| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)}. \tag{3.17}$$

Let $(\underset{\sim}{\phi}_{k-1}, \psi_{k-1}) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$ satisfy

$$\mathscr{B}_{k-1}\left((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{v}, q)\right) = \int_{\Omega} \underset{\sim}{\eta} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}. \tag{3.18}$$

By Theorem 2.10, we have

$$\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)} \leq Ch_k^2 \| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)}. \tag{3.19}$$

But by (3.18), (3.11), the definition of $(\underset{\sim}{\eta}, \tau)$, (3.13), (3.14), (3.19), (3.17), and (2.30), we have

$$\begin{aligned} \|\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)}^2 &= \mathscr{B}_{k-1}\left((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{\eta}, \tau)\right) \\ &= \mathscr{B}_{k-1}\left((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})\right) \\ &\quad - \mathscr{B}_k\left(I_{k-1}^k(\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{\zeta}_k, \xi_k)\right) \\ &= \int_{\Omega} \underset{\sim}{w} \cdot (\underset{\sim}{\phi}_{k-1} - \underset{\sim}{P}_k \underset{\sim}{\phi}_{k-1})\, dx \\ &\leq \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \Big(\|\underset{\sim}{\phi}_{k-1} - \underset{\sim}{\phi}\|_{\underset{\sim}{L}^2(\Omega)} + \|\underset{\sim}{\phi} - \Pi_k^{\perp} \underset{\sim}{\phi}\|_{\underset{\sim}{L}^2(\Omega)} \\ &\qquad\qquad + \|\underset{\sim}{P}_k(\Pi_k^{\perp} \underset{\sim}{\phi} - \underset{\sim}{\phi}_{k-1})\|_{\underset{\sim}{L}^2(\Omega)}\Big) \\ &\leq \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \left(Ch_k^2 \|\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)}\right). \end{aligned}$$

Therefore,

$$\| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} \leq Ch_k^2 \| \underset{\sim}{w} \|_{\underset{\sim}{L}^2(\Omega)}. \tag{3.20}$$

Let $(\underset{\sim}{v}^*, q^*) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$ satisfy

$$(3.21)\quad \mathscr{B}_{k-1}\left((\underset{\sim}{v}^*, q^*), (\underset{\sim}{v}, q)\right) = h_{k-1}^2 \int_\Omega \tau q\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}.$$

By Proposition 2.7 we have

$$(3.22)\qquad \|\underset{\sim}{v}^*\|_{k-1} + \|q^*\|_{L^2(\Omega)} \le h_{k-1}^2 \|\tau\|_{L^2(\Omega)}.$$

From Lemma 3.4 (ii), (3.21) and (3.22), we have

$$\begin{aligned} h_{k-1}^2 \|\tau\|_{L^2(\Omega)}^2 &= \mathscr{B}_{k-1}\left((\underset{\sim}{v}^*, q^*), (\underset{\sim}{\eta}, \tau)\right) \\ &= \mathscr{B}_{k-1}\left((\underset{\sim}{v}^*, q^*), (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})\right) - \mathscr{B}_k\left(I_{k-1}^k(\underset{\sim}{v}^*, q^*), (\underset{\sim}{\zeta}_k, \xi_k)\right) \\ &= \int_\Omega \underset{\sim}{w} \cdot (\underset{\sim}{v}^* - \underset{\sim}{P}_k \underset{\sim}{v}^*)\, dx \\ &\le \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \|\underset{\sim}{v}^* - \underset{\sim}{P}_k \underset{\sim}{v}^*\|_{\underset{\sim}{L}^2(\Omega)} \\ &\le C h_k^3 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)} \|\tau\|_{L^2(\Omega)}. \end{aligned}$$

Hence,

$$(3.23)\qquad h_{k-1} \|\tau\|_{L^2(\Omega)} \le C h_k^2 \|\underset{\sim}{w}\|_{\underset{\sim}{L}^2(\Omega)}.$$

The lemma now follows from (3.20) and (3.23). □

Lemma 3.6. *There exists a positive constant C such that for $k \ge 2$*

$$(3.24)\qquad \|\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)} \le C h_k^2 |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k-1}$$

for all $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$.

Proof. Given any $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$, let $(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1}$ satisfy

$$(3.25)\qquad \begin{aligned} &\mathscr{B}_k\left((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}', q')\right) \\ &\quad = \int_\Omega (\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}) \cdot \underset{\sim}{v}'\, dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V}_k^{\perp} \times Q_{k-1} \end{aligned}$$

and $(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}$ satisfy

$$(3.26)\qquad \begin{aligned} &\mathscr{B}_{k-1}\left((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}', q')\right) \\ &\quad = \int_\Omega (\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}) \cdot \underset{\sim}{v}'\, dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V}_{k-1}^{\perp} \times Q_{k-2}. \end{aligned}$$

Therefore, using (3.1), (3.11), and Lemma 3.5, we have

$$\begin{aligned} \|\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)}^2 &= \int_\Omega (\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}) \cdot \underset{\sim}{v}\, dx - \int_\Omega (\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}) \cdot \underset{\sim}{P}_k \underset{\sim}{v}\, dx \\ &= \mathscr{B}_{k-1}\left((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}, q)\right) - \mathscr{B}_k\left((\underset{\sim}{\zeta}_k, \xi_k), I_{k-1}^k(\underset{\sim}{v}, q)\right) \\ &= \mathscr{B}_{k-1}\left((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)\right) \\ &\le |\!|\!|(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)|\!|\!|_{0,k-1} |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k-1} \\ &\le C h_k^2 \|\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)} |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k-1}. \end{aligned}$$

Hence,

$$\|\underset{\sim}{v} - \underset{\sim}{P}_k \underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)} \le C h_k^2 \|\!|(\underset{\sim}{v}, q)|\!\|_{2,k-1}. \quad \square$$

Lemma 3.7. *Let $k \ge 2$ and $w \in L^2(\Omega)$, $(\underset{\sim}{v}_k, q_k) \in \underset{\sim}{V}_k^\perp \times Q_{k-1}$ satisfy*

$$\mathscr{B}_k\Big((\underset{\sim}{v}_k, q_k), (\underset{\sim}{v}, q)\Big) = \int_\Omega wq\,dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^\perp \times Q_{k-1}, \tag{3.27}$$

and $(\underset{\sim}{v}_{k-1}, q_{k-1}) \in \underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}$ satisfy

$$\mathscr{B}_{k-1}\Big((\underset{\sim}{v}_{k-1}, q_{k-1}), (\underset{\sim}{v}, q)\Big) = \int_\Omega wq\,dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}. \tag{3.28}$$

Then there exists a positive constant C such that

$$\|\!|(\underset{\sim}{v}_k, q_k) - I_{k-1}^k(\underset{\sim}{v}_{k-1}, q_{k-1})|\!\|_{0,k} \le C h_k \|w\|_{L^2(\Omega)}. \tag{3.29}$$

Proof. Recall that

$$\begin{aligned} &\|\!|(\underset{\sim}{v}_k, q_k) - I_{k-1}^k(\underset{\sim}{v}_{k-1}, q_{k-1})|\!\|_{0,k}^2 \\ &\quad = \|\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}^2 + h_k^2 \|q_k - q_{k-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

We first estimate $\|\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}$ by a duality argument.

Since $\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1} \in \underset{\sim}{L}_\perp^2(\Omega)$, the boundary value problem

$$\begin{aligned} -\underset{\sim}{\mathrm{div}}\left\{2\mu \underset{\approx}{\epsilon}(\underset{\sim}{\zeta}) + \lambda \mathrm{tr}\left(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta})\right)\underset{\approx}{\delta}\right\} &= 2\mu(\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1}) \quad \text{in}\,\Omega, \\ \left(2\mu \underset{\approx}{\epsilon}(\underset{\sim}{\zeta}) + \lambda \mathrm{tr}\left(\underset{\approx}{\epsilon}(\underset{\sim}{\zeta})\right)\underset{\approx}{\delta}\right)\underset{\sim}{\nu}_i|_{\Gamma_i} &= 0, \qquad 1 \le i \le n, \end{aligned}$$

has a unique solution $\underset{\sim}{\zeta} \in \underset{\sim}{H}_\perp^2(\Omega)$ because (1.16) and (1.17) are satisfied.

Let $(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k^\perp \times Q_{k-1}$ satisfy

$$\mathscr{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)\Big) = \int_\Omega (\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1}) \cdot \underset{\sim}{v}\,dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^\perp \times Q_{k-1},$$

and $(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) \in \underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}$ satisfy

$$\begin{aligned} &\mathscr{B}_{k-1}\Big((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}, q)\Big) \\ &\quad = \int_\Omega (\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1}) \cdot \underset{\sim}{v}\,dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}. \end{aligned}$$

By Theorem 2.9 and the elliptic regularity estimate (1.20) we have

$$\begin{aligned} \|\xi_k - \xi_{k-1}\|_{L^2(\Omega)} &\le \left\|\xi_k - \frac{\lambda}{2\mu}\mathrm{div}\,\underset{\sim}{\zeta}\right\|_{L^2(\Omega)} + \left\|\xi_{k-1} - \frac{\lambda}{2\mu}\mathrm{div}\,\underset{\sim}{\zeta}\right\|_{L^2(\Omega)} \\ &\le C h_k \|\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}. \end{aligned} \tag{3.30}$$

Let $(\underset{\sim}{\eta}, \tau) = (\underset{\sim}{\zeta}_k, \xi_k) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)$. Using (3.1), (3.11), (3.27), and (3.28), we have

$$
\begin{aligned}
\|\underset{\sim}{v}_k - \underset{\sim}{P}_k \underset{\sim}{v}_{k-1}\|^2_{\underset{\sim}{L}^2(\Omega)} \\
&= \mathscr{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}_k, q_k) - I^k_{k-1}(\underset{\sim}{v}_{k-1}, q_{k-1})\Big) \\
&= \mathscr{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}_k, q_k)\Big) - \mathscr{B}_{k-1}\Big(P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}_{k-1}, q_{k-1})\Big) \\
&= \int_\Omega w\tau\, dx \\
&\le \|w\|_{L^2(\Omega)}\|\tau\|_{L^2(\Omega)}.
\end{aligned} \tag{3.31}
$$

Since $(\underset{\sim}{\eta}, \tau) = (\underset{\sim}{\zeta}_k, \xi_k) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) + \Big((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)\Big)$, it follows from (3.8), (3.30), and Lemma 3.5 that

$$
\begin{aligned}
\|\tau\|_{L^2(\Omega)} &\le \|\xi_k - \xi_{k-1}\|_{L^2(\Omega)} + \frac{1}{h_k}|\!|\!|(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)|\!|\!|_{0,k-1} \\
&\le Ch_k\|\underset{\sim}{v}_k - \underset{\sim}{P}_k\underset{\sim}{v}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}.
\end{aligned} \tag{3.32}
$$

The estimates (3.31) and (3.32) together imply that

$$\|\underset{\sim}{v}_k - \underset{\sim}{P}_k\underset{\sim}{v}_{k-1}\|_{L^2(\Omega)} \le Ch_k\|w\|_{L^2(\Omega)}. \tag{3.33}$$

On the other hand, Proposition 2.7 implies that

$$\|\underset{\sim}{v}_{k-1}\|_{k-1} + \|q_{k-1}\|_{L^2(\Omega)} \le C\|w\|_{L^2(\Omega)} \tag{3.34}$$

and

$$\|\underset{\sim}{v}_k\|_k + \|q_k\|_{L^2(\Omega)} \le C\|w\|_{L^2(\Omega)}. \tag{3.35}$$

Therefore, we have

$$h_k\|q_k - q_{k-1}\|_{L^2(\Omega)} \le Ch_k\|w\|_{L^2(\Omega)}. \tag{3.36}$$

The estimate (3.29) now follows from (3.33) and (3.36). □

4. The multigrid algorithm

We first describe the kth-level iteration scheme. The full multigrid algorithm consists of a nested iteration of these schemes.

The kth-level iteration. The kth-level iteration with initial guess $(\underset{\sim}{y}_0, z_0) \in \underset{\sim}{V}_k^\perp \times Q_{k-1}$ yields $MG(k, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r))$ as an approximate solution to the following equation in $\underset{\sim}{V}_k^\perp \times Q_{k-1}$:

$$B_k^\perp(\underset{\sim}{y}, z) = (\underset{\sim}{w}, r). \tag{4.1}$$

For $k = 1$, $MG(1, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r))$ is the solution obtained from a direct method. In other words,

$$MG(1, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r)) = (B_1^\perp)^{-1}(\underset{\sim}{w}, r).$$

For $k > 1$, there are two steps:

Smoothing step. Let $(\underset{\sim}{y}_l, z_l) \in \underset{\sim}{V}_k \times Q_{k-1}$ be defined recursively by the equations

$$(4.2)\qquad \begin{aligned}(\underset{\sim}{y}_l, z_l) &= (\underset{\sim}{y}_{l-1}, z_{l-1}) \\ &\quad + \frac{1}{\Lambda_k} B_k\Big((\underset{\sim}{w}, r) - B_k(\underset{\sim}{y}_{l-1}, z_{l-1})\Big), \quad 1 \le l \le m,\end{aligned}$$

where m is a positive integer independent of the mesh parameter k and the Lamé constants (μ, λ), and $\Lambda_k := C h_k^{-4}$ (cf. Lemma 3.3) dominates the spectral radius of $(B_k^\perp)^2$.

Correction step. Let $(\overline{\underset{\sim}{w}}, \overline{r}) := I_k^{k-1}((\underset{\sim}{w}, r) - B_k(\underset{\sim}{y}_m, z_m))$. Let $(\underset{\sim}{v}_i, q_i) \in \underset{\sim}{V}_{k-1} \times Q_{k-2}$ $(0 \le i \le 2)$ be defined recursively by

$$(4.3)\qquad \begin{aligned}&(\underset{\sim}{v}_0, q_0) = (\underset{\sim}{0}, 0) \quad \text{and} \\ &(\underset{\sim}{v}_i, q_i) = MG(k-1, (\underset{\sim}{v}_{i-1}, q_{i-1}), (\overline{\underset{\sim}{w}}, \overline{r})), \quad i = 1, 2.\end{aligned}$$

Then $MG(k, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, g))$ is defined to be $(\underset{\sim}{y}_m, z_m) + I_{k-1}^k(\underset{\sim}{v}_2, q_2)$.

Remark. In the smoothing step we use B_k instead of $B_k^\perp$ for the following reason. The space $\underset{\sim}{V}_k \times Q_{k-1}$ has a natural coordinate system, namely the values of $\underset{\sim}{v}$ at the midpoints of $\mathcal{T}_k$ and the values of q on the triangles of $\mathcal{T}_{k-1}$, with respect to which $(\cdot, \cdot)_k$ is represented by a diagonal matrix, and B_k is represented by a sparse banded matrix. In view of Lemma 3.2, the result of the smoothing step $(\underset{\sim}{y}_m, z_m)$ belongs to $\underset{\sim}{V}_k^\perp \times Q_{k-1}$ automatically. Furthermore, the result of the correction step also belongs to $\underset{\sim}{V}_k^\perp \times Q_{k-1}$ by Lemma 3.1 (iii) and (iv). Therefore, in the actual implementation of the multigrid method we use only the natural coordinate system of $\underset{\sim}{V}_k \times Q_{k-1}$ for $k > 1$. We only have to handle the constraints of the $\underset{\sim}{V}_k^\perp$ spaces at the coarsest level, namely when $k = 1$.

The full multigrid algorithm. In the case $k = 1$, the approximate solution $(\underset{\sim}{u}_1^*, p_1^*)$ of (1.26) is obtained by a direct method. The approximate solutions $(\underset{\sim}{u}_k^*, p_k^*)$ $(k \ge 2)$ of (1.26) are obtained recursively from

$$\begin{aligned}&(\underset{\sim}{u}_0^k, p_0^k) = I_{k-1}^k(\underset{\sim}{u}_{k-1}^*, p_{k-1}^*), \\ &(\underset{\sim}{u}_l^k, p_l^k) = MG(k, (\underset{\sim}{u}_{l-1}^k, p_{l-1}^k), (\underset{\sim}{f}_k, 0)), \quad 1 \le l \le r, \\ &(\underset{\sim}{u}_k^*, p_k^*) = (\underset{\sim}{u}_r^k, p_r^k),\end{aligned}$$

where $\underset{\sim}{f}_k \in \underset{\sim}{V}_k^\perp$ satisfies

$$(\underset{\sim}{f}_k, \underset{\sim}{v})_{\underset{\sim}{L}^2(\Omega)} = \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \sum_{i=1}^n \int_{\Gamma_i} \underset{\sim}{g}_i \cdot \underset{\sim}{v}|_{\Gamma_i}\, ds\right]$$

for all $\underset{\sim}{v} \in \underset{\sim}{V}_k^\perp$, and r is a positive integer independent of k and (μ, λ).

Since I_{k-1}^{k}, I_{k}^{k-1}, and B_k are represented by sparse, banded matrices with $\mathcal{O}(n_k)$ nonzero entries, the total work of the full multigrid algorithm is therefore $\mathcal{O}(n_k)$. The proof is the same as the one in [1].

5. Convergence analysis

We follow the methodology of [1]. The details are given in the supplement to this paper. The first step is to discuss the convergence of the two-grid algorithm where the residual equation is solved exactly on the coarser grid. The final output of the two-grid algorithm for (4.1) is therefore $(\underset{\sim}{y}^{\sharp}, z^{\sharp}) := (\underset{\sim}{y}_m, z_m) + I_{k-1}^{k}(\underset{\sim}{v}^{\sharp}, q^{\sharp})$, where

$$\begin{aligned}(\underset{\sim}{v}^{\sharp}, q^{\sharp}) &= (B_{k-1}^{\perp})^{-1}(\overline{\underset{\sim}{w}}, \bar{r}) \\ &= (B_{k-1}^{\perp})^{-1} I_{k}^{k-1}\left((\underset{\sim}{w}, r) - B_k(\underset{\sim}{y}_m, z_m)\right) \\ &= (B_{k-1}^{\perp})^{-1} I_{k}^{k-1} B_k(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m).\end{aligned}$$

The following is the result for the two-grid algorithm.

Theorem (convergence of the two-grid algorithm). *There exists a positive constant C such that for $k > 1$*

$$|||(\underset{\sim}{y} - \underset{\sim}{y}^{\sharp}, z - z^{\sharp})|||_{0,k} \le C m^{-1/2} |||(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|||_{0,k},$$

where $(\underset{\sim}{y}, z)$ solves (4.1), $(\underset{\sim}{y}_0, z_0)$ is the initial guess, and $(\underset{\sim}{y}^{\sharp}, z^{\sharp})$ is the output of the two-grid algorithm. Therefore, for m sufficiently large, the two-grid algorithm is a contraction with contraction number bounded away from one, independent of k.

The convergence theorem for the kth-level iteration follows.

Theorem (convergence of the kth-level iteration). *There exists a positive constant C such that when the kth-level iteration is applied to (4.1), we have*

$$|||(\underset{\sim}{y}, z) - MG(k, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r))|||_{0,k} \le C m^{-1/2} |||(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|||_{0,k},$$

provided that m is chosen to be large enough. Therefore, for m sufficiently large, the kth-level iteration is a contraction with contraction number bounded away from one, independent of k.

A perturbation argument then gives full multigrid convergence.

Theorem (full multigrid convergence). *If m is chosen large enough so that the kth-level iteration is a contraction with respect to $|||\cdot|||_{0,k}$ and the parameter r in the full multigrid algorithm is also chosen large enough, then*

$$\begin{aligned}\|\underset{\sim}{u}_k - \underset{\sim}{u}_k^*\|_{\underset{\sim}{L}^2(\Omega)} &+ h_k\left(\|\underset{\sim}{u}_k - \underset{\sim}{u}_k^*\|_k + \|p_k - p_k^*\|_{L^2(\Omega)}\right) \\ &\le C h_k^2 \left\{\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)}\right\}.\end{aligned}$$

Here, $(\underset{\sim}{u}_k^*, p_k^*)$ is the approximate solution of (1.26) obtained from the full multigrid algorithm.

6. A NUMERICAL EXPERIMENT

The full multigrid algorithm described in §4 was applied to the pure traction boundary value problem (1.15) with $\mu = 1/2$. The domain Ω is the unit square, the vertices are $S_1 = (0, 0)$, $S_2 = (1, 0)$, $S_3 = (1, 1)$ and $S_4 = (0, 1)$, and $\mathcal{T}_0$ consists of $\triangle S_1S_2S_4$ and $\triangle S_2S_3S_4$. The body force $\underset{\sim}{f} = (f_1, f_2)^t$ is given by

$$f_1 = -\pi^2 \sin \pi x_1 \sin \pi x_2 + 2\pi^2 \left(\frac{1}{\lambda} + 1\right) \cos \pi x_1 \sin \pi x_2,$$

$$f_2 = -\pi^2 \cos \pi x_1 \cos \pi x_2 + 2\pi^2 \left(\frac{1}{\lambda} + 1\right) \sin \pi x_1 \cos \pi x_2,$$

and the tractions are given by

$$\underset{\sim}{g}_1 = \left(-\frac{\pi}{\lambda} \cos \pi x_1, 0\right)^t, \qquad \underset{\sim}{g}_2 = \left(\pi \sin \pi x_2, -\frac{\pi}{\lambda} \cos \pi x_2\right)^t,$$

$$\underset{\sim}{g}_3 = \left(-\frac{\pi}{\lambda} \cos \pi x_1, 0\right)^t, \quad \text{and} \quad \underset{\sim}{g}_4 = \left(\pi \sin \pi x_2, -\frac{\pi}{\lambda} \cos \pi x_2\right)^t.$$

The exact solution $\underset{\sim}{u} = (u_1, u_2)^t \in \underset{\sim}{H}^2_\perp$ is

$$u_1 = \left(-\sin \pi x_1 + \frac{1}{\lambda} \cos \pi x_1\right) \sin \pi x_2 + \frac{4}{\pi^2},$$

$$u_2 = \left(-\cos \pi x_1 + \frac{1}{\lambda} \sin \pi x_1\right) \cos \pi x_2.$$

In the following table, $\nu = \lambda/2(\frac{1}{2} + \lambda)$ is the Poisson ratio, h represents the lengths of the horizontal and vertical sides of the triangles in the triangulation, r is the number of nested iterations in the full multigrid algorithm, and the numbers in the third, fifth, and seventh (respectively fourth, sixth, and eighth) columns represent the number m of smoothing steps required to achieve an L^2 relative error of less that 5% (respectively 3%) in the displacement. It is observed that this algorithm fails to converge for this problem for $m \le 4$.

ν	h	$r = 20$		$r = 30$		$r = 40$	
		m(5%)	m(3%)	m(5%)	m(3%)	m(5%)	m(3%)
0.47619	$(0.5)^6$	13	19	9	13	7	10
	$(0.5)^7$	7	9	5	6	5	5
	$(0.5)^8$	5	5	5	5	5	5
0.49751	$(0.5)^6$	20	27	15	19	12	15
	$(0.5)^7$	11	14	8	10	6	8
	$(0.5)^8$	6	8	5	5	5	5
0.49975	$(0.5)^6$	21	28	15	20	12	16
	$(0.5)^7$	12	15	8	11	7	9
	$(0.5)^8$	6	8	5	6	5	5

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Supplement to

A NONCONFORMING MIXED MULTIGRID METHOD FOR THE PURE TRACTION PROBLEM IN PLANAR LINEAR ELASTICITY

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This supplement contains the proofs of the theorems stated in §5.

Lemma S.1. *There holds* $(\underset{\sim}{v}^\sharp, q^\sharp) = P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m)$.

Proof. Note that $(\underset{\sim}{v}^\sharp, q^\sharp)$ and $P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m)$ both belong to $\underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}$. Given any $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}$, it follows from (3.4) and (3.11) that

$$\begin{aligned} \mathcal{B}_{k-1}\left((\underset{\sim}{v}^\sharp, q^\sharp), (\underset{\sim}{v}, q)\right) &= \left(B_{k-1}^\perp(\underset{\sim}{v}^\sharp, q^\sharp), (\underset{\sim}{v}, q)\right)_{k-1} \\ &= \left(I_k^{k-1} B_k(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), (\underset{\sim}{v}, q)\right)_{k-1} \\ &= \left(B_k(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), I_{k-1}^k(\underset{\sim}{v}, q)\right)_k \\ &= \mathcal{B}_k\left((\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), I_{k-1}^k(\underset{\sim}{v}, q)\right) \\ &= \mathcal{B}_{k-1}\left(P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), (\underset{\sim}{v}, q)\right). \end{aligned}$$

The lemma now follows from Proposition 2.7. □

Let the kth-level relaxation operator R_k be defined by

$$R_k := I - \frac{1}{\Lambda_k}\left(B_k^\perp\right)^2. \tag{S.1}$$

(Recall that $\Lambda_k := C\, h_k^{-4}$ dominates the spectral radius of $(B_k^\perp)^2$.)

Note that by the definition of the mesh-dependent norms we have

$$\| R_k(\underset{\sim}{v}, q) \|_{s,k} \le \| (\underset{\sim}{v}, q) \|_{s,k} \quad \forall\, (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^\perp \times Q_{k-1} \text{ and } s \in \mathbb{R}. \tag{S.2}$$

From the smoothing step (4.2), we obtain

$$(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m) = R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0). \tag{S.3}$$

Combining Lemma S.1 and (S.3), we have the following relation between the initial error and the final error of the two-grid algorithm:

$$(\underset{\sim}{y} - \underset{\sim}{y}^\sharp, z - z^\sharp) = (I - I_{k-1}^k P_k^{k-1}) R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0). \tag{S.4}$$

The effect of R_k^m is measured by the following lemma on the smoothing property.

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Lemma S.2 (Smoothing Property). *There exists a positive constant C such that*

$$|\!|\!| R_k^m(\utilde{v},q) |\!|\!|_{2,k} \le C\, h_k^{-2}\, m^{-1/2}\, |\!|\!| (\utilde{v},q) |\!|\!|_{0,k} \quad \forall (\utilde{v},q) \in \utilde{V}_k^{\perp} \times Q_{k-1}. \tag{S.5}$$

Proof. Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n_k}$ be the eigenvalues of $(B_k^{\perp})^2$ and $\boldsymbol{\eta}_i := (\utilde{\phi}_i, \psi_i)$, $1 \le i \le n_k$ be the corresponding eigenvectors satisfying the orthonormal relation $(\boldsymbol{\eta}_i, \boldsymbol{\eta}_j)_k = \delta_{ij}$. We can write $(\utilde{v}, q) = \sum_{i=1}^{n_k} \nu_i \boldsymbol{\eta}_i$. Hence,

$$\begin{aligned} R_k^m(\utilde{v},q) &= \left(I - \frac{1}{\Lambda_k}(B_k^{\perp})^2\right)^m (\utilde{v},q) \\ &= \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^m \nu_i \boldsymbol{\eta}_i. \end{aligned}$$

Hence,

$$\begin{aligned} |\!|\!| R_k^m(\utilde{v},q) |\!|\!|_{2,k}^2 &= \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^{2m} \lambda_i \nu_i^2 \\ &= \Lambda_k \left\{ \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^{2m} \left(\frac{\lambda_i}{\Lambda_k}\right) \nu_i^2 \right\} \\ &= \Lambda_k \left\{ \sup_{0 \le x \le 1} (1-x)^{2m} x \right\} \sum_{i=1}^{n_k} \nu_i^2 \\ &\le C\, h_k^{-4}\, m^{-1}\, |\!|\!| (\utilde{v},q) |\!|\!|_{0,k}^2. \quad \square \end{aligned}$$

The following lemma on the approximation property measures the effect of the operators $I - I_{k-1}^k P_k^{k-1}$.

Lemma S.3 (Approximation Property). *There exists a positive constant C such that for $k > 1$*

$$|\!|\!| (I - I_{k-1}^k P_k^{k-1})(\utilde{v},q) |\!|\!|_{0,k} \le C\, h_k^2\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k} \quad \forall (\utilde{v},q) \in \utilde{V}_k^{\perp} \times Q_{k-1}. \tag{S.6}$$

Proof. Given $(\utilde{v},q) \in \utilde{V}_k^{\perp} \times Q_{k-1}$, let $(\utilde{\eta},\tau) = P_k^{k-1}(\utilde{v},q)$. Hence,

$$(I - I_{k-1}^k P_k^{k-1})(\utilde{v},q) = (\utilde{v} - \utilde{P}_k \utilde{\eta}, q - \tau). \tag{S.7}$$

Recall that $|\!|\!| (\utilde{v} - \utilde{P}_k\utilde{\eta}, q-\tau) |\!|\!|_{0,k}^2 = \|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)}^2 + h_k^2 \|q-\tau\|_{L^2(\Omega)}^2$. We shall estimate $h_k \|q-\tau\|_{L^2(\Omega)}$ and $\|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)}$ by two duality arguments.

Let $(\utilde{v}_k^0, q_k^0) \in \utilde{V}_k^{\perp} \times Q_{k-1}$ satisfy

$$\mathcal{B}_k\left((\utilde{v}_k^0, q_k^0), (\utilde{v}', q')\right) = h_k^2 \int_\Omega (q-\tau)\, q'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_k^{\perp} \times Q_{k-1}, \tag{S.8}$$

and $(\utilde{v}_{k-1}^0, q_{k-1}^0) \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}$ satisfy

$$\mathcal{B}_{k-1}\left((\utilde{v}_{k-1}^0, q_{k-1}^0), (\utilde{v}', q')\right) = h_k^2 \int_\Omega (q-\tau)\, q'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}. \tag{S.9}$$

It follows from (S.8), (S.9), the definition of $(\utilde{\eta},\tau)$, (3.11), (3.9) and Lemma 3.7 that

$$\begin{aligned} h_k^2 \|q-\tau\|_{L^2(\Omega)}^2 &= h_k^2 \int_\Omega (q-\tau)\, q\, dx - h_k^2 \int_\Omega (q-\tau)\, \tau\, dx \\ &= \mathcal{B}_k((\utilde{v}_k^0, q_k^0), (\utilde{v}, q)) - \mathcal{B}_{k-1}((\utilde{v}_{k-1}^0, q_{k-1}^0), (\utilde{\eta}, \tau)) \\ &= \mathcal{B}_k((\utilde{v}_k^0, q_k^0), (\utilde{v}, q)) - \mathcal{B}_{k-1}((\utilde{v}_{k-1}^0, q_{k-1}^0), P_k^{k-1}(\utilde{v}, q)) \\ &= \mathcal{B}_k((\utilde{v}_k^0, q_k^0) - I_{k-1}^k(\utilde{v}_{k-1}^0, q_{k-1}^0), (\utilde{v}, q)) \\ &\le |\!|\!| (\utilde{v}_k^0, q_k^0) - I_{k-1}^k(\utilde{v}_{k-1}^0, q_{k-1}^0) |\!|\!|_{0,k}\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k} \\ &\le C\, h_k^3\, \|q-\tau\|_{L^2(\Omega)}\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k}. \end{aligned}$$

Therefore, we have

$$h_k \|q-\tau\|_{L^2(\Omega)} \le C\, h_k^2\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k}. \tag{S.10}$$

We now turn to the estimate of $\|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)}$. Since $\utilde{v} - \utilde{P}_k\utilde{\eta} \in \utilde{L}_{\perp}^2$, there exists a unique $(\utilde{\zeta}, \xi) \in \utilde{H}_{\perp}^2 \times H^1(\Omega)$ satifying

$$\mathcal{B}\left((\utilde{\zeta}, \xi), (\utilde{v}', q')\right) = \int_\Omega (\utilde{v} - \utilde{P}_k\utilde{\eta}) \cdot \utilde{v}'\, dx \quad \forall (\utilde{v}', q') \in \utilde{H}_{\perp}^1(\Omega) \times L^2(\Omega). \tag{S.11}$$

Let $(\utilde{\zeta}_k, \xi_k) \in \utilde{V}_k^{\perp} \times Q_{k-1}$ satisfy

$$\mathcal{B}_k\left((\utilde{\zeta}_k, \xi_k), (\utilde{v}', q')\right) = \int_\Omega (\utilde{v} - \utilde{P}_k\utilde{\eta}) \cdot \utilde{v}'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_k^{\perp} \times Q_{k-1}, \tag{S.12}$$

and $(\utilde{\zeta}_{k-1}, \xi_{k-1}) \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}$ satisfy

$$\mathcal{B}_{k-1}\left((\utilde{\zeta}_{k-1}, \xi_{k-1}), (\utilde{v}', q')\right) = \int_\Omega (\utilde{v} - \utilde{P}_k\utilde{\eta}) \cdot \utilde{v}'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}. \tag{S.13}$$

The elliptic regularity estimate (1.20) implies that (recall that $\xi = \frac{\lambda}{2\mu} \operatorname{div} \utilde{\zeta}$)

$$\|\utilde{\zeta}\|_{\utilde{H}^2(\Omega)} + \|\xi\|_{H^1(\Omega)} \le C\, \|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)}, \tag{S.14}$$

and the discretization error estimates in Theorems 2.9 and 2.10 imply that

$$\text{(S.15)}\qquad \begin{aligned}&\|\underset{\sim}{\zeta}_k - \underset{\sim}{\zeta}\|_{L^2(\Omega)} + \|\underset{\sim}{\zeta}_{k-1} - \underset{\sim}{\zeta}\|_{L^2(\Omega)}\\ &+ h_k\|\xi_k - \xi\|_{L^2(\Omega)} + h_k\|\xi_{k-1} - \xi\|_{L^2(\Omega)}\\ &\le C\, h_k^2\, \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}.\end{aligned}$$

The interpolation error estimates (2.30), (2.31) and (S.14) imply that

$$\text{(S.16)}\qquad \begin{aligned}&\|\Pi_k^{\perp}\underset{\sim}{\zeta} - \underset{\sim}{\zeta}\|_{L^2(\Omega)} + \|\Pi_{k-1}^{\perp}\underset{\sim}{\zeta} - \underset{\sim}{\zeta}\|_{L^2(\Omega)}\\ &+ h_k\|P_{k-1}\xi - \xi\|_{L^2(\Omega)} + h_k\|P_{k-2}\xi - \xi\|_{L^2(\Omega)}\\ &\le C\, h_k^2\, \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}.\end{aligned}$$

From the definition of $(\underset{\sim}{\eta}, \tau)$, (S.12), (S.13) and (3.11) we have

$$\text{(S.17)}\qquad \begin{aligned}&\|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|^2_{L^2(\Omega)}\\ &= \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)\Big) - \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), I_{k-1}^k(\underset{\sim}{\eta}, \tau)\Big)\\ &= \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k) - (\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi), (\underset{\sim}{v}, q)\Big)\\ &+ \mathcal{B}_k\Big((\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi) - I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi), (\underset{\sim}{v}, q)\Big)\\ &+ \mathcal{B}_k\Big(I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi), (\underset{\sim}{v}, q)\Big) - \int_\Omega(\underset{\sim}{v} - \underset{\sim}{P_k\eta})\cdot \underset{\sim}{P_k\eta}\, dx\\ &= \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k) - (\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi), (\underset{\sim}{v}, q)\Big)\\ &+ \mathcal{B}_k\Big((\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi) - I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi), (\underset{\sim}{v}, q)\Big)\\ &+ \mathcal{B}_{k-1}\Big((\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), P_k^{k-1}(\underset{\sim}{v}, q)\Big)\\ &+ \int_\Omega(\underset{\sim}{v} - \underset{\sim}{P_k\eta})(\underset{\sim}{\eta} - \underset{\sim}{P_k\eta})\, dx.\end{aligned}$$

Combining (3.9), (S.15), (S.16), (3.12), Lemma 3.4 (iii), Lemma 3.6, (S.14) and the definition of $(\underset{\sim}{\eta}, \tau)$, we have

$$\text{(S.18)}\qquad \begin{aligned}&\|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|^2_{L^2(\Omega)}\\ &\le |\!|\!|(\underset{\sim}{\zeta}_k, \xi_k) - (\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi)|\!|\!|_{0,k}\,|\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k}\\ &+ |\!|\!|(\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi) - I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi)|\!|\!|_{0,k}\,|\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k}\\ &+ |\!|\!|(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})|\!|\!|_{0,k-1}\,|\!|\!|P_k^{k-1}(\underset{\sim}{v}, q)|\!|\!|_{2,k-1}\\ &+ \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}\|\underset{\sim}{\eta} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}\\ &\le C\, h_k^2\, \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}|\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k}\\ &+ \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}\|\underset{\sim}{\eta} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}\\ &\le C\, h_k^2\, \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}\left\{|\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k} + |\!|\!|(\underset{\sim}{\eta}, \tau)|\!|\!|_{2,k-1}\right\}\\ &\le C\, h_k^2\, \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)}\,|\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k}\,.\end{aligned}$$

Hence,

$$\text{(S.19)}\qquad \|\underset{\sim}{v} - \underset{\sim}{P_k\eta}\|_{L^2(\Omega)} \le C\, h_k^2\, |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k}\,.$$

The lemma now follows from (S.10) and (S.19). □

The convergence of the two-grid algorithm is obtained by combining the smoothing and approximation properties.

Theorem S.4 (Convergence of the Two-Grid Algorithm). *There exists a positive constant C such that for $k > 1$*

$$\text{(S.20)}\qquad |\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}^\sharp, z - z^\sharp)|\!|\!|_{0,k} \le C\, m^{-1/2}\, |\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{0,k},$$

where $(\underset{\sim}{y}, z)$ solves (4.1), $(\underset{\sim}{y}_0, z_0)$ is the initial guess, and $(\underset{\sim}{y}^\sharp, z^\sharp)$ is the output of the two-grid algorithm. Therefore, for m sufficiently large, the two-grid algorithm is a contraction with contraction number bounded away from one, independent of k.

Proof. Using (S.4), Lemmas S.3 and S.2, we have

$$\begin{aligned}|\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}^\sharp, z - z^\sharp)|\!|\!|_{0,k} &= |\!|\!|(I - I_{k-1}^k P_k^{k-1}) R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{0,k}\\ &\le C\, h_k^2\, |\!|\!|R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{2,k}\\ &\le C\, m^{-1/2}|\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{0,k}. \quad □\end{aligned}$$

A perturbation argument then yields the following theorem on the convergence of the kth-level iteration.

Theorem S.5 (Convergence of the kth-Level Iteration). *There exists a positive constant C such that when the kth-level iteration is applied to* (4.1), *we have*

$$\text{(S.21)}\qquad |\!|\!|(\utilde{y}, z) - MG(k, (\utilde{y}_0, z_0), (\utilde{w}, r))|\!|\!|_{0,k} \le C\, m^{-1/2}\, |\!|\!|(\utilde{y} - \utilde{y}_0, z - z_0)|\!|\!|_{0,k},$$

provided that m is chosen to be large enough. Therefore, for m sufficiently large, the kth-level iteration is a contraction with contraction number bounded away from one, independent of k.

Proof. Let C_* be the maximum of the constants that appear in Lemma 3.4 (i), (3.12) and Theorem S.4. Then (S.21) holds for $C = 2C_*$ and $m \ge 16C_*^6$. The proof is by induction. Since the first-level iteration is a direct solver, (S.21) clearly holds for $k = 1$. Assume (S.21) holds for $k-1$ ($k > 1$). Then from §4 and the definition of $(\utilde{y}^\sharp, z^\sharp)$,

$$(\utilde{y}, z) - MG(k, (\utilde{y}_0, z_0), (\utilde{w}, r)) = (\utilde{y} - \utilde{y}^\sharp, z - z^\sharp) + I_{k-1}^k \left[(\utilde{v}^\sharp - \utilde{v}_2, q^\sharp - q_2)\right].$$

Therefore, Theorem S.4 and Lemma 3.4 (i) imply that

$$\text{(S.22)}\qquad \begin{aligned} &|\!|\!|(\utilde{y}, z) - MG(k, (\utilde{y}_0, z_0), (\utilde{w}, r)|\!|\!|_{0,k} \\ &\le C_* m^{-1/2} |\!|\!|(\utilde{y} - \utilde{y}_0, z - z_0)|\!|\!|_{0,k} + C_* |\!|\!|(\utilde{v}^\sharp - \utilde{v}_2, q^\sharp - q_2)|\!|\!|_{0,k-1}. \end{aligned}$$

Recall that $(\utilde{v}^\sharp, q^\sharp)$ is the exact solution of

$$\text{(S.23)}\qquad B_{k-1}^\perp(\utilde{v}, q) = I_k^{k-1}\left((\utilde{w}, r) - B_k(\utilde{y}_m, z_m)\right),$$

and $(\utilde{v}_2, q_2)$ is the approximate solution of (S.23) obtained by using the $(k-1)$-level iteration twice with initial guess $(\utilde{0}, 0)$. Hence, it follows from the induction hypothesis, Lemma S.1, (3.12), (S.3) and (S.2) that

$$\text{(S.24)}\qquad \begin{aligned} |\!|\!|(\utilde{v}^\sharp - \utilde{v}_2, q^\sharp - q_2)|\!|\!|_{0,k-1} &\le 4C_*^2 m^{-1} |\!|\!|(\utilde{v}^\sharp, q^\sharp)|\!|\!|_{0,k-1} \\ &= 4C_*^2 m^{-1} |\!|\!|P_k^{k-1}(\utilde{y} - \utilde{y}_m, z - z_m)|\!|\!|_{0,k-1} \\ &\le 4C_*^3 m^{-1} |\!|\!|(\utilde{y} - \utilde{y}_m, z - z_m)|\!|\!|_{0,k} \\ &= 4C_*^3 m^{-1} |\!|\!|R_k^m(\utilde{y} - \utilde{y}_0, z - z_0)|\!|\!|_{0,k} \\ &\le 4C_*^3 m^{-1} |\!|\!|(\utilde{y} - \utilde{y}_0, z - z_0)|\!|\!|_{0,k}. \end{aligned}$$

Combining (S.22) and (S.24), we have by the condition on m that

$$\text{(S.25)}\qquad \begin{aligned} &|\!|\!|(\utilde{y}, z) - MG(k, (\utilde{y}_0, z_0), (\utilde{w}, r)|\!|\!|_{0,k} \\ &\le \left(C_* m^{-1/2} + 4C_*^4 m^{-1}\right) |\!|\!|(\utilde{y} - \utilde{y}_0, z - z_0)|\!|\!|_{0,k} \\ &\le 2\,C_*\, m^{-1/2} |\!|\!|(\utilde{y} - \utilde{y}_0, z - z_0)|\!|\!|_{0,k}. \quad \square \end{aligned}$$

We finally turn to the convergence of the full multigrid algorithm. First we establish the following lemma.

Lemma S.6. *There exists a positive constant C such that*

$$\text{(S.26)}\qquad |\!|\!|(\utilde{u}_k, p_k) - I_{k-1}^k(\utilde{u}_{k-1}, p_{k-1})|\!|\!|_{0,k} \le C\, h_k^2 \left\{ \|\utilde{f}\|_{L^2(\Omega)} + \sum_{i=1}^n \|\utilde{g}_i\|_{\utilde{H}^{1/2}(\Gamma_i)} \right\}.$$

Here, $(\utilde{u}_k, p_k)$ and $(\utilde{u}_{k-1}, p_{k-1})$ are the exact solutions of the discretized problem (1.26) *corresponding to the triangulations $\mathcal{T}_k$ and $\mathcal{T}_{k-1}$, respectively.*

Proof. Using (1.20), (1.31), (2.30), (2.31), Theorems 2.9 and 2.10, (3.8) and Lemma 3.4, we have

$$\begin{aligned} &|\!|\!|(\utilde{u}_k, p_k) - I_{k-1}^k(\utilde{u}_{k-1}, p_{k-1})|\!|\!|_{0,k} \\ &\le |\!|\!|(\utilde{u}_k - \Pi_k^\perp \utilde{u}, p_k - P_{k-1}p)|\!|\!|_{0,k} + |\!|\!|(\Pi_k^\perp \utilde{u}, P_{k-1}p) - I_{k-1}^k(\Pi_{k-1}^\perp \utilde{u}, P_{k-2}p)|\!|\!|_{0,k} \\ &\quad + |\!|\!|I_{k-1}^k(\Pi_{k-1}^\perp \utilde{u} - \utilde{u}_{k-1}, P_{k-2}p - p_{k-1})|\!|\!|_{0,k} \\ &\le C\, h_k^2 \left\{ \|\utilde{f}\|_{L^2(\Omega)} + \sum_{i=1}^n \|\utilde{g}_i\|_{\utilde{H}^{1/2}(\Gamma_i)} \right\}. \quad \square \end{aligned}$$

Theorem S.7 (Full Multigrid Convergence). *If m is chosen large enough so that the kth-level iteration is a contraction with respect to $|\!|\!|\cdot|\!|\!|_{0,k}$ and the parameter r in the full multigrid algorithm is also chosen large enough, then*

$$\text{(S.27)}\qquad \begin{aligned} &\|\utilde{u}_k - \utilde{u}_k^*\|_{L^2(\Omega)} + h_k \left(\|\utilde{u}_k - \utilde{u}_k^*\|_k + \|p_k - p_k^*\|_{L^2(\Omega)} \right) \\ &\le C\, h_k^2 \left\{ \|\utilde{f}\|_{L^2(\Omega)} + \sum_{i=1}^n \|\utilde{g}_i\|_{\utilde{H}^{1/2}(\Gamma_i)} \right\}. \end{aligned}$$

Here, $(\utilde{u}_k^*, p_k^*)$ is the approximate solution of (1.26) *obtained from the full multigrid algorithm.*

Proof. By Theorem S.5 we can choose m sufficiently large so that the kth-level iteration ($k = 1, 2, 3, \cdots$) has a contraction number bounded by δ, where $0 < \delta < 1$ is independent of k.

From the full multigrid algorithm, Lemma S.6 and Lemma 3.4 (i), there exists a positive constant C^* such that

$$\text{(S.28)}\qquad \begin{aligned} &|\!|\!|(\utilde{u}_k - \utilde{u}_k^*, p_k - p_k^*)|\!|\!|_{0,k} \\ &\le \delta^r |\!|\!|(\utilde{u}_k, p_k) - I_{k-1}^k(\utilde{u}_{k-1}^*, p_{k-1}^*)|\!|\!|_{0,k} \\ &\le \delta^r \Big\{ |\!|\!|(\utilde{u}_k, p_k) - I_{k-1}^k(\utilde{u}_{k-1}, p_{k-1})|\!|\!|_{0,k} \\ &\qquad + |\!|\!|I_{k-1}^k(\utilde{u}_{k-1} - \utilde{u}_{k-1}^*, p_{k-1} - p_{k-1}^*)|\!|\!|_{0,k} \Big\} \\ &\le C^* \delta^r h_k^2 \left\{ \|\utilde{f}\|_{L^2(\Omega)} + \sum_{i=1}^n \|\utilde{g}_i\|_{\utilde{H}^{1/2}(\Gamma_i)} \right\} \\ &\qquad + C^* \delta^r |\!|\!|(\utilde{u}_{k-1} - \utilde{u}_{k-1}^*, p_{k-1} - p_{k-1}^*)|\!|\!|_{0,k-1}. \end{aligned}$$

Since the first-level iteration is an exact solver, by iterating (S.28) we obtain

$$
\begin{aligned}
\text{(S.29)} \quad & |\!|\!|(\underset{\sim}{u}_k - \underset{\sim}{u}_k^*, p_k - p_k^*)|\!|\!|_{0,k} \\
&\le \left(C^*\delta^r h_k^2 + (C^*)^2\delta^{2r}h_{k-1}^2 + \cdots + (C^*)^{k-1}\delta^{(k-1)r}h_2^2\right) \\
&\qquad \cdot \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\} \\
&= \left(C^*\delta^r + 4(C^*)^2\delta^{2r} + \cdots + 4^{k-2}(C^*)^{k-1}\delta^{(k-1)r}\right) h_k^2 \\
&\qquad \cdot \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\} \\
&\le \left(\frac{C^*\delta^r}{1-4C^*\delta^r}\right) h_k^2 \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\}
\end{aligned}
$$

if $4C^*\delta^r < 1$, or equivalently, if $r \ge |(\ln 4C^*)/(\ln \delta)|$. Therefore, for such choice of r we have by (3.8) that

$$
\text{(S.30)} \quad \|\underset{\sim}{u}_k - \underset{\sim}{u}_k^*\|_{\underset{\sim}{L}^2(\Omega)} + h_k\|p_k - p_k^*\|_{L^2(\Omega)} \le Ch_k^2 \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\}.
$$

The inverse estimate (3.6) then implies

$$
\text{(S.31)} \quad \|\underset{\sim}{u}_k - \underset{\sim}{u}_k^*\|_k \le Ch_k \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\}. \quad \square
$$