IMPRIMITIVE NINTH-DEGREE NUMBER FIELDS WITH SMALL DISCRIMINANTS

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Abstract. We present tables of ninth-degree nonprimitive (i.e., containing a cubic subfield) number fields. Each table corresponds to one signature, except for fields with signature (3,3), for which we give two different tables depending on the signature of the cubic subfield. Details related to the computation of the tables are given, as well as information about the CPU time used, the number of polynomials that we deal with, etc. For each field in the tables, we give its discriminant, the discriminant of its cubic subfields, the relative polynomial generating the field over one of its cubic subfields, the corresponding (irreducible) polynomial over Q, and the Galois group of the Galois closure. Fields having interesting properties are studied in more detail, especially those associated with sextic number fields having a class group divisible by 3.

1. Introduction

The computational time required for the construction of tables of number fields, using the methods known at the present time, grows exponentially with the degree of the fields under consideration.

For this reason, extensive tables of number fields exist only for degrees up to six ([1, 2, 5, 6, 10, 11, 12, 14, 19, 21, 22, 23, 24, 29]).

Martinet’s generalization [20] of the Hunter-Pohst theorem [26] has made it possible to study the relative extensions using the same methods as those used in the primitive case. As a result, many extensive tables have been constructed for sextic fields ([21, 22, 23, 24]), as well as the first minima for discriminants of totally imaginary [9] and totally real [27] octic fields.

All relative extensions known up to date are either relative extensions of a quadratic field (the task in this case is easy because the ground field is abelian), or are quadratic extensions of a number field (and in this case, class field theory can be used, avoiding most of the calculations [27]).

What will be studied here is the case of imprimitive ninth-degree number fields, i.e., relative cubic extensions of a cubic field. To do this, tables of cubic fields with a sufficient amount of arithmetic data are needed. The advantage of working with relative extensions is that the computation only involves relative cubic polynomials (i.e., having only 3 coefficients).

Recently, A.M. Odlyzko advised us that H. Fujita [15] discovered the first three imprimitive totally real number fields of degree nine.

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In this paper, six tables of imprimitive ninth-degree fields are presented. Each table corresponds to one signature, except for signature (3,3), for which two tables are given, one for each conjugacy class of the Frobenius at infinity: Table (3, 3)$_R$ contains number fields having a real cubic subfield; table (3, 3)$_C$ contains number fields having a complex cubic subfield; naturally, the intersection of these two tables is not empty.

Minimal discriminants of imprimitive ninth-degree fields are given for each signature in Table I.

In §2, Martinet's theorem mentioned previously is stated and some explanation concerning its use in the study of the relative extensions is also given. Special emphasis is placed on the algorithms used to obtain faster computations.

The description of the six tables is given in §3. The CPU time needed to compute the table and the number of polynomials studied are also specified.

Section 4 contains a more detailed analysis of several interesting fields found in the tables. In particular, the existence of some nonisomorphic fields having the same discriminant is proven as well as the relation between these fields and certain sextic fields with a class number divisible by 3 or 9.

Finally, in §5, the Galois group of the Galois closure of each field appearing in the tables is computed.

The complete tables are available on floppy disk (source TeX); contact the authors by e-mail.

### 2. Relative cubic extensions of a cubic field

For a number field $L$, its ring of integers is denoted by $\mathbb{Z}_L$ and its discriminant by $d_L$. Throughout the paper, $K$ will denote an imprimitive ninth-degree number field. Because $K$ is assumed imprimitive, it contains, at least, one cubic subfield denoted by $k$. So, $K/k$ is a relative cubic extension, and we denote its relative ideal discriminant by $\delta$.

According to the formula $d_K = (-1)^s |d_k|^3 N_{k/Q}(\delta)$, where $(r, s)$ is the signature of $K$ and $N_{k/Q}$ the norm of the extension $k/Q$, the inequality $|d_k| \leq |d_K|^{1/3}$ is valid for all the possible subfields $k$ of $K$.

There are three $\mathbb{Q}$-embeddings of $k$ into $\mathbb{C}$, denoted by $\sigma_1$, $\sigma_2$, $\sigma_3$; each of them can be extended in three different ways to give a $\mathbb{Q}$-embedding of $K$ into $\mathbb{C}$; we denote by $\sigma'_j$, $j = 1, 2, 3$, the $\mathbb{Q}$-embeddings of $K$ in $\mathbb{C}$ whose restriction to $k$ is $\sigma_i$, $i = 1, 2, 3$. 

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### Table I

<table>
<thead>
<tr>
<th>signature</th>
<th>subfield</th>
<th>minimal discriminant</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9, 0)</td>
<td>real</td>
<td>162403 85609 = 493 \cdot 138041</td>
</tr>
<tr>
<td>(7, 1)</td>
<td>real</td>
<td>-26681 61671 = -493 \cdot 22679</td>
</tr>
<tr>
<td>(5, 2)</td>
<td>real</td>
<td>4678 90073 = 493 \cdot 41 \cdot 97</td>
</tr>
<tr>
<td>(3, 3)$_R$</td>
<td>real</td>
<td>-1645 90951 = -493 \cdot 1399</td>
</tr>
<tr>
<td>(3, 3)$_C$</td>
<td>complex</td>
<td>-1108 52311 = -31^3 \cdot 61^2</td>
</tr>
<tr>
<td>(1, 4)</td>
<td>complex</td>
<td>322 06049 = 23^3 \cdot 2647</td>
</tr>
</tbody>
</table>
For each \( \theta \in K \), the ‘relative trace’ \( Tr_i \) for \( K/k \) is defined as the trace of \( \sigma_i^j(\theta) \in \sigma_i^j(K) \) relative to \( \sigma_i(k) \) for \( i = 1, 2, 3 \). Thus, we have \( Tr_i(\theta) = \sigma_i^1(\theta) + \sigma_i^2(\theta) + \sigma_i^3(\theta) \).

The following theorem is essential:

**Theorem 1** (Martinet [20]). There exists an integer \( \theta \in \mathbb{Z}_K \), \( \theta \notin k \), such that

\[
\sum_{i=1}^{3} \sum_{j=1}^{3} |\sigma_i^j(\theta)|^2 \leq \frac{1}{3} \sum_{i=1}^{3} |Tr_i(\theta)|^2 + \left( \frac{64|d_K|}{81|d_k|} \right)^{1/6}.
\]

Moreover, if \( \theta \) satisfies (*) the same is true for all integers having the form \( \theta + a \) with \( a \in \mathbb{Z}_k \) as well as for \(-\theta\).

If the element \( \theta \in \mathbb{Z}_K \) whose existence is asserted in Theorem 1 is not a primitive element of the extension \( K/Q \), then \( K \) must contain at least two different cubic subfields. Fields having this property can be easily computed directly from tables of cubic fields. Thus, it can be assumed in what follows that \( K = Q(\theta) = k(\theta) \), where \( \theta \) is a root of a monic irreducible relative cubic polynomial \( P(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}_k[X] \).

Let us define

\[
\sigma_i(P(X)) = P_i(X) = X^3 + \sigma_i(a)X^2 + \sigma_i(b)X + \sigma_i(c) = X^3 + a_iX^2 + b_iX + c_i
\]

for each \( i = 1, 2, 3 \). Then \( f(X) = P_1(X)P_2(X)P_3(X) \) is the minimal polynomial of \( \theta \) over \( Q \).

For \( 1 \leq r \leq 9 \), \( r \) odd, and \( s = (9 - r)/2 \), let \( B_r > 0 \) be an upper bound depending on the signature \( (r, s) \) (the reasons for the choice of the value of these constants will be explained in the next section). Let us suppose that \( r \) is fixed and denote \( B_r \) by \( B \). To construct all the fields \( K \) of signature \( (r, s) \) satisfying \( |d_K| \leq B \), it is necessary to consider every cubic field \( k \) having a discriminant \( d_k \) such that \( |d_k| \leq B^{1/3} \). Moreover, the signature \( (r', s') \) of \( k \) must be compatible with the signature of \( K \), i.e., we must have \( s \geq 3s' \). For such a cubic field \( k \), let \( \{1, \alpha, \beta\} \) be an integral basis.

To compute a field \( K \) belonging to the table is equivalent to computing an irreducible polynomial \( P(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}_k[X] \) such that \( K = k(\theta) \) and \( P(\theta) = 0 \). The coefficients \( a, b, c \) of this polynomial can be determined in the following manner.

Define \( T_2(\theta) = \sum_{i=1}^{3} \sum_{j=1}^{3} |\sigma_i^j(\theta)|^2 \). Martinet's theorem shows that every field \( K \) with \( |d_K| \leq B \) contains an integer \( \theta \) such that

\[
T_2(\theta) \leq \frac{1}{3} \sum_{i=1}^{3} |a_i|^2 + C' \quad \text{with} \quad C' = \left( \frac{64B}{81|d_k|} \right)^{1/6}.
\]

The second part of the same theorem asserts that the first coefficient \( a \) of \( P(X) \) can be chosen as \( a = x_1 + x_2 \alpha + x_3 \beta \) with \( x_1, x_2, x_3 \in \{0, 1, -1\} \) and satisfying \( 100x_1 + 10x_2 + x_3 \geq 0 \); there are only 14 possible choices for this coefficient.

If the value of \( a \) has been fixed, the value of \( T_2(\theta) \) is bounded by a real constant \( C = C(a) \) depending only on the given value of \( a \). The following lemma is very useful from a practical point of view.
Lemma 1. For every field $K$ containing a cubic subfield $k$ there exists $a \in \mathbb{Z}_k$ such that

(i) $P(\theta) = 0$ and $K = k(\theta)$;
(ii) \( \frac{1}{2} \sum_{i=1}^{3} |a_i|^2 + C' = C \) is minimum.

Proof. From Theorem 1, the value of $a$ can be chosen among 14 values. If $\theta$ is a root of $P(X)$ generating the extension $K/k$, then the integer $\theta + y$ with $y \in \mathbb{Z}_k$ is also a primitive element of $K/k$. Define $-a' = Tr_{K/k}(\theta + y) = -a + 3y$; because $\sum_{i=1}^{3} |\sigma_i(a')|^2$ is a positive definite quadratic form, there exists $y$ minimizing it (the value of $y$ can be obtained by a back-tracking method) and the value of $a$ has to be replaced by this minimum and the value of $C$ is $C = \frac{1}{2} \sum_{i=1}^{3} |a_i|^2 + C'$.

Once a suitable value of $a$ in $P(X)$ has been determined, there exists in $K$ an integer for which $T_2(\theta) \leq C$. Let us define

$$s_2(\theta) = -2b + a^2 = \sum_{j=1}^{3} \sigma^j(\theta)^2 = y_1 + y_2\alpha + y_3\beta$$

(where $\sigma^j$ for $j = 1, 2, 3$ are the $k$-isomorphisms from $K$ into $\mathbb{C}$), and

$$S_2(\theta) = \sum_{i=1}^{3} \sigma_i(s_2(\theta)) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma^j_i(\theta)^2.$$  

From the inequalities $|S_2(\theta)| \leq \sum_{i=1}^{3} |\sigma_i(s_2(\theta))| \leq T_2(\theta) \leq C$ and

$$\sum_{i=1}^{3} |\sigma_i(s_2(\theta))|^2 \leq \left( \sum_{i=1}^{3} |\sigma_i(s_2(\theta))| \right)^2 \leq C^2,$$

an upper bound is obtained for the values of a positive definite quadratic form in the variables $y_1$, $y_2$, and $y_3$ and among all the integral solutions in this bounded convex body, only those for which the variables $y_i$ have the same parity as the $x_i$'s, where $a^2 = x_1^2 + x_2^2\alpha + x_3^2\beta$, have to be considered. Finally, one can easily deduce all the possible values for the coefficient $b$ in $P(X)$.

Let us now assume that $a$ and $b$ have been fixed, and let $c = z_1 + z_2\alpha + z_3\beta$ be the expression of $c$ in $\mathbb{Z}_k$. From the formula $|c| = \prod_{j=1}^{3} |\sigma^j(\theta)|$ and the inequality between arithmetic and geometric means we obtain

$$|c|^2 \leq \frac{1}{27} \left( \sum_{j=1}^{3} |\sigma^j(\theta)|^2 \right)^3,$$

and we can deduce the inequalities

$$\sum_{i=1}^{3} |\sigma_i(c)|^2 \leq \frac{1}{27} \sum_{i=1}^{3} \left( \sum_{j=1}^{3} |\sigma^j_i(\theta)|^2 \right)^3 \leq \frac{1}{27} \left( \sum_{i=1}^{3} \sum_{j=1}^{3} |\sigma^j_i(\theta)|^2 \right)^3 \leq \frac{1}{27} C^3.$$

The expression is still a bounded positive definite quadratic form in the variables $z_1$, $z_2$, $z_3$, and it is possible to compute all the integral points inside the ellipsoid [28].
Finally, a polynomial \( P(X) \in k[X] \) is obtained for which it is necessary to test if \( P(X) \) defines a number field \( K \) having a suitable signature. If the cubic ground field is a complex one, it is enough to compute the sign of the discriminant of the polynomial \( \sigma(P(X)) \), where \( \sigma \) is the unique real embedding from \( k \) into \( \mathbb{C} \). If the cubic ground field \( k \) is real, the sign of the discriminant has to be determined for each conjugate of \( P(X) \).

The computation of the roots of the polynomials \( P_1(X) \), \( P_2(X) \), \( P_3(X) \) was done using Cardano’s formulae, and at this point the inequality (*) of Theorem 1 was checked. When complex places exist, this test eliminates more than 99% of the polynomials having the appropriate signature. However, in the case of totally real fields \( K \), this computation can be avoided because in this case \( Tr_{K/Q}(\theta^2) \) is a rational integer.

The next step consists in testing for the irreducibility of \( f(X) \). The polynomial \( f(X) \) is eliminated either because \( P_i(X) \) belongs to \( \mathbb{Z}[X] \) or because one of the roots of \( f(X) \) is in \( \mathbb{Z} \). The last condition is easy to verify by using the approximate values of the complex roots already computed. Finally, we search for the possible existence of an irreducible cubic factor of \( f(X) \) in \( \mathbb{Z}[X] \); this is done by factoring \( f(X) \) over \( \mathbb{Z} \), using the PARI system [4].

For the remaining irreducible polynomials having a suitable signature and satisfying the inequality (*) of Theorem 1, the discriminant of \( K \) is computed by using the version written by D. Ford [13] of the ROUND 2 algorithm of Pohst and Zassenhaus [8, p. 305, Algorithm 6.1.8] implemented in the PARI system.

At this stage, the constructed tables of number fields are complete (with the possible exception of the fields obtained as a compositum of two cubic ground fields) but some fields in the tables can be \( \mathbb{Q} \)-isomorphic. In fact, practice proves that for small discriminants this is often true. The following lemma is useful.

Let \( P(X) \) and \( Q(X) \) be two relative polynomials generating respectively the fields \( K \) and \( L \) with the same absolute discriminant. Denote by \( \sigma_i(P) \) and \( \sigma_i(Q) \) for \( i = 1, 2, 3 \) the conjugates of \( P(X) \) and \( Q(X) \), and by \( \sigma_i^j(\theta) \) and \( \sigma_i^j(\eta) \) for \( j = 1, 2, 3 \) their respective roots in \( \mathbb{C} \).

Lemma 2 [3]. The fields \( K \) and \( L \) are \( \mathbb{Q} \)-isomorphic if and only if there exist four permutations \( s, s_1, s_2, s_3 \in S_3 \) such that for all \( h \in \mathbb{N} \) we have

\[
\sum_{j=1}^{3} (\sigma_i^j(\theta))^h \cdot \sigma_{s_i(\ell)}^j(\eta) \in \sigma_i(\mathbb{Z}_k) \quad \text{for } i = 1, 2, 3.
\]

The most tedious case is the totally real one, where this test may require \( 6^4 = 1296 \) trials; the easiest case corresponds to the signature \((1,4)\), where at most 24 trials are necessary. Once a suitable permutation for the roots is obtained, it is easy to compute the equations with rational coefficients (the norm of the index \([\mathbb{Z}_K : \mathbb{Z}[^Q]]\) is the denominator of the coefficients) relating the roots of \( Q(X) \) and the roots of \( P(X) \).

Whenever such a \( \mathbb{Q} \)-isomorphism was not found, we searched for rational prime numbers having different decompositions as a product of prime ideals in \( K \) and \( L \). The existence of different factorizations for a prime number in \( K \) and \( L \) is a sufficient (but not necessary) condition for these fields to be nonisomorphic. In all the cases considered, when the algorithm of Lemma 2

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failed to prove that the fields were isomorphic, prime numbers having different factorizations in these fields were always found, thus providing proof for the isomorphism test.

What remains to be considered is simply the case where the integer \( \theta \in K \), \( \theta \notin k \), whose existence is asserted by Theorem 1, is not a primitive element for \( K/\mathbb{Q} \). Because the field \( K \) can only contain cubic subfields, the field \( k_1 = \mathbb{Q}(\theta) \) is a cubic extension of \( \mathbb{Q} \) and \( k \neq k_1 \). If \( k_1 \) were a conjugate of \( k \), the compositum \( kk_1 \) would be the Galois closure of \( k \) in an algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \); but this is impossible because \( K \) does not contain any sextic subfield. This proves that the relative polynomial \( P(X) \) having \( \theta \) as a root is actually a polynomial with coefficients in \( \mathbb{Z} \).

In this case, to find a primitive element of \( K/\mathbb{Q} \), the following result due to J. Martinet (private communication) was used.

**Theorem 2.** Every ninth-degree extension \( K/k_0 \) of a number field can only contain 0, 1, 2 or 4 cubic extensions of \( k_0 \).

**Proof.** It is sufficient to prove that a field \( K \) containing more than one cubic extension of \( k_0 \) must contain exactly 2 or 4 cubic subfields. Denote by \( k_1 \) and \( k_2 \) two cubic extensions of \( k_0 \) in a given algebraic closure \( \overline{k_0} \); then we have \( K = k_1k_2 \), because the fields \( k_1 \) and \( k_2 \) cannot be \( k_0 \)-isomorphic; let \( d_1 \) and \( d_2 \) be the respective discriminants of \( k_1/k_0 \) and \( k_2/k_0 \) (in \( k_0^{alg} \)); let \( N \) be the Galois closure of \( K/k_0 \) into \( \overline{k_0} \), and \( G \) the Galois group of \( N/k_0 \). Exchanging, if necessary, \( k_1 \) and \( k_2 \), we find that the following cases cover all the different possibilities:

- (A) \( d_1 \) and \( d_2 \) are perfect squares.
- (B) \( d_1 \) is a perfect square and \( d_2 \) is not.
- (C) \( d_1 \) and \( d_2 \) are not perfect squares but \( d_1/d_2 \) is.
- (D) \( d_1 \) and \( d_2 \) are not perfect squares and \( d_1/d_2 \) is not.

The group \( G \) can then be identified with a subgroup of the direct product \( S_3 \times S_3 \). Specifically, in the case (A), \( G \) is isomorphic to \( C_3 \times C_3 \), and the stabilizer of \( K \) is the subgroup \( \langle 1 \rangle \) of \( G \); so, \( G \) has four normal subgroups of order 3, and \( K \) contains four cyclic cubic subfields.

In the case (B), \( G \) is isomorphic to \( C_3 \times S_3 \), and this group can be defined as \( G = \langle \sigma_1, \sigma_2, \tau \rangle \) with the relations \( \sigma_1^3 = \sigma_2^3 = \tau^2 = 1, \sigma_1\sigma_2 = \sigma_2\sigma_1, \sigma_1\tau = \tau\sigma_1^3, \sigma_2\tau = \tau\sigma_2^3 \), where \( \tau \) is the stabilizer of \( K \). Then \( G \) has two subgroups of order 6 containing \( \tau \), say \( \langle \sigma_1, \tau \rangle \) which is normal, and \( \langle \sigma_2, \tau \rangle \) which is not normal; therefore, \( K \) contains two nonisomorphic cubic subfields, one of them being cyclic, but not the other.

In the case (C), \( G \) is isomorphic to \( \langle \sigma_1, \sigma_2, \tau \rangle \) of \( S_3 \times S_3 \), with the relations \( \sigma_1^3 = \sigma_2^3 = \tau^2 = 1, \sigma_1\sigma_2 = \sigma_2\sigma_1, \sigma_1\tau = \tau\sigma_1^3, \sigma_2\tau = \tau\sigma_2^3 \), where \( \tau \) is the element in \( G \) of order 2 leaving \( K \) invariant; therefore, \( G \) has four subgroups isomorphic to \( S_3 \), containing \( \tau \), and they are not normal; so, \( K \) has four nonisomorphic cubic subfields which are noncyclic, associated with a unique common quadratic extension of \( k_0 \), which is the field fixed by the subgroup \( \langle \sigma_1, \sigma_2 \rangle \) of order 9 and index 2.

Finally, in the case (D), the group \( G \) can be written in the form \( \langle \sigma_1, \tau_1 \rangle \times \langle \sigma_2, \tau_2 \rangle \), where \( \langle \sigma_i, \tau_i \rangle \) is isomorphic to \( S_3 \) for \( i = 1, 2 \). We may suppose that the stabilizer of \( K \) is the group \( \langle \tau_i \rangle \), isomorphic to \( V_4 \) (Klein group), and the only subgroups of \( G \) of index 3 containing \( \langle \tau_1, \tau_2 \rangle \) are \( \langle \sigma_1, \tau_1, \tau_2 \rangle \) and

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\(\langle \sigma_2, \tau_1, \tau_2 \rangle\). Thus, \(K\) contains exactly two nonisomorphic cubic extensions of \(k_0\); these fields are not Galois fields, and they are associated with different quadratic extensions of \(k_0\).

**Corollary.** Let \(k_1 = \mathbb{Q}(\theta_1)\) and \(k_2 = \mathbb{Q}(\theta_2)\) be two nonisomorphic cubic subfields of a ninth-degree field \(K\) over \(\mathbb{Q}\). Then, among the numbers \(\theta_1 + \theta_2\), \(\theta_1 - \theta_2\), \(\theta_1 + 2\theta_2\), one (at least) is a primitive element of \(K/\mathbb{Q}\).

3. Description of the tables and their computing time

First, limits for the different tables were fixed. In the case of the totally real fields, the bound was fixed to obtain a table which would include the compositum of the abelian cubic fields of discriminant 49 and 81. Consequently, the corresponding table contains all the imprimitive fields of ninth degree having a discriminant smaller than \(6 \times 10^9\). There are exactly 27 number fields with this property, and all of them are characterized, up to an isomorphism, by their discriminant.

The size of the tables, for the other signatures, was fixed depending on an estimate of the CPU time required. The table corresponding to signature \((7,1)\) contains the 23 imprimitive fields having a discriminant larger than \(-7 \times 10^7\), and they are also characterized by their discriminant.

The table corresponding to signature \((5,2)\) contains the 154 fields having a discriminant smaller than \(5 \times 10^8\), and for 5 values of the discriminant there is a pair of nonisomorphic fields.

In the table corresponding to signature \((3,3)_R\) there are 223 fields having a discriminant larger than \(-4 \times 10^8\) and there are 3 pairs and 3 quadruplets of nonisomorphic fields having the same discriminant.

For the same signature, but in table \((3,3)_C\), there are 200 fields having a discriminant larger than \(-1 \times 10^6\), including 2 pairs of nonisomorphic fields having the same discriminant.

The intersection of tables \((3,3)_R\) and \((3,3)_C\) is not empty. These tables have 7 fields in common, corresponding to ninth-degree fields containing a real and a complex cubic subfield simultaneously.

Finally, for signature \((1,4)\) the table contains the 485 fields having a discriminant smaller than \(5 \times 10^5\), and it includes 9 pairs of nonisomorphic fields with the same discriminant.

The computation involved several phases. During the first phase we wanted to eliminate as many polynomials as possible either by using the signature (Sturm's algorithm) or by explicitly computing the value of \(T_2(\theta)\) from the roots of the relative polynomials. This last procedure is called 'eliminated by trace' in Table II (next page). In the second phase we tested to see if the remaining polynomials were irreducible. We then computed the discriminant of the field, using Dedekind's criterion first, and when we were unable to decide whether or not the field had to be preserved, we used 'ROUND 2' to evaluate the discriminant of the field. In the third phase we considered all the polynomials of degree nine that are the cube of a cubic polynomial in \(\mathbb{Z}[X]\), and we constructed a primitive element of the extension \(K/\mathbb{Q}\) as shown in the corollary to Theorem 2. We observed that in every case the newly found polynomial generated a number field isomorphic to a field already found in the tables.
Finally, we determined whether or not isomorphisms exist between fields having the same discriminant.

All the computations were performed on work stations SPARC 1 or SPARC 2 and, except for the first phase, we used the PARI system.

In the next table precise indications are given concerning the CPU time used in computations as well as the number of polynomials treated. It appears that most of the CPU time used concerns phases I and II; for this reason, the type of the SPARC work station used is also indicated. All the computational time is expressed in minutes.

| Table II |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| signature       | (9,0)           | (7,1)           | (5,2)           | (3, 3)R         | (3, 3)C         | (1,4)           |
| subfields       | real            | real            | real            | real            | complex         | complex         |
| tested polynomials | 155 69964       | 12019 10861     | 8394 23372      | 6849 14826      | 5182 09760      | 2388 86137      |
| eliminated by signature | 154 87260       | 11837 70376     | 6452 51604      | 1023 53452      | 4702 04181      | 222 69560       |
| eliminated by trace | *               | 180 78315       | 1939 58698      | 5219 12888      | 479 68716       | 2165 36090      |
| remaining polynomials | 72674           | 62170           | 213070          | 281848          | 26086           | 72404           |
| time phase I     | 15 ‡            | 729 ‡           | 1742            | 4153            | 1273            | 4728            |
| reducible polynomials | 30116           | 17972           | 16533           | 4334            | 10767 ‡         | 7644 ‡          |
| Dedekind's criterion | 34500           | 35858           | 153190          | 216074          | 17640           | 47682           |
| eliminated by round2 | 5448            | 6182            | 27094           | 36248           | 5260            | 14851           |
| remaining fields  | 2002            | 2058            | 16252           | 24704           | 3176            | 9589            |
| time phase II    | 698 ‡           | 993             | 7242            | 4709 ‡          | 927             | 1471 ‡          |
| fields in the table | 27              | 23              | 154             | 223             | 200             | 485             |
| total time used  | 792             | 1722            | 8984            | 8863            | 2200            | 6262            |

‡These computations were done on a SPARC 2 work station.

4. Description of specific fields in the tables

4.1. In the table corresponding to signature (9,0), the minimum discriminant corresponds to an extension of the cubic field \( k_1 = \mathbb{Q} (\alpha_1) \) of discriminant 49. Here, \( \alpha_1 \) is a zero of the polynomial \( X^3 + X^2 - 2X - 1 \), and the relative discriminant of the extension \( K/k_1 \) is the prime ideal \( p = (138041, \alpha_1 + 113260) \) of \( k_1 \).

The field of discriminant 1 69835 63041 = 19^8 is the maximal real subfield of the cyclotomic field \( \mathbb{Q} (\zeta_{19}) \).

In the same table, the field of discriminant 1 75152 30173 = 7^6 \cdot 53^3 has two cubic subfields: \( k_1 \) and the cubic field of discriminant 2597 = 7^2 \cdot 53.

The last field in the table is the abelian field of discriminant 6 25235 02209 = 3^{12} \cdot 7^6; it is the compositum of the field \( k_1 \) and the cyclic field \( k_2 = \mathbb{Q} (\alpha_2) \) of discriminant 81, where \( \alpha_2 \) is a zero of \( X^3 - 3X - 1 \). This field also has as subfields the two nonisomorphic cubic fields of discriminant 3969 = 3^4 \cdot 7^2.
For 5 fields in this table the norm of the relative discriminant is a square:
\[ d_{K_1} = 16440305941 = 37^2 \cdot 229^3; \quad d_{K_2} = 1698563041 = 19^8; \quad d_{K_3} = 3138105969 = 3^{22}; \quad d_{K_4} = 36763077169 = 7^6 \cdot 13^2 \cdot 43^2 \quad \text{and} \quad d_{K_5} = 53286643921 = 7^6 \cdot 673^2. \] These fields \( K_1, \ldots, K_5 \) are subfields of ray class fields of their corresponding cubic subfields.

4.2. In the table corresponding to signature \((7,1)\) the minimal discriminant is \(-2668161671 = -7^6 \cdot 22679\), and the relative discriminant is the ideal \( p = (22679, \alpha - 28) \) of \( k_1 \).

4.3. The minimal discriminant for imprimitive nonic fields of signature \((5,2)\) is \(467890073 = 7^6 \cdot 41 \cdot 97\), and the relative discriminant is the ideal \( q = (41, \alpha_1 + 11)(97, \alpha_1 - 41) \) of \( k_1 \). We can now state the following result:

**Proposition 1.** For imprimitive number fields of degree 9 and signature \((5, 2)\) having a discriminant in the interval \(467890073 \leq d \leq 5000000000\) there is a pair of nonisomorphic fields with the same discriminant for the following values of \( d:\)
\[ 1209078773 = 7^6 \cdot 43 \cdot 239, \quad 3017579201 = 7^6 \cdot 13 \cdot 1973, \quad 3165816941 = 7^6 \cdot 71 \cdot 379, \quad 4091479273 = 7^6 \cdot 83 \cdot 419 \quad \text{and} \quad 4794549697 = 7^6 \cdot 83 \cdot 491. \]

**Proof.** A look at the table and a straightforward computation of the relative discriminants. \( \square \)

**Corollary.** There are pairs of nonisomorphic number fields of degree 6 and signature \((2, 2)\) with the same discriminant for the following values of the discriminant:
\[ 24675077 = 7^4 \cdot 43 \cdot 239, \quad 61584249 = 7^4 \cdot 13 \cdot 1973, \quad 6460859 = 7^4 \cdot 71 \cdot 379, \quad 83499577 = 7^4 \cdot 83 \cdot 419 \quad \text{and} \quad 97847953 = 7^4 \cdot 83 \cdot 491. \] All these sextic fields have a class number divisible by 3.

**Proof.** The relative quadratic extensions \( L_1 \) and \( L_2 \) of \( k_1 \) attached by Galois theory to the extensions \( K_1/k_1 \) and \( K_2/k_1 \) have different relative discriminants over \( k_1 \). Denote by \( N_1 \) and \( N_2 \) the Galois closure of the respective extensions \( K_1/k_1 \) and \( K_2/k_1 \). The extensions \( N_1/L_1 \) and \( N_2/L_2 \) are relative cyclic extensions unramified at the finite places. Consequently, the class number of \( L_1 \) and \( L_2 \) must be divisible by 3. \( \square \)

The same reasoning can by applied to the sextic fields \( L_i \) attached to the extensions \( K_i/k_1 \) for \( i = 3, \ldots, 10 \).

4.4. In the table corresponding to signature \((3, 3)_{\mathbb{R}}\) the minimal discriminant is \(-164590951 = -7^6 \cdot 1399\) and the field having this discriminant is a relative extension of \( k_1 \) with relative discriminant \((1399, \alpha_1 + 347)\).

**Proposition 2.** Among the imprimitive ninth-degree number fields of signature \((3, 3)\) having a real cubic subfield and a discriminant \( d \) belonging to the interval \(-4000000000 \leq d \leq -164590951\), there exists a pair of nonisomorphic fields with the same discriminant for the following values of \( d:\)
\[ -23156852676 = -39^2 \cdot 7^6, \quad -2838635072 = -2^6 \cdot 7^6 \cdot 13 \cdot 29 \quad \text{and} \quad -3320407727 = -7^6 \cdot 13^2 \cdot 167. \] There also exists a quadruplet of nonisomorphic fields with the same discriminant for the values of \( d:\)
\[ -2045563163 = -7^6 \cdot 17387, \quad -2141094151 = -7^6 \cdot 18199 \quad \text{and} \quad -2386745263 = -7^6 \cdot 20287. \]

**Proof.** Only one of the fields of discriminant \(-39 \cdot 7^6\) has an ideal of norm 43. In one of the fields of discriminant \(-2^6 \cdot 7^6 \cdot 13 \cdot 29\) there is only one prime
ideal over 37 but not in the other field. Similarly, there is only one prime ideal over 19 in one of the fields of discriminant \(-7^2 \cdot 13^2 \cdot 167\) and 3 prime ideals over 19 in the other field.

For \(d = -2045563163\) the decomposition of the primes 11, 17, and 19 shows that the four fields having this discriminant are not isomorphic.

For \(d = -2141094151\), consider the primes 17, 47, and 103.

For \(d = -2386745263\), consider the primes 19, 31, and 37. □

**Corollary.** The sextic number fields

\[
L_1 = \mathbb{Q}(\beta_1), \quad L_2 = \mathbb{Q}(\beta_2), \quad L_3 = \mathbb{Q}(\beta_3)
\]

having the respective discriminants \(d_{L_1} = -41746187 = -7^4 \cdot 17387\), \(d_{L_2} = -43695799 = -7^4 \cdot 18199\), \(d_{L_3} = -48709087 = -7^4 \cdot 20287\), where \(\beta_1, \beta_2, \beta_3\) are zeros of the polynomials

\[
R_1(X) = X^6 - 2X^5 + 20X^4 - 27X^3 + 140X^2 - 98X + 343,
\]
\[
R_2(X) = X^6 - X^5 + 19X^4 - 13X^3 + 133X^2 - 49X + 343,
\]
\[
R_3(X) = X^6 - X^5 + 20X^4 - 9X^3 + 142X^2 - 18X + 377,
\]

have a class group isomorphic to \(C_3 \times C_3\).

**Proof.** These sextic fields are relative quadratic extensions of the cubic field \(k_1\) of discriminant 49 and are associated with the quadruplets of relative cubic extensions of \(k_1\) having the same relative discriminant. The Galois closure for each field of degree nine in a quadruplet is an unramified relative cubic extension (even at infinite places) of one of these sextic number fields. According to class field theory, the existence of 4 unramified relative extensions proves that the class group of these sextic fields must have a 3-rank bigger than 1. The exact value of the class number of these fields was computed using the KANT [17] system and, in every case, the class group was found to be isomorphic to \(C_3 \times C_3\). □

Some fields in this table also have a complex cubic subfield, and they appear in the table corresponding to signature \((3, 3)_C\) when their discriminant belongs to the interval covered by that table. The complete list of these fields is given in Table III.

These fields are of particular interest because they give supplementary information about the class number of the complex cubic subfields. The table corresponding to signature \((3, 3)_C\) has 7 fields in common with the table corresponding to signature \((3, 3)_R\). For 6 of these fields, the norm of the relative discriminant is equal to 1. For 3 other fields in Table III having a discriminant beyond the limits of table \((3, 3)_C\) the discriminant of the field of degree nine is the cube of the discriminant of the complex cubic subfield, and the relative extension has a discriminant of norm equal to 1. It is therefore clear that these complex cubic fields must have a class number divisible by 3.
A comparison with Angell's table [1] of complex cubic fields of discriminant larger than \(-4 \cdot 10^9\)^{1/3} and a class number divisible by 3 shows that only the three nonisomorphic fields of discriminant \(-1228\) do not appear in our Table III. It is nonetheless easy to verify that the ninth-degree fields containing such a cubic subfield are outside the limits of table \((3, 3)_c\). When the real cubic subfield \(k\) of \(K\) (with the notations in Table III) is abelian, \(K\) is a relative cyclic extension of the complex cubic field \(k'\).

It can also be pointed out that between the two nonisomorphic fields of discriminant \(-23156 85267\) only one field contains a complex cubic subfield.

4.5. In the table corresponding to signature \((3, 3)_C\) the minimal discriminant is \(-1108 52311 = -31^{3} \cdot 61^{2}\), and this field is an extension of the complex cubic field \(k_{4}\) of discriminant equal to \(-31\). The relative discriminant of the extension \(K/k_{4}\) is the ideal \((61, \alpha_{4} - 24)^{2}\), where \(\alpha_{4}\) is a zero of the polynomial \(X^3 + X - 1\) and \(k_{4} = \mathbb{Q}(\alpha_{4})\).

**Proposition 3.** Among the imprimitive number fields of degree 9 and signature \((3, 3)\) containing a complex cubic subfield whose discriminant belongs to the interval \(-1000000000 \leq d \leq -1108 52311\), there exists a pair of nonisomorphic
Corollary. There exists a pair of nonisomorphic sextic number fields with signature \((2, 2)\) and discriminants \(22764457 = 23^3 \cdot 1871\) and \(38084297 = 23^2 \cdot 71993\). These sextic fields have a class number divisible by 3.

Proof. These sextic fields are relative quadratic extensions of the complex cubic field \(\mathbb{Q}(\alpha)\) of discriminant \(-23\) generated over \(\mathbb{Q}\) by a zero \(\alpha\) of the polynomial \(X^3 + X^2 - 1\), and they are associated with the two pairs of ninth-degree fields whose discriminants were given in Proposition 3. It is easy to verify that the two sextic fields of discriminant \(22764457\) are generated respectively by a zero of the polynomials \(X^6 - 2X^5 - 22X^4 + 21X^3 + 114X^2 + 185X - 809\) and \(X^6 - 29X^4 - 8X^3 + 50X^2 + 286X^2 - 292X - 1127\).

The generating polynomials for the sextic fields of discriminant \(38084297\) are \(X^6 - X^5 - 31X^4 + 25X^3 + 313X^2 - 197X - 1219\) and \(X^6 - 2X^5 - 27X^4 + 50X^3 - 292X^2 - 292X - 1127\).

As previously established in similar proofs, the class number of these fields must be divisible by 3. \(\square\)

4.6. The minimal discriminant for ninth-degree number fields of signature \((1, 4)\) containing a cubic subfield is \(32206049 = 23^3 \cdot 2647\). The field \(K\) having this discriminant is a relative cubic extension of \(k_3\), and the relative discriminant of \(K/k_3\) is the prime ideal \((2647, \alpha_3 + 143)\).

This field as well as the next two fields in the table, of discriminants \(33860761 = 11^2 \cdot 23^4\) and \(35028793 = 23^3 \cdot 2879\), are Euclidean for the norm, and they were discovered by Leutbecher [18].

Proposition 4. Among the imprimitive ninth-degree number fields of signature \((1, 4)\) whose discriminant belongs to the interval \(32206049 \leq d \leq 50000000\), there are 9 pairs of nonisomorphic fields having the same discriminant. The corresponding values for these discriminants are: \(94987769 = 23^3 \cdot 37 \cdot 211\), \(154022053 = 23^3 \cdot 12659\), \(229749461 = 23^4 \cdot 821\), \(246844096 = 2^6 \cdot 23^3 \cdot 317\), \(329845952 = 2^6 \cdot 31^3 \cdot 173\), \(351516797 = 23^3 \cdot 167 \cdot 173\), \(361883081 = 7^2 \cdot 23^3 \cdot 607\), \(369876800 = 2^6 \cdot 5^2 \cdot 19 \cdot 23^3\), and \(474481257 = 3 \cdot 31^3 \cdot 5309\).

Proof. For each pair of fields having the same discriminant, we can produce a prime number having a different decomposition in the two fields as a product of prime ideals. \(\square\)

Corollary. There exists a pair of nonisomorphic totally complex sextic fields with the same discriminant for the following values of \(d\): \(-4129903 = -23^2 \cdot 37 \cdot 211\), \(-669611 = -23^2 \cdot 12659\), \(-9989107 = -23^3 \cdot 821\), \(-10640192 = -2^6 \cdot 31^3 \cdot 173\), \(-10732352 = -2^6 \cdot 23^3 \cdot 317\), \(-15283339 = -23^3 \cdot 167 \cdot 173\) and \(-15305847 = -3 \cdot 31^2 \cdot 5309\). The class number of all these sextic fields is divisible by 3.

Proof. The proof is similar to that already established. \(\square\)
Remark. The two sextic fields associated with the fields of degree 9 and discriminant $3618\,83081$ do not have the same discriminant. For one of these nonic fields the relative discriminant over $k_3$ is the ideal $(7, \alpha_3^2 + 4\alpha_3 + 5)(607, \alpha_3 + 184)$, and the absolute discriminant of the associated sextic field is $-157\,34047$. This field is generated over $\mathbb{Q}$ by a zero of the polynomial $X^6 - 2X^5 + 21X^4 - 45X^3 + 192X^2 - 188X + 488$. The relative discriminant for the other nonic field having the same absolute discriminant is the ideal $(7, \alpha_3 + 4)^2(607, \alpha_3 + 184)$, and the relative quadratic extension over $k_3$ associated with it has an absolute discriminant equal to $-321\,1103$ and it is generated by a zero of the polynomial $X^6 - 3X^5 + 7X^4 - 9X^3 + 11X^2 - 7X + 1$.

Similarly, for one of the nonic fields with discriminant $3698\,76800$, the relative discriminant over $k_3$ is the ideal $(2)^2(5, \alpha_3^2 - \alpha_3 + 2)(19, \alpha_3 + 3)$, and the associated sextic field, of discriminant $-160\,81600$, is generated by a zero of $X^6 - 4X^4 - 65X^2 + 475$. For the other nonic field with the same discriminant, the relative discriminant over $k_3$ is the ideal $(2)^2(5, \alpha_3 + 2)^2(19, \alpha_3 + 3)$, and the associated sextic field, of discriminant $-10051$, is generated by a zero of $X^6 - 3X^5 + 4X^4 - 4X^3 + 3X^2 - X + 1$.

We conclude this section with the following theorem whose proof is clear:

**Theorem 3.** Let $K$ denote a number field of degree 9 appearing in the tables for which the relative discriminant $\delta$ of the extension $K/k$ is not divisible by the square of any prime ideal of $k$. Let $L$ denote the sextic field containing $k$. If $|d_L| = d_k^2 \cdot N_{k/\mathbb{Q}}(\delta)$, then the class number of $L$ is divisible by 3.

5. Galois groups

Let $K = \mathbb{Q}(\theta)$ be an imprimitive ninth-degree number field, and denote by $\Gamma$ the Galois group of the Galois closure of $K/\mathbb{Q}$. The action of $\Gamma$ on $\theta$ allows us to consider $\Gamma$ as a transitive ninth-degree permutation group. When this is the case, we will say that $K$ is of type $\Gamma$.

All the transitive groups up to degree 11 are known [7] and, up to conjugacy, there are 34 ninth-degree transitive groups. Among them, 23 are imprimitive, and 12 are imprimitive and even. In this section, we use the notations of [7] to designate these groups. For example, we denote by $T_3^+$ group $T_3$ in [7] and the upper sign $+$ means that this group is even and, consequently, the discriminant of a field $K$ of type $T_3^+$ is a square.

The following proposition is due to Y. Eichenlaub (private communication).

**Proposition 5.** In each signature, the possible Galois groups for imprimitive ninth-degree number fields are:

- (9, 0) : $T_1^+, T_2^+, T_3^+, T_4, T_5^+, T_6^+, T_7^+, T_8, T_9^+, T_{10}^+, T_{11}^+, T_{12}, T_{13}$
- $T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{21}, T_{22}, T_{23}, T_{24}, T_{25}, T_{26}, T_{27}, T_{28}, T_{29}, T_{30}, T_{31}$
- (7, 1) : $T_{28}, T_{31}$
- (5, 2) : $T_{24}^+, T_{25}, T_{26}, T_{27}$
- (3, 3) : $T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{21}, T_{22}, T_{23}, T_{24}, T_{25}, T_{26}, T_{27}, T_{28}, T_{29}, T_{30}, T_{31}$
- (1, 4) : $T_3^+, T_5^+, T_7^+, T_8, T_9^+, T_{10}^+, T_{11}^+, T_{12}^+, T_{13}^+, T_{14}^+, T_{15}^+, T_{16}^+, T_{17}^+, T_{18}^+, T_{19}^+, T_{20}^+, T_{21}^+, T_{22}^+, T_{23}^+, T_{24}^+, T_{25}^+, T_{26}^+, T_{27}^+, T_{28}^+, T_{29}^+, T_{30}^+, T_{31}^+$
To compute the Galois group for each ninth-degree polynomial in the tables, we use the classical relative resolvent method ([16] and [30]), based on the following property: Let us denote by $G$ a transitive subgroup of degree $n$ of $S_n$, and by $H$ a subgroup of $G$. An $H$-polynomial for $G$ is a polynomial $F(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$ such that

$$H = \{\sigma \in G | F(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = F(X_1, \ldots, X_n)\}.$$ 

If $\theta_1, \ldots, \theta_n$ denote the conjugates of $\theta$ in $\mathbb{C}$, then the $G$-resolvent related to $H$ and associated with $F$ is the polynomial

$$R_{(G,H,F)}(Y) = \prod_{\sigma \in A} (Y - F(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)})),$$

where $A$ is a complete set of the right cosets of $G$ modulo $H$ (the right coset associated with $\pi \in G$ is $\pi H$).

**Proposition 6.** Suppose that for a fixed order of the roots $(\theta_1, \ldots, \theta_n)$ of the irreducible polynomial defining the extension $K/\mathbb{Q}$, the Galois group $\Gamma$ is included in $G$. Then:

1. $\sigma \in G$ and $F(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}) \notin \mathbb{Z}$ implies $\sigma^{-1} \Gamma \sigma \not\subset H$,
2. if $F(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}) \in \mathbb{Z}$ is a simple root of $R_{(G,H,F)}$, then $\sigma^{-1} \Gamma \sigma \subset H$.

The Galois group of the fields in the tables are, according to the signature:

- $(9,0)$: two fields of type $T_1^+ = C_9$, one of type $T_2^+ = C_9^2$, one of type $T_4 = C_3 \times S_3$, two of type $T_1^+ = C_9$, one of type $T_20$, 19 of type $T_{28}$ and one of type $T_{31}$.
- $(7,1)$: all the fields of type $T_{28}$.
- $(5,2)$: 148 fields of type $T_{28}$ and 6 of type $T_{31}$.
- $(3,3)_R$: 15 fields of type $T_4 = C_3 \times S_3$, one of type $T_8 = C_3^2 \times C_3^2$, one of type $T_{13}$, two of type $T_{18}$, 3 of type $T_{22}$, 189 of type $T_{28}$, and 12 of type $T_{31}$.
- $(3,3)_C$: 7 fields of type $T_4 = C_3 \times S_3$, 5 of type $T_{12}$, 31 of type $T_{20}$, one of type $T_{29}$, and 156 of type $T_{31}$. The seven fields with type $T_4$ are those belonging to the tables $(3,3)_R$ and $(3,3)_C$.
- $(1,4)$: one field of type $T_8 = C_3^2 \times C_3^2$, two of type $T_{18}$, one of type $T_{24}$, two of type $T_{30}^+$, and 479 of type $T_{31}$.

The next diagram gives the graph of all transitive imprimitive ninth-degree odd groups. In the right column the order of the groups is indicated, and a line joining two groups indicates an inclusion up to conjugacy by elements of $S_9$. The number on the line gives the number of conjugate copies pairwise distinct from the smallest group included in the biggest one. The maximal transitive imprimitive ninth-degree group is $T_{31}$, of order 1296. For each line in the graph of the transitive imprimitive even (resp. odd)
groups, the relative $G$-polynomials are computed using the method of [16] (cf. [25]).

We also have:

**Proposition 7.** For the following ninth-degree transitive imprimitive groups, the minimal discriminants are, according to the signature:

- $(9,0)$: $1\ 69835\ 63041$ for the type $T'_9$, $6\ 25235\ 02209$ for the type $T'_2$, $1\ 75152\ 30173$ for the type $T_4$, $3\ 67630\ 77169$ for the type $T'_1$, $1\ 64403\ 05941$ for the type $T_{20}$, $1\ 62403\ 85609$ for the type $T_8$ and $5\ 30389\ 58912$ for the type $T_{31}$.

- $(7,1)$: $-26681\ 61671$ for the type $T_{28}$.

- $(5,2)$: $4678\ 90073$ for the type $T_{28}$ and $12999\ 58592$ for the type $T_{31}$.

- $(3,3)$: $-2032\ 97472$ for the type $T_4$, $-35391\ 49227$ for the type $T_8$, $-3573\ 66875$ for the type $T_{12}$, $-23156\ 85267$ for the type $T_{13}$, $-11119\ 34656$ for the type $T_{18}$, $-1108\ 52311$ for the type $T_{20}$, $-12724\ 91584$ for the type $T_{22}$, $-1645\ 90951$ for the type $T_{28}$, $-5624\ 19575$ for the type $T_{29}$ and $-1471\ 84199$ for the type $T_{31}$.

- $(1,4)$: $3624\ 67097$ for the type $T_8$, $2394\ 83061$ for the type $T_{18}$, $4497\ 28821$ for the type $T_{24}$, $338\ 60761$ for the type $T_{30}$ and $322\ 06049$ for the type $T_{31}$.
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