

QUINTUPLES WITH SQUARE TRIPLETS

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ABSTRACT. By applying modern computing power to an 1848 formula of C. Gill, we find five positive integers, every three of which sum to a square.

Diophantus (V, 17) gave a very simple method for getting four distinct positive integers such that the sum of every three is a perfect square (see [1, pp. 275–276] and for more background, also for the problem of integers with every pair summing to a square, see [3, problem D15]). For example, every triple from $\{1, 22, 41, 58\}$ sums to a square. In 1848, C. Gill [2, pp. 60–64] improved this by finding five distinct integers every three of which sum to a square. However, his methods did not guarantee positive integers; a typical example is: 639226, 684073, 1384201, -660599 , 23401.

Gill's method uses trigonometric functions to obtain a very complicated function $G(m/n, p/q)$ that churns out solutions (rational 5-tuples) for any pair of rational arguments. Of course, the rational output can just be scaled up to integers. Gill suggests that his formula ought to produce some positive solutions, "but the complexity of the formulas do not encourage the attempt". Modern software allows one, after replacing the trigonometric functions by their equivalent rational expressions, to easily implement a search for a positive solution, and a few minutes work with *Mathematica* showed that $G(23, 11)$ is such (it yields the smallest example I have found; the search was based on integer arguments only). Thus, we obtain the following five integers near 10^{20} , every triple from which sums to a square:

26072 32331 15686 61931
43744 83974 22825 91947
1 18132 65441 36751 38222
1 86378 73280 75870 76747
5 19650 11481 49050 02347

Although the computation is straightforward, Gill's formula— $G(m/n, p/q)$ —is quite massive when expressed purely in terms of m/n and p/q . The interested reader can see the gigantic expression by programming the description below, using some high-precision symbolic software, and examining v , w , x , y , and z . The variable names (such as sAB) derive from Gill's use of trigono-

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metric functions (such as $\sin(A - B)$). Here is a description of G :

$$\begin{aligned} \{cB, sB\} &= \frac{\{p^2 - q^2, 2pq\}}{p^2 + q^2} & \{cC, sC\} &= \frac{\{m^2 - n^2, 2mn\}}{m^2 + n^2} \\ s_2 &= 2sC cC & c_2 &= 2cC^2 - 1 \\ \alpha &= 1 - sB sC + cB c_2 & \beta &= \frac{\alpha^2 - (cB cC - cC)^2}{2\alpha cB sC} \\ cA &= 1 - 2/(1 + \beta^2) & sA &= (1 - cA)\beta \\ sAB &= sA cB - cA sB & cAB &= sA sB + cA cB \\ a &= (cC(cA + cB))^2 & b &= (sAB cC + cA s_2)^2 \\ c &= (sAB cC - cB s_2)^2 & d &= (cC - sB s_2 + cC cAB)^2 \\ e &= (cC - sA s_2 + cC cAB)^2 & f &= (cA + cB c_2 + sC sAB)^2 \\ g &= (-cB - cA c_2 + sC sAB)^2 \\ v &= (a + b + d - 2f)/3 & w &= v + f - d & x &= w - b + d \\ y &= x - a + b & z &= (c + e + g - 2a)/3 \end{aligned}$$

Then $G(m/n, p/q) = \{v, w, x, y, z\}$ is a rational quintuple with square triplet sums (and $a - g$ are seven of the ten squares that arise in this way).

Using *Mathematica's* Reduce[] function, one discovers that the quintuple-with-square-triplets problem is equivalent (over the rationals or integers) to the quintuple-with-square-pairs problem, in the sense that one has a solution if and only if the other does; this was partially realized by Gill in 1848. However, this equivalence does not preserve positiveness. So even though a positive solution to the quintuple-with-square-pairs problem has been found (Nicolas; see [3]), it does not directly yield a positive quintuple with square triplets.

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