

## ON YOKOI'S CONJECTURE

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**ABSTRACT.** In this paper we obtain a lower bound for those discriminants of real quadratic fields  $\mathbb{Q}(\sqrt{D})$  with  $D = m^2 + 4$  and  $h(D) = 1$ .

In 1986 Yokoi [11] posed the following conjecture.

**Conjecture.** Let  $D = m^2 + 4$  be a square-free rational integer and  $m$  be a positive integer. Then there exist exactly six real quadratic fields  $\mathbb{Q}(\sqrt{D})$  with  $h(D) = 1$ , i.e.,  $(D, m) = (5, 1), (13, 3), (29, 5), (53, 7), (173, 13), (293, 17)$ .

In 1987, by using Tatzawa [10], Huyn Kwang Kim et al. [2] proved that there exists at most one discriminant  $D = m^2 + 4 \geq e^{16}$  with  $h(D) = 1$ . Later, Mollin and Williams [5] generalized this result to arbitrary ERD types, i.e., those radicands of the form  $D = l^2 + r$  where  $r|4l$ . Furthermore, they extended their techniques in [6], where they were able to determine (with the only possible exception ruled out by GRH) all real quadratic fields with class number one and continued fraction period length of the principal class less than 25.

In this paper, by combining the ideas of Stark [9] and Hecke [1], we obtain the following lower bound.

**Theorem.** Let  $m > 17$  be a positive integer and  $D = m^2 + 4$  be a square-free rational integer. If  $h(D) = 1$ , then we have  $D > \exp(3.7 \times 10^8)$ .

For proving the result in the case  $293 < D \leq 10^{13}$ , we use a computer and the following lemma, which is an immediate consequence of Theorem 2.1 in Mollin and Williams [4].

**Lemma 1** (see also [3]). Let  $D = m^2 + 4$  be a square-free integer and  $m$  be a positive integer. Then the following four statements are equivalent:

- (i)  $h(D) = 1$ .
- (ii)  $f_D(x) = -x^2 + x + (D-1)/4 \not\equiv 0 \pmod{p}$  for all integers  $x$  and primes  $p$  such that  $0 \leq x < p < \sqrt{D-1}/2$ .
- (iii)  $f_D(x)$  is prime for all integers  $x$  with  $1 < x < \sqrt{D-1}/2$ .
- (iv)  $\left(\frac{D}{p}\right) = -1$  for all primes  $p < \sqrt{D-1}/2$ .

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**Corollary 1.** *If  $D = m^2 + 4$  is square-free and  $D$  is composite, then we have  $h(D) > 1$ .*

*Proof.* Case I:  $D$  has at least three different odd prime divisors. Let  $p$  denote the smallest divisor. Then we have  $p < D^{\frac{1}{3}} < \sqrt{D-1}/2$ . Taking  $x = (p+1)/2$ , we obtain  $f_D(\frac{p+1}{2}) = \frac{D-p^2}{4} \equiv 0 \pmod{p}$ , which contradicts (ii) of Lemma 1.

Case II:  $D = pq$  ( $p < q$ ,  $p$  and  $q$  are odd primes). We can easily show that  $p < \sqrt{D-1} - 1$ . Taking  $x = (p+1)/2$  leads to  $f_D(\frac{p+1}{2}) = p(q-p)/4$ , which cannot be a prime. By (iii) of Lemma 1, we also have  $h(D) > 1$ .  $\square$

**Corollary 2.** *If  $D = m^2 + 4$  is square-free and  $m$  is composite, then we have  $h(D) > 1$ .*

*Proof.* Taking  $x = (m-1)/2$  in (iii) of Lemma 1 leads to the result immediately.  $\square$

We now turn to the proof in the case  $293 < D \leq 10^{13}$ .

**Lemma 2.** *Let  $D = m^2 + 4$  be a square-free integer and  $m$  be an odd prime such that  $293 < D \leq 10^{13}$ . Then we have  $h(D) > 1$ .*

*Sketch of the Proof.* We have  $m < 10^{6.5}$  for  $D \leq 10^{13}$ . Let  $k$  be a natural number to be chosen later. Let  $S$  be the set of the first  $k$  primes  $q_j$  with  $q_1 = 5$ . For such  $q_j \in S$ , tabulate all those integers  $m_{ij}$  satisfying

$$0 \leq m_{ij} \leq q_j - 1, \quad \left( \frac{m_{ij}^2 + 4}{q_j} \right) = 1,$$

where  $\left( \frac{\cdot}{\cdot} \right)$  denotes the Legendre symbol. This can be easily done by using the tables from p. 437 to p. 444 of Riesel [8].

Let  $m_0$  be an integer such that  $17 < m_0 < 10^{6.5}$ . If there exist a prime  $q_j \in S$  and some  $m_{ij}$  such that

$$q_j < \sqrt{m_0^2 + 3}/2, \quad m_0 \equiv m_{ij} \pmod{q_j},$$

then, by (iv) of Lemma 1, we have  $h(D_0) > 1$  for  $D_0 = m_0^2 + 4$ . Thus, such an  $m_0$  could be eliminated. It can be easily seen that approximately half of the  $m$ 's are eliminated for each  $q_j$ . To insure success, we take  $k = 100$  instead of the least  $k = 22$  satisfying  $2^k > 10^{6.5}$ . The corresponding  $m_{ij}$  to  $q_j > 101$  must be calculated by trial. We use a personal computer to sieve out as many  $m$  with  $17 < m < 10^{6.5}$  as possible. The computation shows that for  $17 < m < 10^{6.5}$  and prime  $D = m^2 + 4$ , there exist some  $q_j < \sqrt{m^2 + 3}/2$  and  $q_j \in S$  such that  $\left( \frac{D}{q_j} \right) = 1$ . This completes the proof.  $\square$

The next lemma gives the first 20 convergents for the ratio of the imaginary parts of the first two nontrivial zeros in the upper half-plane of  $\zeta(s)$ .

**Lemma 3.** *Let  $a = 1.487\ 262\ 003\ 298\ 890\ 048$ . Then its expansion in continued fraction is  $a = [1; 2, 19, 7, 1, 10, 1, 20, 1, 1, 26, 1, 1, 6, 1, 3, 1, 1, 1, 8, 3, 1, 2, 1, 1, 25, 1, 1, 7, 5, 1, 3, 1, 2, 2]$  and its first 20 convergents are  $\alpha_1 = 1, \alpha_2 = \frac{3}{2}, \alpha_3 = \frac{58}{39}, \alpha_4 = \frac{409}{275}, \alpha_5 = \frac{467}{314}, \alpha_6 = \frac{5079}{3415}, \alpha_7 = \frac{5546}{3729}, \alpha_8 = \frac{115999}{77995}, \alpha_9 = \frac{121545}{81724}, \alpha_{10} = \frac{237544}{159719}, \alpha_{11} = \frac{6297689}{4234418}, \alpha_{12} = \frac{6535233}{4394137},$*

$$\alpha_{13} = \frac{12832922}{8628555}, \alpha_{14} = \frac{83532765}{56165467}, \alpha_{15} = \frac{96365687}{64794022}, \alpha_{16} = \frac{372629826}{250547533}, \alpha_{17} = \frac{468995513}{315341555},$$

$$\alpha_{18} = \frac{841625339}{565889088}, \alpha_{19} = \frac{1310620852}{881230643}, \alpha_{20} = \frac{11326592155}{7615734232}.$$

The following integral is one of our major analytic tools, which was first used in 1917 by Hecke [1] to obtain a Kronecker limit formula for real quadratic fields.

**Lemma 4.** *Define*

$$(1) \quad c(s) = \int_{-\infty}^{\infty} \frac{dv}{(e^v + e^{-v})^s}, \quad \text{Re } s > 0.$$

Then  $c(s)$  can be continued to a meromorphic function over the  $s$ -plane with the only singularity  $s = 0$  (a pole of order 1). Besides, for any  $s$  we have

$$(2) \quad c(s) = \Gamma\left(\frac{s}{2}\right)^2 / (2\Gamma(s)).$$

*Proof.* For  $\text{Re } s > 0$  we have

$$c(s) = \int_0^{\frac{1}{2}} t^{\frac{1}{2}s-1} (1-t)^{\frac{1}{2}s-1} dt = \frac{1}{2} B\left(\frac{1}{2}s, \frac{1}{2}s\right) = \Gamma\left(\frac{1}{2}s\right)^2 / (2\Gamma(s)),$$

which leads to the desired result.  $\square$

The following lemma is similar to the one used in [9]. Here, however,  $c(s)$  is used to turn our real quadratic fields into imaginary ones, which can be treated by Stark's method.

**Lemma 5.** *Let  $D$  be a square-free integer with  $h(D) = 1$ . Define the  $L$ -function*

$$(3) \quad L_D(s) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) n^{-s} \quad (\text{Re } s > 1),$$

where  $\left(\frac{D}{n}\right)$  denotes the Kronecker symbol. Then, for any  $s$ , we have

$$(4) \quad \zeta(s)L_D(s)c(s) = \zeta(2s)c(\varepsilon, s) + \sqrt{\pi}D^{\frac{1}{2}-s} \frac{\zeta(2s-1)\Gamma(s-\frac{1}{2})}{\Gamma(s)} c(\varepsilon, 1-s) + R_0(s),$$

where  $\varepsilon = (m + \sqrt{D})/2$  is the fundamental unit of the real quadratic field  $\mathbb{Q}(\sqrt{D})$ ,

$$(5) \quad R_0(s) = \int_{-\ln \varepsilon}^{\ln \varepsilon} R_v(s) dv,$$

$$(6) \quad R_v(s) = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \left(y - [y] - \frac{1}{2}\right) \frac{d}{dy} (Q_v(k, y)^{-s}) dy,$$

$$(7) \quad Q_v(k, y) = Ak^2 + Bky + Cy^2,$$

$$(8) \quad A = (1 + \sqrt{D})^2 e^v / 4 + (1 - \sqrt{D})^2 e^{-v} / 4,$$

$$B = (1 + \sqrt{D})e^v + (1 - \sqrt{D})e^{-v}, \quad C = e^v + e^{-v},$$

$$(9) \quad B^2 - 4AC = -4D,$$

$$(10) \quad c(\varepsilon, z) = \int_{-\ln \varepsilon}^{\ln \varepsilon} \frac{dv}{(e^v + e^{-v})^z}.$$

*Proof.* We only need to prove this for  $\text{Re } s > 1$ . Obviously,  $f(x, y) = x^2 + xy - (D - 1)y^2/4$  is a quadratic form with discriminant  $D$ . Let  $\mathfrak{u}$  be the integral ideal corresponding to  $f(x, y)$ . Then we can take  $\mathfrak{u} = [1, \omega]$ , where  $\omega = (1 + \sqrt{D})/2$ . Some  $h(D) = 1$ , we can easily show that

$$(11) \quad \zeta(s)L_D(s) = \frac{1}{2} \sum'_{\lambda \in \mathfrak{u}/\varepsilon} 1/|\lambda\lambda'|^s,$$

where  $\sum'$  means the summation is taken over all nonzero ideals  $\lambda$ ,

$$(12) \quad \left. \begin{aligned} \lambda &= k\omega + r, & \lambda' &= k\omega' + r, \\ \omega &= (1 + \sqrt{D})/2, & \omega' &= (1 - \sqrt{D})/2, \end{aligned} \right\} k, r \in \mathbb{Z}$$

and  $\zeta(s)$  denotes the Riemann zeta function.

By the definition of  $c(s)$  we obtain that

$$(13) \quad \begin{aligned} \zeta(s)L_D(s)c(s) &= \frac{1}{2} \sum'_{\lambda \in \mathfrak{u}/\varepsilon} \int_{-\infty}^{\infty} \frac{dv}{(\lambda^2 e^v + \lambda'^2 e^{-v})^s} \\ &= \frac{1}{2} \sum'_{\lambda \in \mathfrak{u}} \int_{-\ln \varepsilon}^{\ln \varepsilon} \frac{dv}{Q_v(k, r)^s} = \int_{-\ln \varepsilon}^{\ln \varepsilon} M_v(k, r) dv, \end{aligned}$$

where

$$(14) \quad M_v(k, r) = \frac{1}{2} \sum_{(k, r) \neq (0, 0)} \frac{1}{Q_v(k, r)^s},$$

$Q_v(k, r)$  is defined in (7), and (9) can be easily verified.

From Euler's summation formula it follows that

$$(15) \quad \begin{aligned} M_v(k, r) &= \frac{\zeta(2s)}{(e^v + e^{-v})^s} + \sum_{k=1}^{\infty} \sum_{r=-\infty}^{\infty} Q_v(k, r)^{-s} \\ &= \frac{\zeta(2s)}{(e^v + e^{-v})^s} + \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{dy}{Q_v(k, y)^s} \\ &\quad + \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \left( y - [y] - \frac{1}{2} \right) \frac{d}{dy} (Q_v(k, y)^{-s}) dy. \end{aligned}$$

A direct calculation gives

$$(16) \quad \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{dy}{Q_v(k, y)^s} = \frac{\sqrt{\pi} D^{\frac{1}{2}-s}}{(e^v + e^{-v})^{1-s}} \frac{\zeta(2s - 1)\Gamma(s - \frac{1}{2})}{\Gamma(s)}.$$

This completes the proof.  $\square$

Combining Lemma 5 with our reiteration method leads to the following result, which secures the theorem.

**Lemma 6.** *Let  $D = m^2 + 4 > 10^{13}$  be a square-free integer with  $h(D) = 1$ . Then we have*

$$(17) \quad D > \exp(371815978).$$

*Proof.* Let  $B_j(u)$  denote the  $j$ th Bernoulli polynomial. We have

$$(18) \quad B_1(u) = u - \frac{1}{2}, \quad B_2(u) = u^2 - u + \frac{1}{6}.$$

By (2.6) of Rademacher [7] and the substitution  $t = (2Cy + Bk)/(2\sqrt{D}k)$  we obtain

$$(19) \quad \int_{-\infty}^{\infty} B_1(y - [y]) \frac{d}{dy} (Q_v(k, y)^{-s}) dy = \frac{C^{s+1}}{D^{s+\frac{1}{2}} k^{2s+1}} (J_1(s) + J_2(s)),$$

where

$$(20) \quad J_1(s) = \frac{1}{48} \int_{-\infty}^{\infty} \frac{d^2}{dt^2} ((t^2 + 1)^{-s}) dt,$$

$$(21) \quad J_2(s) = \int_{-\infty}^{\infty} \frac{B_2(y - [y]) - \frac{1}{24}}{2} \frac{d^2}{dt^2} ((t^2 + 1)^{-s}) dt.$$

We can easily obtain

$$(22) \quad J_1(s) = \frac{\sqrt{\pi} \Gamma(s + \frac{1}{2})}{24 \Gamma(s)}$$

and

$$(23) \quad -\frac{1}{8} \leq B_2(y - [y]) - \frac{1}{24} < \frac{1}{8}.$$

Let  $\nu_j$  be the ordinate of the  $j$ th nontrivial zero in the upper half-plane of the Riemann zeta function. In [6] it was shown that

$$(24) \quad \nu_1 = 14.134\ 725\ 141\ 734\ 693\ 790\ 457 + 10^{-21}\theta,$$

$$(25) \quad \nu_2 = 21.022\ 039\ 638\ 771\ 554\ 992\ 628 + 10^{-21}\theta, \quad (|\theta| \leq 1).$$

Thus we have

$$(26) \quad \left| J_2\left(\frac{1}{2} + i\nu_j\right) \right| \leq \frac{1}{8} \int_0^\infty \left| \frac{d}{dt^2} ((t^2 + 1)^{-s}) \right| dt$$

$$\leq \frac{1}{6} \left| \frac{1}{2} + i\nu_j \right| \left| \frac{3}{2} + i\nu_j \right| < \begin{cases} 33.5065, & j = 1, \\ 73.8644, & j = 2. \end{cases}$$

From (5), (19), (22) and (26) it follows that

$$(27) \quad R_0\left(\frac{1}{2} + i\nu_j\right) = \frac{\sqrt{\pi} \Gamma(1 + i\nu_j)}{24 D^{1+i\nu_j} \Gamma\left(\frac{1}{2} + i\nu_j\right)} \zeta(2 + 2i\nu_j) c\left(\varepsilon, -\frac{3}{2} - i\nu_j\right)$$

$$+ \frac{\alpha_j \zeta(2) c\left(\varepsilon, -\frac{\varepsilon}{2}\right) \theta}{D} \quad (|\theta| \leq 1),$$

where

$$(28) \quad \alpha_j = \begin{cases} 33.5065, & j = 1, \\ 73.8644, & j = 2. \end{cases}$$

It can be readily shown that

$$(29) \quad c\left(\varepsilon, \frac{1}{2} + i\nu_j\right) = c\left(\frac{1}{2} + i\nu_j\right) - 2 \int_{\ln \varepsilon}^\infty \frac{dv}{(e^v + e^{-v})^{\frac{1}{2} + i\nu_j}}$$

$$= \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}i\nu_j\right)^2}{2\Gamma\left(\frac{1}{2} + i\nu_j\right)} + \frac{4\theta}{\sqrt{\varepsilon}} \quad (|\theta| \leq 1).$$

From (5) it follows that

$$(30) \quad R_0 \left( \frac{1}{2} + i\nu_j \right) = -\zeta(1 + 2i\nu_j) c \left( \varepsilon, \frac{1}{2} + i\nu_j \right) - \sqrt{\pi} D^{-i\nu_j} \frac{\zeta(2i\nu_j) \Gamma(i\nu_j)}{\Gamma(\frac{1}{2} + i\nu_j)} c \left( \varepsilon, \frac{1}{2} - i\nu_j \right).$$

By well-known formulae for the Riemann zeta function and gamma function we have

$$(31) \quad \zeta(2i\nu_j) \Gamma(i\nu_j) = \pi^{-\frac{1}{2} + 2i\nu_j} \zeta(1 - 2i\nu_j) \Gamma \left( \frac{1}{2} - i\nu_j \right).$$

From (27), (30), (29) and (31) it follows that

$$(32) \quad \left( \frac{D}{\pi^2} \right)^{i\nu_j} = -\frac{\Gamma(\frac{1}{4} - \frac{1}{2}i\nu_j)^2 \zeta(1 - 2i\nu_j)}{\Gamma(\frac{1}{4} + \frac{1}{2}i\nu_j)^2 \zeta(1 + 2i\nu_j)} + \sum_{k=1}^3 J_k^* \left( \frac{1}{2} + i\nu_j \right), \quad j = 1, 2,$$

where

$$(33) \quad J_1^* \left( \frac{1}{2} + i\nu_j \right) = -\frac{16}{\sqrt{\varepsilon}} \left( \frac{D}{\pi^2} \right)^{i\nu_j} \frac{\Gamma(\frac{1}{2} + i\nu_j)}{\Gamma(\frac{1}{4} + \frac{1}{2}i\nu_j)^2} \theta, \quad j = 1, 2 \ (|\theta| \leq 1),$$

$$(34) \quad J_2^* \left( \frac{1}{2} + i\nu_j \right) = -\frac{\pi^{\frac{1}{2} - 2i\nu_j} \Gamma(1 + i\nu_j) \zeta(2 + 2i\nu_j) c(\varepsilon, -\frac{3}{2} - i\nu_j)}{12D \Gamma(\frac{1}{4} + \frac{1}{2}i\nu_j)^2 \zeta(1 + 2i\nu_j)}, \quad j = 1, 2,$$

$$(35) \quad J_3^* \left( \frac{1}{2} + i\nu_j \right) = -\frac{1}{3} \left( \frac{\pi^2}{D} \right)^{1 - i\nu_j} \alpha_j \frac{\Gamma(\frac{1}{2} + i\nu_j) c(\varepsilon, -\frac{3}{2}) \theta}{\Gamma(\frac{1}{4} + \frac{1}{2}i\nu_j)^2 \zeta(1 + 2i\nu_j)}, \quad j = 1, 2 \ (|\theta| \leq 1).$$

A direct calculation gives

$$(36) \quad \left| \Gamma \left( \frac{1}{2} + i\nu_j \right) \right| > \begin{cases} 5.708835 \times 10^{-10}, & j = 1, \\ 1.413149 \times 10^{-14}, & j = 2, \end{cases}$$

$$(37) \quad \left| \Gamma \left( \frac{3}{4} + \frac{1}{2}i\nu_j \right) \right| < \begin{cases} \exp(-9.6937176), & j = 1, \\ \exp(-15.0036975), & j = 2, \end{cases}$$

$$(38) \quad |\Gamma(1 + i\nu_j)| < \begin{cases} 2.158099 \times 10^{-9}, & j = 1, \\ 5.261146 \times 10^{-4}, & j = 2, \end{cases}$$

$$(39) \quad |\zeta(2 + 2i\nu_j)| < \begin{cases} 1.4229, & j = 1, \\ 0.9162, & j = 2. \end{cases}$$

By (36), (37) and the multiplication formula for the gamma function we have

$$(40) \quad \left| \Gamma \left( \frac{1}{4} + \frac{1}{2}i\nu_j \right) \right|^2 > \begin{cases} 5.3843214 \times 10^{-10}, & j = 1, \\ 8.8395796 \times 10^{-15}, & j = 2. \end{cases}$$

By [9] we have

$$(41) \quad |\zeta(1 + 2i\nu_j)| > \begin{cases} 1.948757, & j = 1, \\ 0.830962, & j = 2. \end{cases}$$

For  $D > 10^{13}$  we easily have

$$(42) \quad c\left(\varepsilon, -\frac{3}{2}\right) < \frac{4}{3} \left\{ \left(1 + \frac{4}{\varepsilon^2}\right)^{3/2} + \left(\left(1 + \frac{1}{\varepsilon}\right)^{3/2} - 1\right) 2^{-3/2} + (2^{3/2} - 1)e^{-3/4} \right\} D^{3/4} < 1.33336607D^{3/4}.$$

Therefore, for  $D > 10^{13}$  we have

$$(43) \quad \left| J_1^* \left( \frac{1}{2} + i\nu_j \right) \right| < \begin{cases} 16.9643217D^{-1/4}, & j = 1, \\ 20.6914635D^{-1/4}, & j = 2, \end{cases}$$

$$(44) \quad \left| J_2^* \left( \frac{1}{2} + i\nu_j \right) \right| < \begin{cases} 0.57636882D^{-1/4}, & j = 1, \\ 1.2924129D^{-1/4}, & j = 2, \end{cases}$$

$$(45) \quad \left| J_3^* \left( \frac{1}{2} + i\nu_j \right) \right| < \begin{cases} 79.967973D^{-1/4}, & j = 1, \\ 504.258735D^{-1/4}, & j = 2. \end{cases}$$

From (32), (43), (44) and (45) it follows that for  $D > 10^{13}$  we have

$$(46) \quad \left(\frac{D}{\pi^2}\right)^{i\nu_j} = -\frac{\Gamma\left(\frac{1}{4} - \frac{1}{2}i\nu_j\right)^2 \zeta(1 - 2i\nu_j)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}i\nu_j\right)^2 \zeta(1 + 2i\nu_j)}$$

$$(47) \quad + \begin{cases} 97.5086636\theta D^{-\frac{1}{4}}, & j = 1, \\ 526.242612\theta D^{-\frac{1}{4}}, & j = 2, \end{cases} \quad (|\theta| \leq 1).$$

Taking the arguments on both sides of (46) and (47), and noticing that  $D > 10^{13}$ , we obtain

$$(48) \quad \nu_j \ln(D/\pi^2) = a_j + 2\pi x_j + \begin{cases} 0.054833152\theta, & j = 1, \\ 0.295927968\theta, & j = 2, \end{cases}$$

where  $x_j$  are nonnegative integers and

$$(50) \quad a_j \equiv \pi - 4 \arg \Gamma\left(\frac{1}{4} + \frac{1}{2}i\nu_j\right) - 2 \arg \zeta(1 + 2i\nu_j) \pmod{2\pi},$$

$$0 \leq a_j < 2\pi, \quad j = 1, 2.$$

From (48) and (49) it follows that

$$(51) \quad x_2 = \frac{\nu_2}{\nu_1} x_1 + a + R^*,$$

where

$$(52) \quad a = \frac{1}{2\pi} \left( \frac{\nu_2}{\nu_1} a_1 - a_2 \right)$$

and

$$(53) \quad |R^*| \leq \frac{1}{2\pi} \left( \frac{\nu_2}{\nu_1} (0.054833152) + 0.295927968 \right) < 0.0600776857.$$

In [9] it is shown that

$$(54) \quad \frac{\nu_2}{\nu_1} = 1.487\ 262\ 003\ 892\ 890\ 048 + 10^{-18}\theta,$$

$$\begin{aligned}
 (55) \quad & \frac{1}{\pi} \arg \Gamma \left( \frac{1}{2} + i\nu_j \right) = \begin{cases} 7.418\ 512\ 651\ 985\ 173 + 2 \times 10^{-13}\theta, & j = 1, \\ 13.688\ 619\ 111\ 000\ 235 + 2 \times 10^{-13}\theta, & j = 2, \end{cases} \\
 (56) \quad & \\
 (57) \quad & \frac{1}{\pi} \arg \zeta(1+2i\nu_j) = \begin{cases} -0.108\ 452\ 737\ 083\ 095 + 10^{-10}\theta \pmod{2}, & j = 1, \\ 0.067\ 103\ 865\ 503\ 910 + 10^{-10}\theta \pmod{2}, & j = 2. \end{cases} \\
 (58) \quad &
 \end{aligned}$$

It can be easily seen that

$$(59) \quad \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}i\nu_j\right)}{\Gamma\left(\frac{1}{4} - \frac{1}{2}i\nu_j\right)} = 2^{\frac{1}{2}-i\nu_j} \Gamma\left(\frac{1}{2} + i\nu_j\right) \sin\left(\left(\frac{1}{4} - \frac{1}{2}i\nu_j\right)\pi\right) / \sqrt{\pi},$$

from which it follows that

$$(60) \quad \arg \Gamma\left(\frac{1}{4} + \frac{1}{2}i\nu_j\right) = \frac{1}{2} \left( \arg \Gamma\left(\frac{1}{2} + i\nu_j\right) - \nu_j \ln 2 + \arg \left\{ \sin\left(\left(\frac{1}{4} - \frac{1}{2}i\nu_j\right)\pi\right) \right\} \right).$$

By (60), (55) and (56) we have

$$\begin{aligned}
 (61) \quad & \frac{2}{\pi} \arg \Gamma\left(\frac{1}{4} + \frac{1}{2}i\nu_j\right) \\
 (62) \quad & = \begin{cases} 4.049\ 889\ 087\ 345\ 757\ 85 + 2.1 \times 10^{-13}\theta, & j = 1. \\ 8.800\ 408\ 778\ 867\ 03 + 2.1 \times 10^{-13}\theta, & j = 2. \end{cases}
 \end{aligned}$$

From (50), (57), (58), (61) and (62) we obtain

$$\begin{aligned}
 (63) \quad & \frac{a_j}{2\pi} = \begin{cases} 0.558\ 563\ 649\ 737\ 337\ 15 + 1.003 \times 10^{10}\theta, & j = 1. \\ 0.632\ 487\ 355\ 629\ 06 + 1.003 \times 10^{-10}\theta, & j = 2, \end{cases} \\
 (64) \quad & \\
 (65) \quad & a = 0.198\ 243\ 137\ 455\ 44 + 1.76 \times 10^{-10}\theta.
 \end{aligned}$$

Our next device is the following reiteration. By (48) and  $D > 10^{13}$  we first have  $x_1 > 61$ . Then, by using this reiteration, we push  $x_1$  to a much greater value which corresponds to a much greater lower bound for  $D$ .

Let us now prove that

$$(66) \quad x_1 \geq 84.$$

Taking  $x_1 = 7$  in (51) gives

$$(67) \quad 10.410\ 834\ 027\ 250\ 230\ 336 + 7 \times 10^{-18}\theta = 7 \frac{\nu_2}{\nu_1} + a + R^* + 1.77 \times 10^{-10}\theta.$$

By (51) and (67) we have

$$(68) \quad x_2 - 10 = \frac{\nu_2}{\nu_1}(x_1 - 7) - b_1 + R_1,$$

where

$$(69) \quad b_1 = 0.4108, \quad |R_1| < 0.1201893989.$$

Take

$$(70) \quad p_1 = 58, \quad q_1 = 39,$$

for which we have

$$(71) \quad \left| \frac{\nu_2}{\nu_1} - \frac{p_1}{q_1} \right| < 8.2517 \times 10^{-5}.$$

Assume that  $Q$  and  $R$  ( $0 \leq R < q_1$ ) are two integers such that

$$(72) \quad Q + \frac{R}{q_1} = \frac{p_1}{q_1}(x_1 - 7).$$

By (68) and (72) we have

$$(73) \quad x_2 - Q - 10 = \left( \frac{\nu_2}{\nu_1} - \frac{p_1}{q_1} \right) (x_1 - 7) + \left( \frac{R}{q_1} - b_1 \right) + 0.1201893989\theta.$$

If  $x_1 \leq 83$ , by (71) and (73) we have

$$(74) \quad |x_2 - \theta - 10| \leq 76 \times 8.2517 \times 10^{-5} + (1 - b_1) + 0.121 < 1.$$

On the other hand, we have

$$(75) \quad b_1 q_1 = (0.4108)(39) = 16.0212.$$

Thus, for  $R \notin [12, 20]$  we have

$$(76) \quad |x_2 - Q - 10| > 0.12766 - 76 \times 8.2517 \times 10^{-5} - 0.121 > 0.$$

which contradicts (74).

Furthermore, we easily have

$$(77) \quad 3q_1 - 2p_1 = 1,$$

from which and (72) it follows that

$$(78) \quad x_1 \equiv 7 + 37R \pmod{q_1}.$$

For  $61 < x_1 \leq 83$  and  $R \in [12, 20]$  a direct calculation shows that there is no such pair of  $x_1$  and  $R$  satisfying (78). This proves (66).

Now we prove that

$$(79) \quad x_1 \geq 92.$$

By (66) and (48) we have

$$(80) \quad D > 2.07447887 \times 10^{17}.$$

By (80) and (46), (47) we easily reduce (48) and (49) to

$$(81) \quad \nu_j \ln(D/\pi^2) = a_j + 2\pi x_j + \begin{cases} 0.0045689453\theta, & j = 1, \\ 0.0246580517\theta, & j = 2, \end{cases}$$

and (73) because

$$(83) \quad x_2 - Q - 10 = \left( \frac{\nu_2}{\nu_1} - \frac{p_2}{q_2} \right) (x_1 - 7) + \left( \frac{R}{q_2} - b_1 \right) + 0.0100459145\theta,$$

where we choose

$$(84) \quad p_2 = 409, \quad q_2 = 275.$$

We easily have

$$(85) \quad \left| \frac{\nu_2}{\nu_1} - \frac{p_2}{q_2} \right| < 1.0724 \times 10^{-5}$$

and

$$(86) \quad 39p_2 - 58q_2 = 1, \quad b_1 q_2 = 112.97.$$

If (79) is false, we should have

$$(87) \quad 84 \leq x_1 \leq 91.$$

By (83), (85) and (87) we have

$$(88) \quad |x_2 - Q - 10| \leq (84)(1.0724 \times 10^{-5}) + (1 - b_1) + 0.01005 < 1.$$

For  $R \notin [110, 115]$  we have

$$(89) \quad \left| \frac{R}{q_2} - b_1 \right| \geq \frac{3.03}{q_2} > 0.01101818.$$

Hence, for  $R \notin [110, 115]$  we have

$$|x_2 - Q - 10| \geq 0.01101818 - (84)(1.0724 \times 10^{-5}) - 0.010046 > 0,$$

which contradicts (88). Here we have assumed that  $Q$  and  $R$  ( $0 \leq R < q_2$ ) are two integers defined by

$$(90) \quad Q + \frac{R}{q_2} = \frac{p_2}{q_2}(x_1 - 7).$$

By (86) and (90) we have

$$(91) \quad x_1 \equiv 7 + 39R \pmod{q_2}.$$

A direct calculation shows that there is no such pair of  $x_1$  and  $R$ ,  $84 \leq x_1 \leq 91$ ,  $R \in [110, 115]$ . This proves (79).

By (79) we have

$$(92) \quad \nu_j \ln(D/\pi^2) = a_j + 2\pi x_j + \begin{cases} 0.0018763923\theta, & j = 1, \\ 0.0101266647\theta, & j = 2. \end{cases}$$

By taking

$$(94) \quad p_3 = 467, \quad q_3 = 314$$

and defining  $Q$  and  $R$  ( $0 \leq R < q_3$ ) by

$$(95) \quad Q + \frac{R}{q_3} = \frac{p_3}{q_3}(x_1 - 7),$$

we have

$$(96) \quad x_2 - Q - 10 - \left( \frac{\nu_2}{\nu_1} - \frac{p_3}{q_3} \right) (x_1 - 7) + \left( \frac{R}{q_3} - b_1 \right) + 0.0041457482\theta.$$

This easily leads to

$$(97) \quad x_1 \geq 289.$$

By (97) and taking

$$(98) \quad p_4 = 468995513, \quad q_4 = 315341555,$$

we are led to

$$(99) \quad x_1 \geq 315341562.$$

By (99) and taking

$$(100) \quad p_5 = 1310620852, \quad q_5 = 881230643,$$

we are led to

$$(101) \quad x_1 \geq 836441460.$$

By (101) we easily have

$$D > \exp(371815978),$$

which is the desired conclusion.  $\square$

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