

## TURÁN'S PURE POWER SUM PROBLEM

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ABSTRACT. Let  $1 = z_1 \geq |z_2| \geq \cdots \geq |z_n|$  be  $n$  complex numbers, and consider the power sums  $s_\nu = z_1^\nu + z_2^\nu + \cdots + z_n^\nu$ ,  $1 \leq \nu \leq n$ . Put  $R_n = \min \max_{1 \leq \nu \leq n} |s_\nu|$ , where the minimum is over all possible complex numbers satisfying the above. Turán conjectured that  $R_n > A$ , for  $A$  some positive absolute constant. Atkinson proved this conjecture by showing  $R_n > 1/6$ . It is now known that  $1/2 < R_n < 1$ , for  $n \geq 2$ . Determining whether  $R_n \rightarrow 1$  or approaches some other limiting value as  $n \rightarrow \infty$  is still an open problem. Our calculations show that an upper bound for  $R_n$  decreases for  $n \leq 55$ , suggesting that  $R_n$  decreases to a limiting value less than 0.7 as  $n \rightarrow \infty$ .

### 1. INTRODUCTION

Let  $z_1, z_2, \dots, z_n$  be complex numbers and let

$$(1) \quad s_\nu = \sum_{k=1}^n z_k^\nu$$

be the  $\nu$ th (pure) power sum of these  $n$  complex numbers. Turán initiated the study of the problem of determining how small  $|s_\nu|$  can be for all  $\nu$  in some range and  $z_k$  satisfying various conditions. Depending on the conditions imposed on the range of values of  $\nu$  and  $z_k$ , and also on the norm used in obtaining the lower bound, many interesting problems arise, see [8, Chapters 1–4]. We are concerned with the problem of determining how small  $|s_\nu|$  can be for all  $1 \leq \nu \leq m$  and for any configuration of complex numbers satisfying

$$(2) \quad 1 = z_1 \geq |z_2| \geq \cdots \geq |z_n|.$$

This condition on the  $z_k$ 's is not particularly restrictive since many more general problems can quickly be reduced to this case. We let

$$(3) \quad M_n(m) = \max_{1 \leq \nu \leq m} |s_\nu|, \quad R_n(m) = \min_{1 = z_1 \geq |z_2| \geq \cdots \geq |z_n|} M_n(m),$$

i.e.,  $R_n(m)$  is the min-max of the absolute value of the first  $m$  power sums of  $n$  complex numbers satisfying (2). Since the maximum is nondecreasing with  $m$ , we

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see that  $R_n(m)$  is nondecreasing in  $m$ . Taking  $z_k = e(k/n)$ , for  $1 \leq k \leq n$ , where  $e(u) = e^{2\pi i u}$ , we see that  $s_\nu = 0$  for  $1 \leq \nu \leq n-1$ . Therefore,

$$R_n(m) = 0 \quad \text{for } 1 \leq m \leq n-1.$$

The first case of interest is thus when  $m = n$ , the min-max of  $n$  complex numbers over the first  $n$  power sums. For this case, denote

$$(4) \quad M_n = M_n(n), \quad R_n = R_n(n).$$

Turán conjectured in 1938 that  $R_n$  is bounded below by an absolute constant for all  $n$ . This conjecture was proved by Atkinson [1] in 1961, where he obtained

$$(5) \quad R_n > 1/6.$$

Atkinson subsequently improved his result to  $R_n > 1/3$  and also obtained a somewhat larger lower bound for some  $n$  [2]. Very recently, Biró [3] found an elegant elementary argument which shows

$$(6) \quad R_n > 1/2.$$

On the other hand, Komlós, Sárközy, and Szemerédi [4] showed that

$$(7) \quad R_n < 1 - \frac{1}{250n}$$

for all sufficiently large  $n$ . In view of these results the main question to be examined is whether  $R_n \rightarrow 1$  as  $n \rightarrow \infty$  or whether it decreases to some limiting value. Both possibilities have been suggested, and it appears Turán himself has at various times conjectured both possibilities [8, p. 30], [9, p. XVII]. The only known values are  $R_1 = 1$ ,  $R_2 = 0.87403\dots$ , and  $R_3 = 0.8247830\dots$ . Here  $R_1 = 1$  is trivial, and  $R_2 = \sqrt{3 - \sqrt{5}}$  may be obtained by a relatively easy Lagrange multiplier argument. The value  $(R_2)^2$  satisfies the equation

$$x^2 - 6x + 4 = 0.$$

Determination of  $R_3$  is much harder. Lawrynowicz [6] determined  $R_3$  algebraically in 1960, and obtained the numerical value given above in 1967 [7]. The value  $(R_3)^2$  satisfies the equation<sup>1</sup>

$$\begin{aligned} & x^{15} - 81x^{14} + 2613x^{13} - 43629x^{12} + 417429x^{11} - 2450985x^{10} + 9516137x^9 \\ & - 26203659x^8 + 53016480x^7 - 83714418x^6 + 112601340x^5 - 140002992x^4 \\ & + 156204288x^3 - 124361568x^2 + 55427328x - 10077696 = 0. \end{aligned}$$

In this paper we report on some computations which extend our knowledge of  $R_n$  and suggest that  $R_n$  decreases to a limiting value. Our results are summarized in the following theorem.

<sup>1</sup>Lawrynowicz determined  $(R_3)^2$  in terms of a different algebraic number which satisfied an equation of degree 15. The equation above was obtained for us by David Bailey, who used the Ferguson-Forcade algorithm and his multiprecision arithmetic package for finding integer relations between numbers.

TABLE 1

$n$	$M_n$	$n$	$M_n$
2	0.87403204889764214159	27	0.71351379274002663079
3	0.82478303091462249657	28	0.71280297643602584513
4	0.79787183891219785566	29	0.71213492854485684518
5	0.78068751175852318911	30	0.71150571445562842882
6	0.76866240212865063009	31	0.71091187832481901227
7	0.75972275127528535640	32	0.71035037153531655407
8	0.75278476350214275366	33	0.70981849370598932067
9	0.74722423731338641938	34	0.70931384374342863335
10	0.74265504914476718905	35	0.70883427898651897355
11	0.73882487313790486797	36	0.70837788091679418294
12	0.73556147974443498507	37	0.70794292622936772579
13	0.73274307211717818350	38	0.70752786230650853148
14	0.73028094927147070166	39	0.70713128632741405830
15	0.72810889170059241298	40	0.70675192739708693051
16	0.72617640456896152304	41	0.70638863119452338965
17	0.72444427147342536314	42	0.70604034673315181320
18	0.72288154379177815772	43	0.70570611490021873219
19	0.72146345068685877134	44	0.70538505850083375768
20	0.72016991610628663886	45	0.70507637357986472730
21	0.71898448586498316988	46	0.70477932183327719645
22	0.71789353784931532652	47	0.70449322395173041892
23	0.71688569151654277288	48	0.70421745376474166007
24	0.71595136015246079880	49	0.70395143307465529630
25	0.71508240701664073135	50	0.70369462708689897930
26	0.71427187818188997713	55	0.70253266322361406713

**Theorem.** *There exist configurations of points satisfying equation (2) with  $M_n$  given in Table 1. These numbers are accurate to 20 digits.*

## 2. THE COMPUTATIONAL SCHEME

Our method for computing  $R_n$  is due to Ławrynowicz [6]. He proved that among the extremal configurations of points  $z_k$  which takes on the value  $R_n$ , there is always a configuration with

$$(8) \quad |s_1| = |s_2| = \cdots = |s_n|.$$

This is possibly the only type of configuration where  $R_n$  is attained, but this is not known. Consequently, we can formulate the computing of  $R_n$  as a Lagrange multiplier problem. To compute  $R_n$ , let  $z_1 = 1$ ,  $z_k = x_k + iy_k$  for  $2 \leq k \leq n$ , and

$$(9) \quad \begin{aligned} & F(x_2, x_3, \dots, x_n, y_2, \dots, y_n, \lambda_1, \lambda_2, \dots, \lambda_{n-1}) \\ &= |s_1|^2 - \sum_{k=1}^{n-1} \lambda_k (|s_1|^2 - |s_{k+1}|^2). \end{aligned}$$

Then the equation  $R_n = |s_1|$  is satisfied at a critical point of  $F$ . To obtain this critical point, we need to solve the system of  $3n - 3$  equations

$$(10) \quad \frac{\partial F}{\partial x_{j+1}} = 0, \quad \frac{\partial F}{\partial y_{j+1}} = 0, \quad \frac{\partial F}{\partial \lambda_j} = 0, \quad \text{for } j = 1, 2, \dots, n-1.$$

Each equation is a polynomial in  $3n - 3$  variables of degree  $\leq 2n$  in each variable. For example, when  $n = 2$  we have  $z_1 = 1$ ,  $z_2 = x_2 + iy_2$ ,  $|s_1|^2 = (1 + x_2)^2 + y_2^2$ ,  $|s_2|^2 = (1 + x_2^2 - y_2^2)^2 + 4x_2^2 y_2^2$ , and our system is

$$(11a) \quad \frac{\partial F}{\partial x_2} = 2(1 + x_2) - \lambda_1 \left( 2(1 + x_2) - 4x_2(1 + x_2^2 - y_2^2) - 8x_2 y_2^2 \right) = 0,$$

$$(11b) \quad \frac{\partial F}{\partial y_2} = 2y_2 - \lambda_1 \left( 2y_2 + 4y_2(1 + x_2^2 - y_2^2) - 8x_2^2 y_2 \right) = 0,$$

$$(11c) \quad \frac{\partial F}{\partial \lambda_1} = (1 + x_2)^2 + y_2^2 - (1 + x_2^2 - y_2^2)^2 - 4x_2^2 y_2^2 = 0.$$

This system may be solved by hand; equations (11a) and (11b) imply  $1 + 2x_2 = x_2^2 + y_2^2$ . From equation (11c) we get  $x_2 = (1 \pm \sqrt{5})/4$ , and  $y_2 = \pm\sqrt{3}x_2$ . The value  $y_2 = 1 + \sqrt{5}$  is outside the unit circle. Therefore the value for  $R_2$  is  $\sqrt{3 - \sqrt{5}}$ , which is obtained when  $x_2 = (1 - \sqrt{5})/4$ , and  $y_2 = \pm\sqrt{3}x_2$ .

This example illustrates one of the procedures we use to compute  $R_n$ . To find the critical points of (10), Newton's method is used. At each critical point we compute  $|s_1|$ ; the smallest such value is  $R_n$ . As will be described below, we cannot actually carry this procedure out completely. What we actually compute is a configuration which is likely to give the value  $R_n$ , and this configuration of points, independent of the computational scheme, can be checked to give a small value for  $M_n$ , and thus provides a rigorously proven upper bound for  $R_n$ .

### 3. THE COMPUTATION

All of our computations make use of Mathematica. The system of equations (10) is solved numerically using a Newton iterative scheme which converges quadratically to the solution that is closest to the given initial guess. To obtain all solutions of (10) with  $M_n < 1$ , random initial guesses are used in Newton's method. The program is run with thousands of different random initial guesses and continued until no further new critical points are obtained in over several thousand runs. For  $n = 3$ , only two critical points are found, one of which gives Ławrynowicz's solution, and the other is outside the unit circle with the value 1.374... Proceeding in this fashion, we compute the critical points for  $n = 4, 5, 6$ . The resulting smallest values for  $M_n$  at the critical points are listed in Table 2. By the time  $n = 6$ , 56 critical points with  $M_n < 1$  and 125 critical points with  $M_n < 1.45$  are found. This procedure does not ensure that the values in Table 2 are a complete list of all critical points. At  $n = 7$ , the scheme of using random initial guesses in Newton's algorithm for solving (10) is infeasible for finding  $R_n$ . First, the Newton algorithm will only converge infrequently, and second, there are large numbers of critical points. This algorithm, for  $n = 7$ , took several months of CPU time before yielding the value given in Table 1.

TABLE 2

Values of  $M_n < 1$  at critical points (not all 56 values found for  $n = 6$  shown)

$n = 3$	$n = 4$	$n = 5$	$n = 6$
0.8247830309	0.7978718389	0.7806875117	0.7686624021
	0.8389236336	0.8337671859	0.8111179657
	0.8762366068	0.8690990094	0.8285874602
	0.9050025366	0.8712303154	0.8393034012
	0.9074678426	0.8727150203	0.8432191421
	0.9504184932	0.8825304010	0.8483174001
	0.9577375152	0.8829360128	0.8503894070
	0.9746757164	0.8829360128	0.8537885153
	0.9885588908	0.9069016543	0.8556601896
		0.9333716319	0.8613214008
		0.9413655298	0.8737623150
		0.9443800869	0.8740381632
		0.9550142456	0.8894480590
		0.9677037216	0.8958343035
		0.9828023416	0.8958688134
		0.9889082339	0.8972435100
		0.9922111204	0.9178211517
			0.9202763035
			...
			0.9996198736
			0.9996426733
			0.9998003871
			0.9998070146

We can draw two conclusions from these calculations. First, because of the small number of critical points with  $M_4 \leq 1$ , it may be possible but very laborious to prove that the upper bound for  $R_4$  in Table 1 is actually equal to  $R_4$ . Secondly, because the next smallest value of  $M_n$  found for  $n = 4, 5, 6$  is considerably larger than the smallest value, it is very likely that the numbers in Table 1 are the correct values for  $R_4, R_5$ , and  $R_6$ .

To go beyond  $n = 6$  we use information obtained from our extremal configurations for  $n = 3, 4, 5$  and  $6$ . These are shown in Figures 1a and 1b. The power sums display a very regular pattern and they lie on a circle as required by equation (8). In every case we find that they move from  $s_1$  to  $s_n$  in order, clockwise, starting from the top position. Further, as  $n$  increases the points shift in a regular pattern. By computing difference tables we obtained a guess for the case  $n = 7$ . That is, we obtain  $s_k, 1 \leq k \leq 7$ . One can then obtain the  $z$ 's from the  $s$ 's by solving the following equations:

$$(12) \quad p(z) = \prod_{k=1}^n (z - z_k) = \sum_{k=0}^n a_k z^{n-k}, \quad z_1 = 1, \quad a_0 = 1,$$

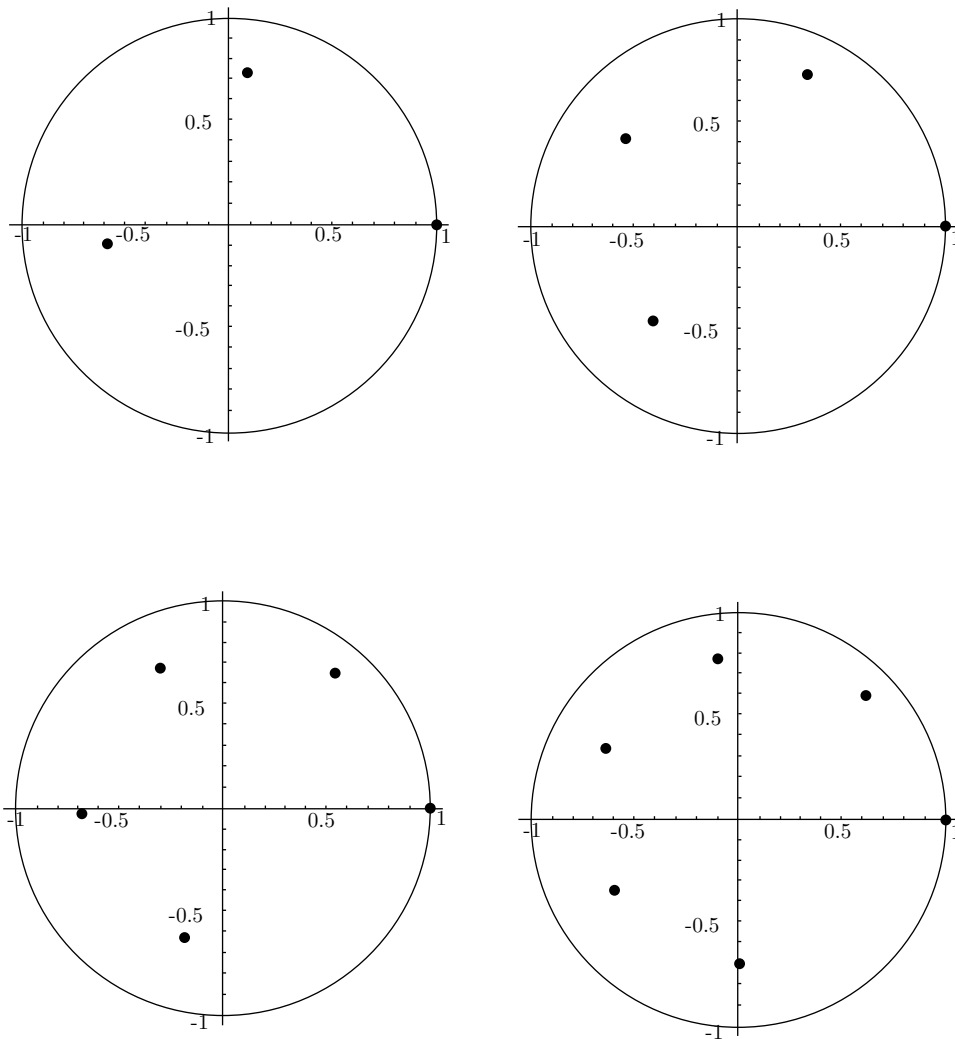


FIGURE 1a. Extremal configuration of  $z_k$ 's for  $n = 3, 4, 5, 6$

and

$$(13) \quad a_k = (-1)^k \frac{1}{k!} \det \begin{pmatrix} s_1 & 1 & 0 & 0 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & 0 & 0 & \dots & 0 \\ s_3 & s_2 & s_1 & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ s_{k-1} & s_{k-2} & s_{k-3} & s_{k-4} & s_{k-5} & \dots & k \\ s_k & s_{k-1} & s_{k-2} & s_{k-3} & s_{k-4} & \dots & s_1 \end{pmatrix}.$$

The  $a_k$ 's are computed by (13) using the  $s$ 's determined by our difference procedure. From (12),  $p(z)$  is obtained. Newton's method is then used to find the roots  $z_k$  of  $p(z)$ . These  $z_k$ 's are used as the initial starting guess for the Newton method used

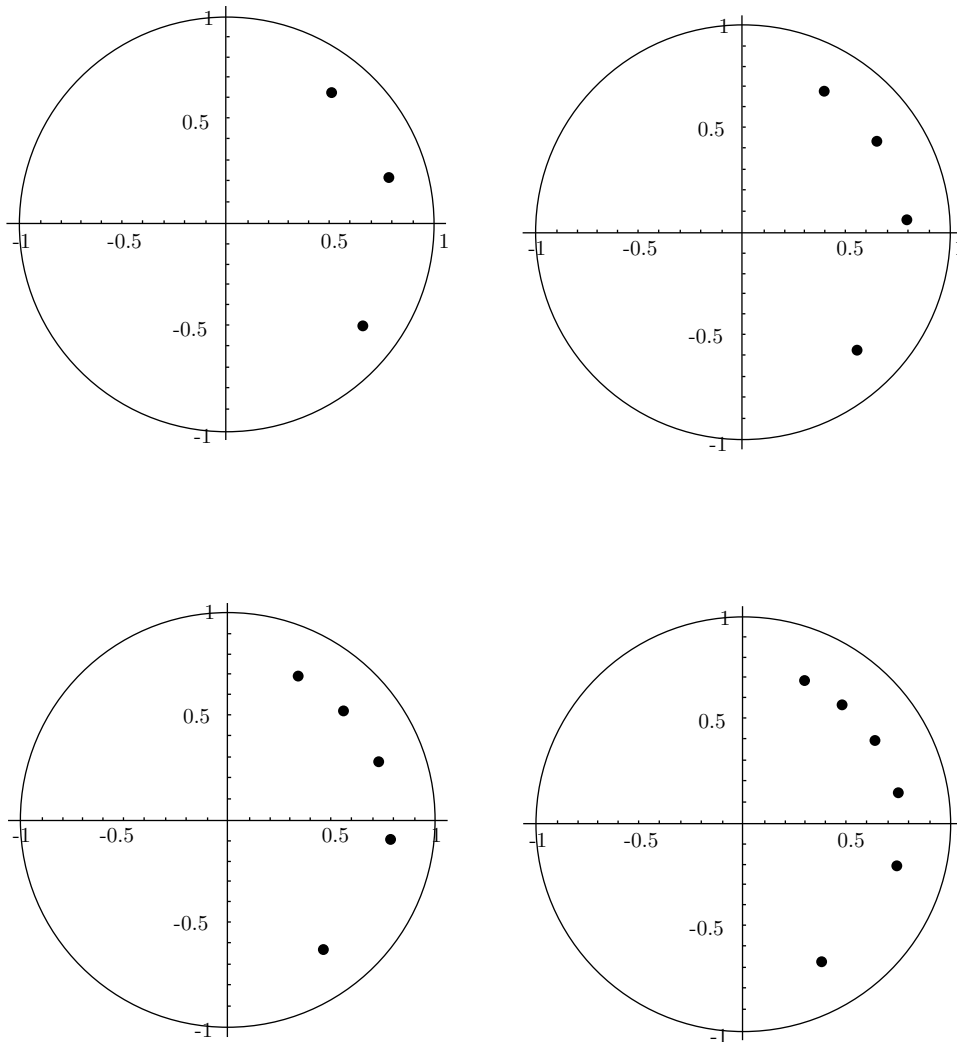


FIGURE 1b. Extremal configuration of  $s_k$ 's for  $n = 3, 4, 5, 6$

to solve (10). In practice, we found that Newton's method with this initial guess converged immediately to the answer. This procedure was used to compute Table 1 up to  $n = 36$ . At this point, the Newton method used to obtain the  $z$ 's from the  $s$ 's in (12) and (13) takes a considerable amount of CPU time. Instead, since the  $z$  configuration is now very regular, we use an interpolation routine to guess the new  $z$ 's from our previous results directly, bypassing equations (12) and (13). By this point it is also possible to obtain at once guesses for the  $z_k$ 's not just for the next  $n$ , but for the next five  $n$  with sufficient accuracy. These points are used as initial guesses in the Newton algorithm to solve (10). Using this procedure we compute up to  $M_{50}$ .

When we compute the power sums directly from our guess of  $z_k$ 's, without first solving (10), we obtain slightly higher values for  $M_n$ . In particular, using

the configurations for  $n \leq 50$  we obtain  $R_{51} \leq 0.70344665$ ,  $R_{52} \leq 0.7032078$ ,  $R_{53} \leq 0.702982$ ,  $R_{54} \leq 0.70281$ , and  $R_{55} \leq 0.702871$ . In comparison, for the case of  $n = 55$ , if the configuration obtained above is used as the initial guess to Newton's algorithm to solve (10) the value  $R_{55} \leq 0.702532\dots$  in Table 1 is obtained. The first 50 configurations give guesses for the  $z_k$ 's with  $M_n < 1$  for  $n \leq 62$ .

#### 4. CONCLUSIONS

From our calculations, it is likely that  $R_n$  decreases to a limiting value. In Figure 2 we display some of the extremal solutions found by our calculations. The extremal configuration of  $z$ 's are converging to a limiting configuration. On the other hand, the distribution of the power sums has not developed sufficiently to predict the ultimate pattern it will take.

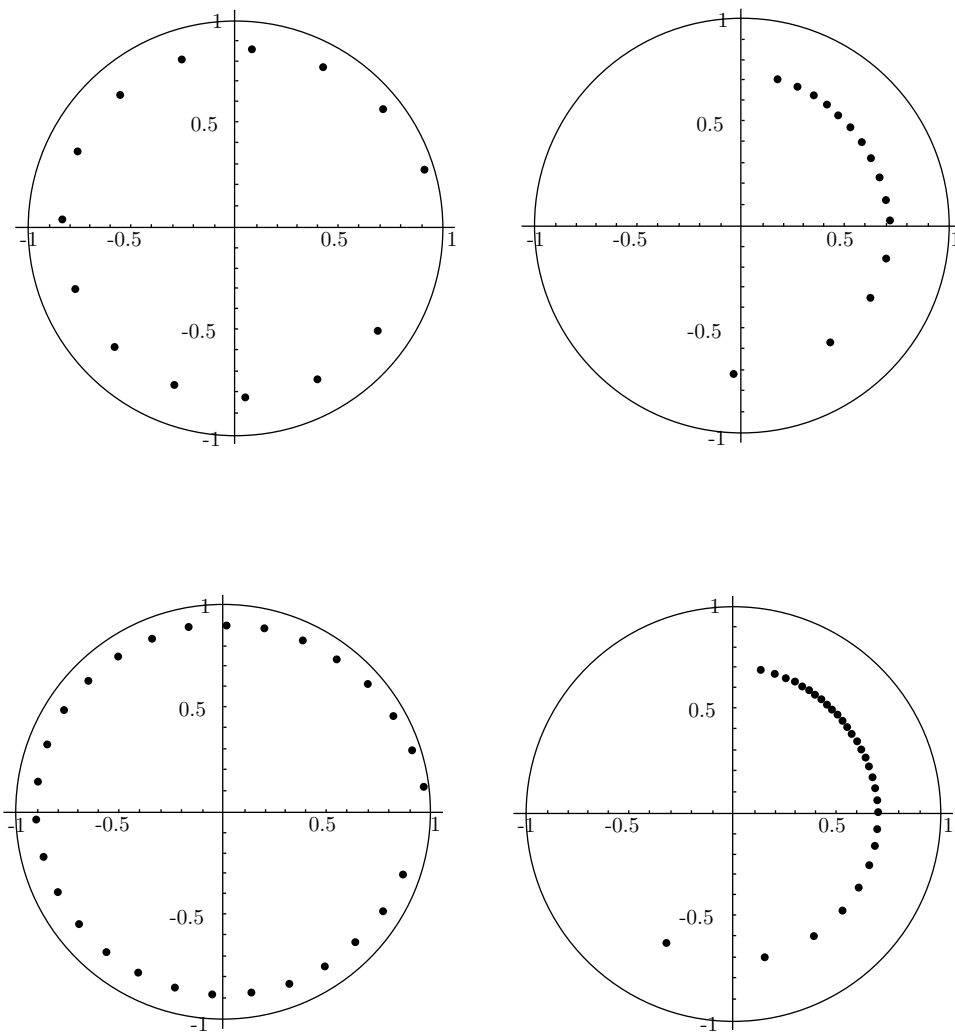


FIGURE 2a. Extremal configuration of  $z_k$ 's and  $s_k$ 's for  $n = 15, 30$



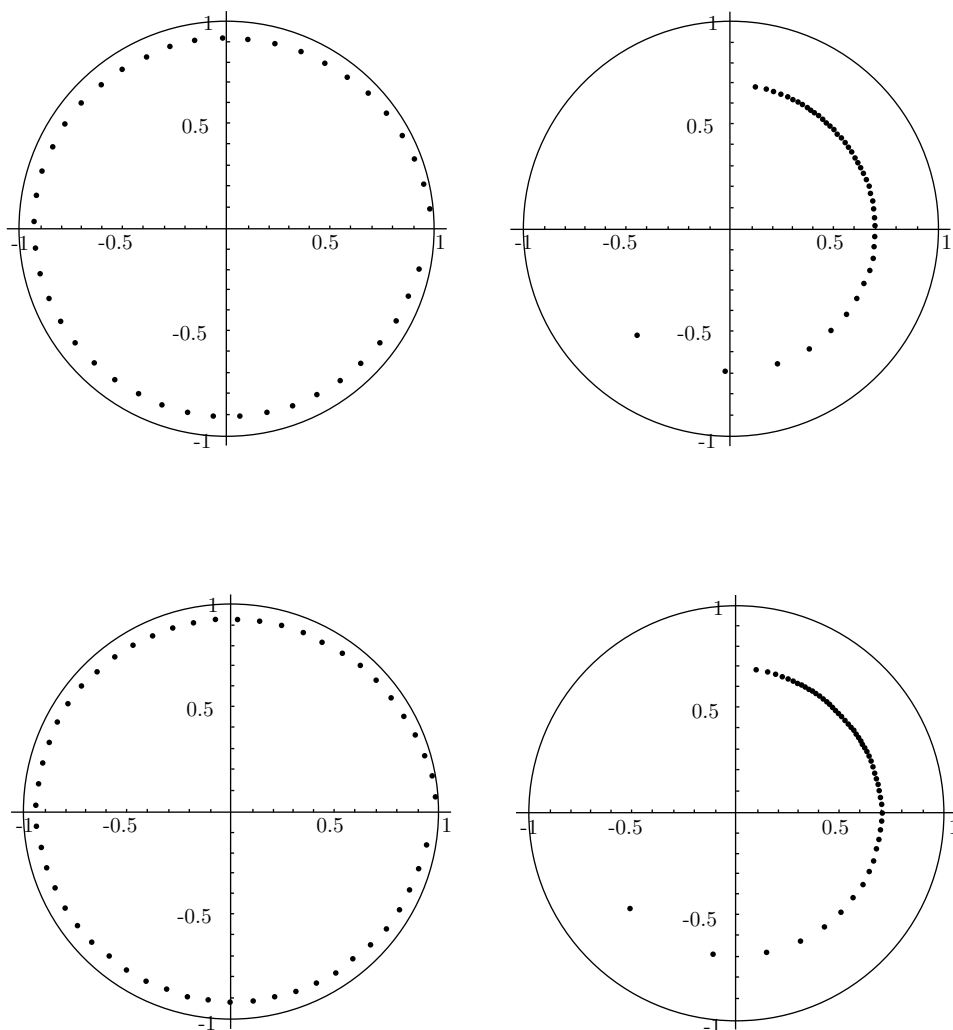


FIGURE 2b. Extremal configuration of  $z_k$ 's and  $s_k$ 's for  $n = 45, 55$

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