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THE QUASI-LAGUERRE ITERATION

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ABSTRACT. The quasi-Laguerre iteration has been successfully established, by the same authors, in the spirit of Laguerre's iteration for solving the eigenvalues of symmetric tridiagonal matrices. The improvement in efficiency over Laguerre's iteration is drastic. This paper supplements the theoretical background of this new iteration, including the proofs of the convergence properties.

1. INTRODUCTION

When the Laguerre iteration [4]

$$L_{\pm}(x) = x + \frac{n}{\left(-\frac{f'(x)}{f(x)}\right) \pm \sqrt{(n-1)\left[(n-1)\left(-\frac{f'(x)}{f(x)}\right)^2 - n\left(\frac{f''(x)}{f(x)}\right)\right]}} \tag{1.1}$$

is used to solve a polynomial f with all its zeros being real, such as the characteristic polynomial of a real symmetric matrix, the most important advantages are its global and monotonic convergence. While its ultimate convergence rate is cubic, the requirement of evaluating f'', which is relatively time consuming, constitutes a major disadvantage of this iteration in terms of its efficiency. A new iteration, which we called the quasi-Laguerre iteration, has been established in [1] which avoids the evaluation of f'' but still maintains global and monotone convergence when applied to polynomials with all real zeros. The purpose of this paper is to supplement the theoretical background of this new iteration.

Formula (1.1) can be derived in diverse ways. The best one seems to be to answer the following question [3]:

Question 1. Among all polynomials $p(x)$ of degree n with n real zeros and with $p(x_0) = f(x_0) \neq 0, p'(x_0) = f'(x_0)$ and $p''(x_0) = f''(x_0)$ at a specified real x_0, which one has a zero closest to x_0? and where?

In general, of all those polynomials, $L_+(x_0)$ in (1.1) gives the closest zero from the right and $L_-(x_0)$ gives the closest one from the left.

To avoid the evaluation of f'', the above optimization problem can be revised as follows:

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Question 2. Given two specified reals $a < b$, among all polynomials $p(x)$ of degree n with n real zeros, none of which lie in $[a, b]$, and with

$$\frac{p'(a)}{p(a)} = \frac{f'(a)}{f(a)} \quad \text{and} \quad \frac{p'(b)}{p(b)} = \frac{f'(b)}{f(b)}, \tag{1.2}$$

which one has a zero closest to a from the right or from the left? and where?

Even further, to account for multiple zeros, we reformulate Question 2 in a more general form:

Question 3. Given two specified reals $a < b$, for polynomials satisfying the conditions in Question 2, consider their mth ($m < n$) zero to the right (or to the left) of a. Which polynomial has the closest one to a from the right (or from the left)? and where?

This optimization problem has been solved in [1] and the solution has a closed form. To convert the solution of this optimization problem to an iterative scheme, let z_M be a zero of f with multiplicity $M \geq m$. To approximate z_M from its left, suppose z_M is on the right-hand side of two starting points $x_m^{(0)} < x_m^{(1)}$ and none of the zeros of f lie between $x_m^{(0)}$ and z_M. Then, for $k \geq 1$, we may let $a = x_m^{(k-1)}$, $b = x_m^{(k)}$ and let $x_m^{(k+1)}$ be the closest mth zero to the right of $x_m^{(k)}$ of polynomials satisfying the conditions in Question 2. Similarly, when two starting points $x_m^{(1)} < x_m^{(0)}$ are available on the right-hand side of z_M with no zeros of f lying in $(z_M, x_m^{(0)})$, then for $k \geq 1$, we let $a = x_m^{(k)}$, $b = x_m^{(k-1)}$ and let $x_m^{(k+1)}$ be the closest mth zero to the left of $x_m^{(k)}$ of polynomials satisfying the conditions in Question 2.

The sequence $\{x_m^{(k)}\}$ so constructed is obviously a monotone sequence. It will be shown in §3 that this sequence converges to z_M with ultimate convergence rate $\sqrt{2}+1$ when $m = M$, and the convergence is linear if $m < M$. The linear convergence ratio,

$$\lim_{k\to\infty} \frac{x_m^{(k+1)} - z_M}{x_m^{(k)} - z_M},$$

will also be given.

In practice, the multiplicity of any zero of f is, in general, not revealed ahead of time. Therefore, we can only use $m = 1$ in our iterations. We will show in §4 that when a multiple zero occurs and when the linear convergence ratio becomes evident, the convergence can be speeded up substantially by a special device.

This newly derived quasi-Laguerre iteration was employed to approximate eigenvalues of symmetric tridiagonal matrices with remarkable results. The details of the practical implementation and comprehensive numerical experiments on diverse types of matrices can be found in [1]. Our algorithm considerably improves the speed of Laguerre's iteration for the same purpose [1].

During the writing of this paper, we became aware of an earlier unpublished work of L. Foster [2] in which a class of globally convergent iterations, including our quasi-Laguerre iteration for the case of simple roots, were studied. Our work here is based on a different approach and achieves much more general results.

2. The quasi-Laguerre iteration

For the reader's convenience, we briefly outline the basic algorithm for the quasi-Laguerre iteration in this section. A more detailed derivation can be found in [1] and we only introduce the relevant part that is essential for the presentation of the convergence analysis given in §3.

We are given a polynomial f of degree n with all of its zeros being real. For $a < b$ with $f(a)f(b) \neq 0$, let $\mathcal{F}$ be the class of polynomials $p(x)$ of degree n that satisfy the following conditions:

(i) all zeros of $p(x)$ are real;
(ii) none of the zeros of $p(x)$ are in $[a, b]$;
(iii) $p(a)p(b) \neq 0$ and

$$\frac{p'(a)}{p(a)} = \frac{f'(a)}{f(a)} \equiv q(a), \quad \frac{p'(b)}{p(b)} = \frac{f'(b)}{f(b)} \equiv q(b). \tag{2.1}$$

In answering Question 3, stated in §1, we are led to the quadratic equation [1]

$$mr(b,a)Y^2 - \left[r(a,b)r(b,a) - n^2 + 2mn\right] Y + mr(a,b) = 0, \tag{2.2}$$

where

$$r(a,b) \equiv n + (b-a)q(a) \quad \text{and} \quad r(b,a) = n + (a-b)q(b). \tag{2.3}$$

Its solutions are given by

$$Y_{m\pm} = \frac{r(a,b)r(b,a) - n^2 + 2mn \pm \sqrt{[r(a,b)r(b,a) - n^2]\,[r(a,b)r(b,a) - (n-2m)^2]}}{2mr(b,a)}.$$

Now, as in [1], let

(2.4)

$$u_{m\pm} = a - \frac{b-a}{Y_{m\mp} - 1} = a - \frac{2mr(b,a)(b-a)}{r(a,b)r(b,a) - n^2 + 2mn - 2mr(b,a) \mp \sqrt{[r(a,b)r(b,a) - n^2][r(a,b)r(b,a) - (n-2m)^2]}}$$
$$= a + \frac{2m[n-(b-a)q(b)]}{-(b-a)R - 2mq(b) \pm \sqrt{R[(b-a)^2R + 4m(n-m)]}},$$

where

$$R = n\left(\frac{q(a)-q(b)}{b-a}\right) - q(a)q(b),$$

and also define

$$v_{1\pm} = a - \frac{b-a}{Y_{(n-m)\pm} - 1}. \tag{2.5}$$

The solutions of the optimization problems in Question 3 can then be described as follows (see [1] for details):

(i) If f has at least m (counting multiplicities) zeros to the left of a, then the zero u_{m-} in (2.4) of the polynomial

$$p_1(x) = C(x - u_{m-})^m (x - v_{1-})^{n-m} \tag{2.6}$$

is the closest mth zero to the left of a among all polynomials in $\mathcal{F}$. And it is clear that

$$z_{m-} \le u_{m-} \le \cdots \le u_{2-} \le u_{1-} < a < b, \tag{2.7}$$

where z_{m-} is the mth zero of f to the left of a.

(ii) If f has at least m (counting multiplicities) zeros to the right of b, then the zero u_{m+} in (2.4) of the polynomial

$$p_2(x) = C(x - u_{m+})^m (x - v_{1+})^{n-m} \tag{2.8}$$

is the closest mth zero to the right of b among all polynomials in $\mathcal{F}$. It is also clear that

$$a < b < u_{1+} \le u_{2+} \le \cdots \le u_{m+} \le z_{m+}, \tag{2.9}$$

where z_{m+} is the mth zero of f to the right of b.

Let a polynomial f of degree n be given with all its zeros being real. Let z_M be a zero of f with multiplicity $M \ge 1$. To approximate z_M from its left, we shall use u_{m+} in (2.4) with $m \le M$ to generate a monotonically increasing sequence $\{x_{m+}^{(k)}\}$ which converges to z_M as $k \to \infty$. Similarly, to approximate z_M from its right, a monotonically decreasing sequence $\{x_{m-}^{(k)}\}$ converging to z_M, as $k \to \infty$, can be generated by using u_{m-} in (2.4).

To be more precise, suppose z_M is on the right-hand side of two starting points $x_{m+}^{(0)} < x_{m+}^{(1)}$ and none of the zeros of f lie between $x_{m+}^{(0)}$ and z_M. Then, for $k \ge 1$, we let

$$x_{m+}^{(k+1)} = x_{m+}^{(k)} + \frac{2m\left[n - (x_{m+}^{(k-1)} - x_{m+}^{(k)})q(x_{m+}^{(k-1)})\right]}{-(x_{m+}^{(k-1)} - x_{m+}^{(k)})R - 2mq(x_{m+}^{(k-1)}) + \sqrt{R\left[(x_{m+}^{(k-1)} - x_{m+}^{(k)})^2 R + 4m(n-m)\right]}}, \tag{2.10}$$

where

$$q(x_{m+}^{(k)}) = \frac{f'(x_{m+}^{(k)})}{f(x_{m+}^{(k)})}, \quad q(x_{m+}^{(k-1)}) = \frac{f'(x_{m+}^{(k-1)})}{f(x_{m+}^{(k-1)})}$$

and

$$R = n\left(\frac{q(x_{m+}^{(k-1)}) - q(x_{m+}^{(k)})}{x_{m+}^{(k)} - x_{m+}^{(k-1)}}\right) - q(x_{m+}^{(k-1)})q(x_{m+}^{(k)}).$$

In other words, we replace u_{m+}, a and b in (2.4) by $x_{m+}^{(k+1)}, x_{m+}^{(k)}$ and $x_{m+}^{(k-1)}$, respectively.

From what has been presented earlier (and derived in more detail in [1]), it is easy to see that

$$x_{m+}^{(k)} < x_{m+}^{(k+1)} \le z_M$$

and no zeros of f fall between $x_{m+}^{(k+1)}$ and z_M. So the sequence $\{x_{m+}^{(k)}\}$ satisfies

$$x_{m+}^{(0)} < x_{m+}^{(1)} < \cdots < x_{m+}^{(k)} < \cdots \le z_M.$$

Similarly, when two starting points $x_{m-}^{(1)} < x_{m-}^{(0)}$ are available on the right-hand side of z_M with no zeros of f lying in $(z_M, x_{m-}^{(0)})$, then for $k \ge 1$, let $a = x_{m-}^{(k)}$,

$b = x_{m-}^{(k-1)}$ and $u_{m-} = x_{m-}^{(k+1)}$ in (2.4). Namely,

(2.11)
$$x_{m-}^{(k+1)} = x_{m-}^{(k)} + \frac{2m\left[n - (x_{m-}^{(k-1)} - x_{m-}^{(k)})q(x_{m-}^{(k-1)})\right]}{-(x_{m-}^{(k-1)} - x_{m-}^{(k)})R - 2mq(x_{m-}^{(k-1)}) - \sqrt{R\left[(x_{m-}^{(k-1)} - x_{m-}^{(k)})^2 R + 4m(n-m)\right]}},$$

where

$$q(x_{m-}^{(k)}) = \frac{f'(x_{m-}^{(k)})}{f(x_{m-}^{(k)})}, \quad q(x_{m-}^{(k-1)}) = \frac{f'(x_{m-}^{(k-1)})}{f(x_{m-}^{(k-1)})}$$

and

$$R = n\left(\frac{q(x_{m-}^{(k)}) - q(x_{m-}^{(k-1)})}{x_{m-}^{(k-1)} - x_{m-}^{(k)}}\right) - q(x_{m-}^{(k)})q(x_{m-}^{(k-1)}).$$

Again it is clear that

$$z_M \le x_{m-}^{(k+1)} < x_{m-}^{(k)}$$

and none of the zeros of f lie between z_M and $x_{m-}^{(k+1)}$. Moreover, the sequence $\{x_{m-}^{(k)}\}$ satisfies

$$z_M \le \cdots < x_{m-}^{(k)} < \cdots < x_{m-}^{(1)} < x_{m-}^{(0)}.$$

We call the process of generating the sequences $\{x_{m\pm}^{(k)}\}$ defined by (2.10) and (2.11) the **quasi-Laguerre iterations**.

3. Convergence analysis

We first prove the following theorem.

Theorem 3.1. *Both sequences $\{x_{m\pm}^{(k)}\}$ converge to z_M monotonically.*

Proof. We will prove the theorem for the sequence $\{x_{m-}^{(k)}\}_{k=1}^{\infty}$. Similar arguments hold for the sequence $\{x_{m+}^{(k)}\}_{k=1}^{\infty}$. We shall write y_k for $x_{m-}^{(k)}$ when there is no ambiguity. Obviously, by the way it is constructed, $\{y_k\}$ is a decreasing sequence bounded below by z_M. Then it must converge to a certain number $x^* \ge z_M$. Let $z^{(1)}, \dots, z^{(n-M)}$ be the rest of the zeros of f besides z_M. Suppose $x^* \ne z_M$; then $f(x^*) \ne 0$. Recall that

$$q(y) = \frac{f'(y)}{f(y)} = \sum_{i=1}^{n-M} \frac{1}{y - z^{(i)}} + \frac{M}{y - z_M}.$$

Thus,

$$\begin{aligned}
R &= n\left(\frac{q(y_k)-q(y_{k-1})}{y_{k-1}-y_k}\right) - q(y_k)q(y_{k-1}) \\
&= \frac{n}{y_{k-1}-y_k}\left[\sum_{i=1}^{n-M}\frac{1}{y_k-z^{(i)}} + \frac{M}{y_k-z_M} - \sum_{i=1}^{n-M}\frac{1}{y_{k-1}-z^{(i)}} - \frac{M}{y_{k-1}-z_M}\right] \\
&\quad - q(y_k)q(y_{k-1}) \\
&= n\left[\sum_{i=1}^{n-M}\frac{1}{(y_k-z^{(i)})(y_{k-1}-z^{(i)})} + \frac{M}{(y_k-z_M)(y_{k-1}-z_M)}\right] - q(y_k)q(y_{k-1}) \\
&\to n\left[\sum_{i=1}^{n-M}\frac{1}{(x^*-z^{(i)})^2} + \frac{M}{(x^*-z_M)^2}\right] - [q(x^*)]^2 \equiv \hat{R} < \infty.
\end{aligned}$$

From (2.11),

$$y_{k+1} = y_k - \frac{2m[n-(y_{k-1}-y_k)q(y_{k-1})]}{(y_{k-1}-y_k)R + 2mq(y_{k-1}) + \sqrt{R[(y_{k-1}-y_k)^2R + 4m(n-m)]}}.$$

Taking $y_k \longrightarrow x^*$, we have

$$x^* = x^* - \frac{2mn}{2mq(x^*) + \sqrt{4m(n-m)\hat{R}}} \neq x^*,$$

which leads to a contradiction. □

The next theorem provides the rate of convergence of both sequences $\{x_{m\pm}^{(k)}\}$.

Theorem 3.2. (i) *When $m < M$, the convergence of $\{x_{m\pm}^{(k)}\}$ to z_M is linear and the convergence ratio*

$$\omega \equiv \lim_{k\to\infty}\frac{x_{m\pm}^{(k+1)} - z_M}{x_{m\pm}^{(k)} - z_M}$$

is the only real solution in $(0,1)$ of

$$\frac{n-M-m}{n-M}x^3 - x^2 - x + \frac{M-m}{M} = 0.$$

(ii) *When $m = M$, the convergence of $\{x_{m\pm}^{(k)}\}$ to z_M is superlinear with convergence rate $\sqrt{2}+1$.*

Proof. Again, we will prove the theorem for the sequence $\{x_{m-}^{(k)}\}_{k=1}^{\infty}$ and write y_k for $x_{m-}^{(k)}$ when there is no ambiguity. Similar arguments hold for the sequence $\{x_{m+}^{(k)}\}_{k=1}^{\infty}$.

From (2.11),

(3.1)
$$y_{k+1} = y_k - \frac{2m[n-(y_{k-1}-y_k)q(y_{k-1})]}{(y_{k-1}-y_k)R + 2mq(y_{k-1}) + \sqrt{R[(y_{k-1}-y_k)^2R + 4m(n-m)]}},$$

where

$$q(y_{k-1}) = \frac{f'(y_{k-1})}{f(y_{k-1})}, \quad q(y_k) = \frac{f'(y_k)}{f(y_k)}, \quad \text{and}$$

$$R = n\left(\frac{q(y_k) - q(y_{k-1})}{y_{k-1} - y_k}\right) - q(y_k)q(y_{k-1}).$$

Let $z^{(1)}, \dots, z^{(n-M)}$ be the rest of the zeros of f besides z_M. Then,

$$q(y_{k-1}) = \frac{f'(y_{k-1})}{f(y_{k-1})} = \frac{M}{y_{k-1} - z_M} + Q(y_{k-1}) \quad \text{with} \quad Q(y_{k-1}) \equiv \sum_{i=1}^{n-M} \frac{1}{y_{k-1} - z^{(i)}},$$

$$q(y_k) = \frac{f'(y_k)}{f(y_k)} = \frac{M}{y_k - z_M} + Q(y_k) \quad \text{with} \quad Q(y_k) \equiv \sum_{i=1}^{n-M} \frac{1}{y_k - z^{(i)}}$$

and

$$\begin{aligned} R &= n\left(\frac{q(y_k) - q(y_{k-1})}{y_{k-1} - y_k}\right) - q(y_k)q(y_{k-1}) \\ &= \frac{M(n-M)}{(y_k - z_M)(y_{k-1} - z_M)} - \frac{MQ(y_{k-1})}{y_k - z_M} - \frac{MQ(y_k)}{y_{k-1} - z_M} - Q(y_k)Q(y_{k-1}) + nR_1 \end{aligned}$$

with

$$R_1 \equiv \sum_{i=1}^{n-M} \frac{1}{(y_{k-1} - z^{(i)})(y_k - z^{(i)})}.$$

Since $y_k \to z_M$ as $k \to \infty$, we have

$$Q(y_{k-1}) = \sum_{i=1}^{n-M} \frac{1}{y_{k-1} - z^{(i)}} \to Q \equiv \sum_{i=1}^{n-M} \frac{1}{z_M - z^{(i)}} < \infty,$$

$$Q(y_k) = \sum_{i=1}^{n-M} \frac{1}{y_k - z^{(i)}} \to Q$$

and

$$R_1 = \sum_{i=1}^{n-M} \frac{1}{(y_{k-1} - z^{(i)})(y_k - z^{(i)})} \to R^* \equiv \sum_{i=1}^{n-M} \frac{1}{(z_M - z^{(i)})^2} < \infty. \tag{3.2}$$

Let $\epsilon_k = \frac{y_k - z_M}{y_{k-1} - z_M}$ for all $k \geq 1$; then $\frac{y_{k-1} - y_k}{y_{k-1} - z_M} = 1 - \epsilon_k$. Obviously, $0 < \epsilon_k < 1$ and $0 < 1 - \epsilon_k < 1$. From (3.1),

(3.3)

$$\frac{y_{k+1} - z_M}{y_k - z_M} = 1 - \frac{2m\left[n - (y_{k-1} - y_k)q(y_{k-1})\right]}{(y_k - z_M)\left\{(y_{k-1} - y_k)R + 2mq(y_{k-1}) + \sqrt{R[(y_{k-1} - y_k)^2 R + 4m(n-m)]}\right\}}.$$

Now,

(a) $$\begin{aligned} (y_{k-1} - y_k)q(y_{k-1}) &= (y_{k-1} - y_k)\left[\frac{M}{y_{k-1} - z_M} + Q(y_{k-1})\right] \\ &= (y_{k-1} - z_M)\left[1 - \frac{y_k - z_M}{y_{k-1} - z_M}\right]\left[\frac{M}{y_{k-1} - z_M} + Q(y_{k-1})\right] \\ &= (1 - \epsilon_k)[M + Q(y_{k-1})(y_{k-1} - z_M)] = M(1 - \epsilon_k) + O(y_{k-1} - z_M), \end{aligned}$$

(b) $$\begin{aligned} &(y_k - z_M)(y_{k-1} - z_M)R = M(n-M) - Q(y_k)Q(y_{k-1})(y_k - z_M)(y_{k-1} - z_M) \\ &-(y_{k-1} - z_M)MQ(y_{k-1}) - (y_k - z_M)MQ(y_k) + n(y_k - z_M)(y_{k-1} - z_M)R_1 \\ &= M(n-M) + O(y_{k-1} - z_M), \end{aligned}$$

(c) $(y_k - z_M)(y_{k-1} - y_k)R = (1-\epsilon_k)(y_k - z_M)(y_{k-1} - z_M)R$
$= (1-\epsilon_k)M(n-M) + O(y_{k-1} - z_M)$ (because of (b) above),

(d) $(y_k - z_M)^2 R = \epsilon_k (y_k - z_M)(y_{k-1} - z_M)R = \epsilon_k M(n-M) + O(y_{k-1} - z_M)$
(because of (b) above),

(e) $(y_k - z_M)^2 R\left[(y_{k-1} - y_k)^2 R + 4m(n-m)\right]$
$= [(y_k - z_M)(y_{k-1} - y_k)R]^2 + 4m(n-m)(y_k - z_M)^2 R$
$= (1-\epsilon_k)^2 M^2 (n-M)^2 + 4m(n-m)\epsilon_k M(n-M) + O(y_{k-1} - z_M)$
(because of (c) and (d) above).

Substituting (a), (c), (e) into (3.3) yields

$$\begin{aligned}\epsilon_{k+1} &= \frac{y_{k+1} - z_M}{y_k - z_M} = 1 - \\ &\frac{2m[n-(1-\epsilon_k)M] + O(y_{k-1} - z_M)}{M(n-M)(1-\epsilon_k) + 2mM\epsilon_k + M(n-M)\sqrt{(1-\epsilon_k)^2 + \frac{4m(n-m)}{M(n-M)}\epsilon_k} + O(y_{k-1} - z_M)} \\ &= G(\epsilon_k) + O(y_{k-1} - z_M),\end{aligned} \tag{3.4}$$

where

$$G(\epsilon) = 1 - \frac{2m[n-(1-\epsilon)M]}{M(n-M)(1-\epsilon) + 2mM\epsilon + M(n-M)\sqrt{(1-\epsilon)^2 + \frac{4m(n-m)}{M(n-M)}\epsilon}}.$$

We claim that $G(\epsilon)$ can be rewritten as

$$G(\epsilon) = \frac{2(M-m)}{M\left(1 + \epsilon + \sqrt{(1-\epsilon)^2 + \frac{4m(n-m)}{M(n-M)}\epsilon}\right)}. \tag{3.5}$$

To see this, let $\triangle = \sqrt{(1-\epsilon)^2 + \frac{4m(n-m)}{M(n-M)}\epsilon}$ and

$$\begin{aligned} A &= m[n-(1-\epsilon)M], \\ B &= (n-M)(1-\epsilon) + 2m\epsilon + (n-M)\triangle, \\ C &= M - m, \\ D &= (1+\epsilon) + \triangle. \end{aligned}$$

Then, $G(\epsilon) = 1 - \dfrac{2A}{MB}$, and

$$\begin{aligned} AD &= m(n-M)(1+\epsilon) + mM\epsilon(1+\epsilon) + [m(n-M) + mM\epsilon]\triangle, \\ BC &= (M-m)(n-M)(1-\epsilon) + 2(M-m)m\epsilon + (M-m)(n-M)\triangle. \end{aligned}$$

So,

$$\begin{aligned} AD + BC = {} & mM\epsilon^2 + (M^2 - Mn + 2mn + Mm - 2m^2)\epsilon \\ & + M(n-M) + [M(n-M) + mM\epsilon]\triangle. \end{aligned}$$

Meanwhile,

$$\begin{aligned}MBD &= M(n-M)(1-\epsilon^2) + 2mM\epsilon + 2mM\epsilon^2 + M(n-M)(1-\epsilon)^2 \\ &\quad + 4m(n-m)\epsilon + [M(n-M)(1+\epsilon) + M(n-M)(1-\epsilon) + 2mM\epsilon]\triangle \\ &= 2mM\epsilon^2 + (2M^2 - 2Mn + 4mn + 2Mm - 4m^2)\epsilon \\ &\quad + 2M(n-M) + [2M(n-M) + 2mM\epsilon]\triangle \\ &= 2(AD+BC).\end{aligned}$$

Thus,

$$\frac{1}{2}M[1-G(\epsilon)] = \frac{A}{B} = \frac{A}{B} + \frac{C}{D} - \frac{C}{D} = \frac{AD+BC}{BD} - \frac{C}{D} = \frac{M}{2} - \frac{C}{D}.$$

Therefore,

$$G(\epsilon) = \frac{2C}{MD} = \frac{2(M-m)}{M\left(1+\epsilon+\sqrt{(1-\epsilon)^2 + \frac{4m(n-m)}{M(n-M)}\epsilon}\right)}.$$

In this form, it can be easily shown that $G(\epsilon)$ is monotonically decreasing. To prove this, let

$$h(\epsilon) = 1+\epsilon+\sqrt{(1-\epsilon)^2 + 4\gamma\epsilon}, \quad \text{where} \quad \gamma = \frac{m(n-m)}{M(n-M)}.$$

For $\epsilon \in (0,1)$,

$$h'(\epsilon) = 1 + \frac{\epsilon - 1 + 2\gamma}{\sqrt{(1-\epsilon)^2 + 4\gamma\epsilon}} > 0,$$

so, from (3.5), $G(\epsilon)$ is monotonically decreasing.

Now,

(i) For $m < M$, let

$$\beta = \frac{M-m}{M} \in (0,1) \quad \text{and} \quad \gamma = \frac{m(n-m)}{M(n-M)} > 0.$$

Since $G(\epsilon)$ is monotonically decreasing for $\epsilon \in [0,1]$ and $\epsilon_k \in (0,1)$ for all $k \geq 1$, we have

$$0 < \frac{\beta}{1+\sqrt{\gamma}} = G(1) < G(\epsilon_k) < G(0) = \beta < 1.$$

Hence, when k is sufficiently large,

$$0 < \frac{G(1)}{2} < \epsilon_{k+1} = G(\epsilon_k) + O(y_{k-1} - z_M) < \frac{G(0)+1}{2} < 1.$$

Consequently, the convergence of $\{y_k\}_{k=1}^{\infty}$ to z_M is linear.

Moreover, (3.4) implies that

$$|\epsilon_{k+1} - G(\epsilon_k)| = |O(y_{k-1} - z_M)| \leq C(y_{k-1} - z_M) \quad \text{for } k \geq 1 \tag{3.6}$$

for some positive number C independent of k. Since $y_k \to z_M$, there exists an integer $N > 1$ such that

$$G(1) - C(y_{N-1} - z_M) > 0 \quad \text{and} \quad G(0) + C(y_{N-1} - z_M) < 1. \tag{3.7}$$

Let

$$c_1 = G(1) - C(y_{N-1} - z_M), \quad d_1 = G(0) + C(y_{N-1} - z_M)$$

and

$$c_{k+1} = G(d_k) - C(y_{N+k-1} - z_M), \quad d_{k+1} = G(c_k) + C(y_{N+k-1} - z_M), \quad \text{for } k \geq 1.$$

Then,

(i) $\{c_k\}$ is monotonically increasing and $\{d_k\}$ is monotonically decreasing, namely,

$$0 < c_1 < c_2 < c_3 < \cdots < c_k < \cdots < d_k < \cdots < d_3 < d_2 < d_1 < 1.$$

(ii) $c_k < \epsilon_{N+k} < d_k \quad$ for $k \geq 1$.

This can be shown by mathematical induction as follows. For $k = 1$,

$$\begin{aligned} 0 < c_1 &= G(1) - C(y_{N-1} - z_M) \quad \text{(because of (3.7))} \\ &< G(\epsilon_N) - C(y_{N-1} - z_M) \quad \text{(because } G \text{ is decreasing and } 0<\epsilon_N<1) \\ &< \epsilon_{N+1} < G(\epsilon_N) + C(y_{N-1} - z_M) \quad \text{(because of (3.6))} \\ &< G(0) + C(y_{N-1} - z_M) = d_1 < 1 \quad \text{(because of (3.7))}. \end{aligned}$$

So, both (i) and (ii) hold. Suppose the conclusions hold for $k \geq 1$, that is,

(3.8)
$$0 < c_1 < \cdots < c_{k-1} < c_k, \quad d_k < d_{k-1} < \cdots < d_1 < 1 \quad \text{and} \quad c_k < \epsilon_{N+k} < d_k.$$

It follows from the monotonicity of both G and the sequence $\{y_k\}$ that

$$\begin{aligned} c_{k+1} &= G(d_k) - C(y_{N+k-1} - z_M) > G(d_{k-1}) - C(y_{N+k-2} - z_M) = c_k, \\ d_{k+1} &= G(c_k) + C(y_{N+k-1} - z_M) < G(c_{k-1}) + C(y_{N+k-2} - z_M) = d_k. \end{aligned}$$

Furthermore,

$$\begin{aligned} c_{k+1} &= G(d_k) - C(y_{N+k-1} - z_M) < G(\epsilon_{N+k}) - C(y_{N+k-1} - z_M) \quad \text{(because of (3.8))} \\ &< \epsilon_{N+k+1} < G(\epsilon_{N+k}) + C(y_{N+k-1} - z_M) \quad \text{(because of (3.6))} \\ &< G(c_k) + C(y_{N+k-1} - z_M) = d_{k+1} \quad \text{(because of (3.8))}. \end{aligned}$$

Hence,

$$0 < c_1 < \cdots < c_k < c_{k+1}, \quad d_{k+1} < d_k < \cdots < d_1 < 1 \quad \text{and}$$
$$c_{k+1} < \epsilon_{N+k+1} < d_{k+1}.$$

We now let $c_k \to c$ and $d_k \to d$. Then $G(c) = d$ and $G(d) = c$. Since $G(\epsilon)$ is monotonically decreasing, $G^2 = G \circ G$ is monotonically increasing and can have at most one fixed point. So, $c = d$, and consequently $G(\omega) = \omega$, where

$$\omega = \lim_{k\to\infty} \epsilon_k = \lim_{k\to\infty} \frac{y_k - z_M}{y_{k-1} - z_M}.$$

Thus,

$$\omega = \frac{2(M-m)}{M\left(1 + \omega + \sqrt{(1-\omega)^2 + \frac{4m(n-m)}{M(n-M)}\omega}\right)}$$

or, ω is a solution to the equation

$$g(x) \equiv \frac{n-M-m}{n-M}x^3 - x^2 - x + \frac{M-m}{M} = 0. \tag{3.9}$$

There is only one real solution to (3.9) in $(0, 1)$, since

$$\begin{aligned} g(0) &= \frac{M-m}{M} > 0, \\ g(1) &= \frac{n-M-m}{n-M} - 1 - 1 + \frac{M-m}{M} = \left(\frac{n-M-m}{n-M} - 1\right) + \left(\frac{M-m}{M} - 1\right) \\ &= \frac{-m}{n-M} + \frac{-m}{M} = \frac{-mn}{M(n-M)} < 0, \end{aligned}$$

and

$$g'(x) = 3\frac{n-M-m}{n-M}x^2 - 2x - 1 < 0 \quad \text{for } x \in (0,1).$$

(ii) For $m = M$, from (3.5), $G(\epsilon_k) = 0$ for $k \geq 1$; accordingly, from (3.4),

$$\epsilon_{k+1} = O(y_{k+1} - z_M) \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, the convergence of $\{y_k\}_{k=1}^{\infty}$ to z_M is superlinear. Further, from (2.6), the polynomial $p(x) = C(x - y_{k+1})^m (x - v)^{n-m}$, where C is a constant and v is a certain real number not in (y_k, y_{k-1}), satisfies

$$\frac{p'(y_k)}{p(y_k)} = \frac{f'(y_k)}{f(y_k)} \quad \text{and} \quad \frac{p'(y_{k-1})}{p(y_{k-1})} = \frac{f'(y_{k-1})}{f(y_{k-1})}.$$

That is,

$$\frac{M}{y_k - y_{k+1}} + \frac{n-M}{y_k - v} = \frac{M}{y_k - z_M} + \sum_{i=1}^{n-M} \frac{1}{y_k - z^{(i)}} \tag{3.10}$$

and

$$\frac{M}{y_{k-1} - y_{k+1}} + \frac{n-M}{y_{k-1} - v} = \frac{M}{y_{k-1} - z_M} + \sum_{i=1}^{n-M} \frac{1}{y_{k-1} - z^{(i)}}. \tag{3.11}$$

Or,

$$\frac{M(y_{k+1} - z_M)}{(y_k - z_M)(y_k - y_{k+1})} = -\frac{n-M}{y_k - v} + \sum_{i=1}^{n-M} \frac{1}{y_k - z^{(i)}}, \tag{3.12}$$

$$\frac{M(y_{k+1} - z_M)}{(y_{k-1} - z_M)(y_{k-1} - y_{k+1})} = -\frac{n-M}{y_{k-1} - v} + \sum_{i=1}^{n-M} \frac{1}{y_{k-1} - z^{(i)}}. \tag{3.13}$$

Subtracting (3.13) from (3.12) and dividing the result by $(y_{k-1} - y_k)$, yields

$$\begin{aligned} &\frac{M(y_{k+1} - z_M)(y_{k-1} + y_k - y_{k+1} - z_M)}{(y_k - z_M)(y_k - y_{k+1})(y_{k-1} - z_M)(y_{k-1} - y_{k+1})} \\ &= -\frac{n-M}{(y_k - v)(y_{k-1} - v)} + \sum_{i=1}^{n-M} \frac{1}{(y_k - z^{(i)})(y_{k-1} - z^{(i)})}. \end{aligned} \tag{3.14}$$

Note that $R^* = \sum_{i=1}^{n-M} \frac{1}{(z_M - z^{(i)})^2}$ in (3.2). Hence, for sufficiently large k,

$$0 < \frac{(y_{k+1} - z_M)(y_{k-1} + y_k - y_{k+1} - z_M)}{(y_k - z_M)(y_k - y_{k+1})(y_{k-1} - z_M)(y_{k-1} - y_{k+1})} < \frac{2R^*}{M},$$

since $(y_k - v)(y_{k-1} - v) > 0$. Furthermore, the left-hand side of (3.12) satisfies

$$\begin{aligned} 0 &< \frac{M(y_{k+1} - z_M)}{(y_k - z_M)(y_k - y_{k+1})} \\ &= \frac{M(y_{k+1} - z_M)(y_{k-1} - y_{k+1} + y_k - z_M)}{(y_k - z_M)(y_k - y_{k+1})(y_{k-1} - z_M)(y_{k-1} - y_{k+1})} \cdot \frac{(y_{k-1} - z_M)(y_{k-1} - y_{k+1})}{y_{k-1} - y_{k+1} + y_k - z_M} \\ &< 2R^*(y_{k-1} - z_M) \to 0 \qquad \text{as} \quad k \to \infty. \end{aligned}$$

Thus, the right-hand side of (3.12) gives

$$\frac{n-M}{y_k-v} \to Q = \sum_{i=1}^{n-M} \frac{1}{z_M - z^{(i)}} \quad \text{as} \quad k \to \infty.$$

Therefore, when $k \to \infty$, the right-hand side of (3.14) approaches

(3.15)

$$E \equiv R^* - \frac{Q^2}{n-M} = \sum_{i=1}^{n-M} \frac{1}{(z_M - z^{(i)})^2} - \frac{1}{n-M}\left(\sum_{i=1}^{n-M} \frac{1}{z_M - z^{(i)}}\right)^2 \geq 0.$$

Let

$$a = \underbrace{(1, \ldots, 1)}_{n-M} \quad \text{and} \quad b = \left(\frac{1}{z_M - z^{(1)}}, \ldots, \frac{1}{z_M - z^{(n-M)}}\right).$$

Then, by the Cauchy-Schwarz inequality, $|\langle a, b\rangle| \leq \|a\|\|b\|$, or,

$$\frac{1}{n-M}\left(\sum_{i=1}^{n-M} \frac{1}{z_M - z^{(i)}}\right)^2 \leq \sum_{i=1}^{n-M} \frac{1}{(z_M - z^{(i)})^2}.$$

Hence, $E = 0$ in (3.15) implies $z^{(1)} = z^{(2)} = \cdots = z^{(n-M)}$. That is, $f(x)$ has only two distinct zeros: $z^{(1)}$ and z_M with multiplicities $n-M$ and M, respectively. By (3.10) and (3.11), one can easily see that in this case $y_2 = z_M$ with starting points y_0 and y_1. So, the iteration converges in one step. Thus, we only need to consider the case where $E > 0$ in (3.15). Now,

$$\begin{aligned}
&\lim_{k\to\infty} \frac{y_{k+1} - z_M}{(y_k - z_M)^2(y_{k-1} - z_M)} \\
&= \lim_{k\to\infty} \frac{(y_{k+1} - z_M)(y_{k-1} + y_k - y_{k+1} - z_M)}{(y_k - z_M)(y_k - y_{k+1})(y_{k-1} - z_M)(y_{k-1} - y_{k+1})} \\
&\quad \cdot \frac{(y_k - y_{k+1})(y_{k-1} - y_{k+1})}{(y_k - z_M)(y_{k-1} - y_{k+1} + y_k - z_M)} \\
&= \frac{E}{M} \cdot \lim_{k\to\infty} \frac{y_k - y_{k+1}}{y_k - z_M} \cdot \frac{1}{\frac{y_{k-1} - y_{k+1} + y_k - z_M}{y_{k-1} - y_{k+1}}} = \frac{E}{M} \cdot \lim_{k\to\infty} \frac{\frac{y_k - y_{k+1}}{y_k - z_M}}{1 + \frac{y_k - z_M}{y_{k-1} - y_{k+1}}} \\
&= \frac{E}{M} > 0
\end{aligned}$$

since

$$\lim_{k\to\infty} \frac{y_k - y_{k+1}}{y_k - z_M} = 1 - \lim_{k\to\infty} \frac{y_{k+1} - z_M}{y_k - z_M} = 1$$

and

$$\lim_{k\to\infty} \frac{y_k - z_M}{y_{k-1} - y_{k+1}} = \lim_{k\to\infty} \frac{\frac{y_k - z_M}{y_{k-1} - z_M}}{1 - \frac{y_{k+1} - z_M}{y_{k-1} - z_M}} = 0.$$

By the following lemma, the convergence rate of $\{y_k\}_{k=1}^{\infty}$ to z_M is indeed $\sqrt{2}+1$. $\square$

Lemma 3.3. *Suppose a sequence $\{y_k\}_{k=1}^{\infty}$ converges to z. Let $e_k = |y_k - z|$. If*

$$e_{k+1} = M_k e_k^2 e_{k-1} \quad \textit{and} \quad \lim_{k\to\infty} M_k = M^*,$$

where $0 < M^* < \infty$*, then the convergence rate of* $\{y_k\}$ *is* $\sqrt{2}+1$*. More precisely,*

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^{\sqrt{2}+1}} = (M^*)^{1/\sqrt{2}}.$$

Proof. First we assume $M^* = 1$, that is, $\lim_{k\to\infty} M_k = 1$. Let

$$d_k = \ln e_{k+1} - (\sqrt{2}+1)\ln e_k, \qquad k \geq 1.$$

From $e_{k+1} = M_k e_k^2 e_{k-1}$ we obtain

$$\ln e_{k+1} = 2\ln e_k + \ln e_{k-1} + \ln M_k.$$

So,

$$\begin{aligned} d_k &= 2\ln e_k + \ln e_{k-1} + \ln M_k - (\sqrt{2}+1)\ln e_k \\ &= (1-\sqrt{2})\ln e_k + \ln e_{k-1} + \ln M_k \\ &= (1-\sqrt{2})[\ln e_k - (\sqrt{2}+1)\ln e_{k-1}] + \ln M_k \\ &= (1-\sqrt{2})d_{k-1} + \ln M_k. \end{aligned}$$

Since $0 > 1-\sqrt{2} > -1$ and $\ln M_k \longrightarrow 0$ as $k \longrightarrow \infty$, we have $d_k \longrightarrow 0$ as $k \longrightarrow \infty$. That is,

$$\frac{e_{k+1}}{e_k^{\sqrt{2}+1}} \longrightarrow 1 \quad \text{as } k \longrightarrow \infty.$$

Now, if $M^* \neq 1$, let

$$e_k' = \sqrt{M^*}e_k \qquad \text{and} \qquad M_k' = \frac{M_k}{M^*}.$$

Then,

$$e_{k+1}' = \sqrt{M^*}e_{k+1} = \frac{M_k}{M^*}(\sqrt{M^*}e_k)^2(\sqrt{M^*}e_{k-1}) = M_k'(e_k')^2 e_{k-1}'$$

and

$$\lim_{k\to\infty} M_k' = 1.$$

So,

$$\lim_{k\to\infty} \frac{e_{k+1}'}{(e_k')^{\sqrt{2}+1}} = 1$$

and

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^{\sqrt{2}+1}} = \frac{(\sqrt{M^*})^{\sqrt{2}+1}}{\sqrt{M^*}} = (M^*)^{1/\sqrt{2}}. \quad \square$$

4. Practical consideration on clusters of zeros

In practice, when our quasi-Laguerre iteration is used to approximate a real zero x^* of a real polynomial f, the multiplicity of x^* is, in general, not revealed ahead of time. Therefore, one can only use $m = 1$ in (2.10) or (2.11) which may cause slow convergence when the multiplicity of x^* is larger than 1. Even when x^* is simple, slow convergence may still occur when x^* leads a cluster of close zeros of f. To overcome this difficulty, we propose the following procedure, which works very efficiently in practice:

By Theorem 3.2, when the quasi-Laguerre iteration for a simple zero ($m = 1$) is used to approximate an M-fold zero x^* ($M \geq 2$), or a cluster of M simple zeros, of a polynomial f of degree n, it converges linearly with a ratio $q_{n,M}$ which is the unique solution in $(0,1)$ of

$$\frac{n-M-1}{n-M}x^3 - x^2 - x + \frac{M-1}{M} = 0.$$

The following theorem gives the properties of $q_{n,M}$ for $M \geq 2$ and $n \geq M+1$.

Theorem 4.1. (i) *For a fixed M, $q_{n,M}$ increases as n increases $(n > M)$.*
(ii) *For a given n, $q_{n,M} \geq q_{n,2}$ if $2 \leq M \leq n-2$.*
(iii) *When $M = n-1$, then $q_{n,n-1}$ increases as n increases $(n \geq 3)$.*

Proof. (i) For a fixed M, write

$$g_n(x) \equiv \frac{n-M-1}{n-M}x^3 - x^2 - x + \frac{M-1}{M}.$$

Then, when $n_1 > n_2$,

$$g_{n_1}(x) - g_{n_2}(x) = \left(\frac{1}{n_2-M} - \frac{1}{n_1-M}\right)x^3 > 0 \quad \text{for } x \in (0,1).$$

So, $q_{n_1,M} > q_{n_2,M}$.

(ii) For a given n, write

$$g_M(x) \equiv \frac{n-M-1}{n-M}x^3 - x^2 - x + \frac{M-1}{M}.$$

When $2 \leq M \leq \frac{n}{2}$, then

$$\begin{aligned} g_M(x) - g_2(x) &= \left(\frac{1}{n-2} - \frac{1}{n-M}\right)x^3 + \frac{1}{2} - \frac{1}{M} \geq \frac{1}{2} + \frac{1}{n-2} - \frac{1}{M} - \frac{1}{n-M} \\ &= \frac{n}{2(n-2)} - \frac{n}{M(n-M)} \geq 0 \quad \text{for } x \in (0,1). \end{aligned}$$

So, $q_{n,M} \geq q_{n,2}$.

If $\frac{n}{2} < M \leq n-2$, then $2 \leq n-M < \frac{n}{2}$, and hence, $q_{n,n-M} \geq q_{n,2}$. On the other hand,

$$g_M(x) - g_{n-M}(x) = \left(\frac{1}{n-M} - \frac{1}{M}\right)(1-x^3) > 0 \quad \text{for } x \in (0,1).$$

Then $q_{n,M} > q_{n,n-M}$. Thus, $q_{n,M} > q_{n,n-M} \geq q_{n,2}$.

(iii) If $M = n-1$, write

$$g_n(x) \equiv \frac{n-M-1}{n-M}x^3 - x^2 - x + \frac{M-1}{M} = -x^2 - x + 1 - \frac{1}{n-1}.$$

Then, when $n_1 > n_2$,

$$g_{n_1}(x) - g_{n_2}(x) = \frac{1}{n_2-1} - \frac{1}{n_1-1} > 0 \quad \text{for } x \in (0,1).$$

So, $q_{n_1,n_1-1} > q_{n_2,n_2-1}$. □

For $n \geq 3$ and $M \geq 2$, by Theorem 4.1 (iii),

$$q_{n,M} \geq q_{3,2} \quad \text{when } M = n-1,$$

and by Theorem 4.1 (i), (ii),

$$q_{n,M} \geq q_{n,2} \geq q_{3,2} \quad \text{when } M \leq n-2.$$

In any case, $q_{3,2}$, the zero of $x^2+x-0.5=0$ in $(0,1)$, is the smallest linear ratio for multiple zeros. Let $x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots$ be the monotone sequence of the quasi-Laguerre iterates. When $\{x^{(k)}\}$ converges superlinearly, then

$$q_k \equiv \frac{|x^{(k+1)}-x^{(k)}|}{|x^{(k)}-x^{(k-1)}|} \longrightarrow 0 \quad \text{as } k \to \infty,$$

so, q_k should be smaller than $q_{3,2}$ when k is sufficiently large. Therefore, when $q_k > q_{3,2}$, linear convergence seems apparent. In this case, for sufficiently large k,

$$\frac{|x^{(k+1)}-x^*|}{|x^{(k)}-x^*|} \approx q_{n,M} \qquad \text{or} \qquad |x^{(k+1)}-x^*| \approx q|x^{(k)}-x^*| \quad \text{where } q \equiv q_{n,M},$$

and

$$q_k = \frac{|x^{(k+1)}-x^{(k)}|}{|x^{(k)}-x^{(k-1)}|} = \frac{|x^{(k)}-x^*|-|x^{(k+1)}-x^*|}{|x^{(k-1)}-x^*|-|x^{(k)}-x^*|} \approx \frac{(1-q)|x^{(k)}-x^*|}{(1-q)|x^{(k-1)}-x^*|} \approx q.$$

In practice, the q_k's can be very close to q after several iterations. If $q_j = q$ for $j \ge k$, then

$$|x^{(j+1)}-x^*| = q|x^{(j)}-x^*| \quad \text{for} \;\; j \ge k,$$

and

$$x^* = x^{(k)} + \frac{1}{1-q}(x^{(k+1)}-x^{(k)}).$$

So, at the kth iterate $x^{(k)}$, if linear convergence is revealed, namely $q_k > q_{3,2}$, then instead of performing the regular quasi-Laguerre iteration

$$x^{(k+1)} = x^{(k)} + \delta_k,$$

where δ_k is the correction $x^{(k+1)}-x^{(k)}$ to $x^{(k)}$ calculated by the quasi-Laguerre iteration with $m=1$, one can accelerate the iteration by setting

$$x^{(k+1)} = x^{(k)} + \frac{\delta_k}{1-q_k}.$$

To illustrate the efficiency of this acceleration device, consider a Wilkinson matrix W_{99}^+ with dimension $n=99$. This is a symmetric tridiagonal matrix whose eigenvalues are mostly pairs of numerically indistinguishable real numbers [5, pp. 308 – 309]. Let f be the characteristic polynomial of W_{99}^+; namely,

$$f(z) = \det(W_{99}^+ - zI).$$

The zeros of f are the eigenvalues of W_{99}^+. The pair $z_1 = 11.0 + 5 \times 10^{-15}$ and $z_2 = 11.0 - 5 \times 10^{-15}$ is a pair of close zeros of f. To approximate z_1, the starting points $x^{(0)} = 11.5$ and $x^{(1)} = 11.27$ (one step Laguerre's iteration from $x^{(0)}$) are used. For $k = 1, 2, \ldots$, $\delta_k = x^{(k+1)} - x^{(k)}$ is evaluated using the quasi-Laguerre formula with $m=1$. If $q_k < q_{3,2}$, then most likely z_1 is neither a multiple zero nor a member of any cluster. Then $x^{(k+1)}$ is accepted and we continue onto the next iterate. Otherwise, let $c = x^{(k)} + \dfrac{\delta_k}{1-q}$ and check for possible overshoot by evaluating the Sturm sequence at c. If c overshoots, then reset $c = x^{(k)} + \dfrac{\delta_k(1-q^l)}{1-q}$ for $l = 8, 4, 2, 1$ until overshooting disappears. Then c is accepted as $x^{(k+1)}$ and we continue onto the next iterate.

TABLE 1. Acceleration of the quasi-Laguerre iteration for W_{99}^{+}. For each number in the table, the digits before a space are correct

k	$x^{(k)}$ (original)	$q_k = \frac{\lvert x^{(k+1)} - x^{(k)}\rvert}{\lvert x^{(k)} - x^{(k-1)}\rvert}$	$x^{(k)}$ (accelerated)
1	11 .500000000000000		11. 500000000000000
2	11 .270700712327294		11. 270700712327294
3	11 .149428542073966	0. 706618	11.0 13287361699255
4	11.0 63735435125402	0.4 37604	11.000 124299930121
5	11.0 26235830973979	0.4 17533	11.0000 61856842477
6	11.0 10578521008206	0.40 3427	11.000000 198368395
7	11.00 4261938886133	0.40 3112	11.0000000 99023529
8	11.00 1715645778917	0.4025 55	11.0000000000 12077
9	11.000 690622605815	0.4025 56	11.00000000000 6040
10	11.000 277993423158	0.40252 7	11.00000000000 2304
11	11.000 111899071095	0.40252 5	11.000000000000 156
12	11.0000 45041905516	0.40252 3	11.0000000000000 64
13	11.0000 18130368451	0.402522	
14	11.00000 7297872838	0.402522	
15	11.00000 2937554150	0.402522	
16	11.00000 1182429996	0.402522	
17	11.000000 475954003	0.402522	
18	11.000000 191581922	0.402522	
19	11.0000000 77115924	0.402522	
20	11.0000000 31040850	0.402522	
21	11.0000000 12494622	0.402522	
22	11.00000000 5029360	0.402522	
23	11.00000000 2024429	0.402522	
24	11.000000000 814877	0.402522	
25	11.000000000 328006	0.402522	
26	11.000000000 132029	0.402520	
27	11.0000000000 53145	0.40252 1	
28	11.0000000000 21393	0.40252 6	
29	11.00000000000 8612	0.40252 1	
30	11.00000000000 3467	0.402 477	
31	11.00000000000 1396	0.40 1680	
32	11.000000000000 565	0. 399604	
33	11.000000000000 233	0. 381937	
34	11.000000000000 107	0. 305425	

Table 1 gives the comparison between the original quasi-Laguerre iteration and the accelerated quasi-Laguerre iteration. Notice that the q_k's in the third column of Table 1 are identical up to six digits with $q = q_{99,2} \approx 0.402522$.

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