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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 1991 Mathematics Subject Classification can be found in the annual subject index of *Mathematical Reviews* starting with the December 1990 issue.

1[65N30, 65N15]—*Superconvergence in Galerkin finite element methods*, by Lars B. Wahlbin, Lecture Notes in Mathematics, Vol. 1605, Springer, Berlin, 1995, xii+166 pp., $23\frac{1}{2}$ cm, softcover, $33.00

These are the notes from a graduate seminar on superconvergence of finite element methods for second order scalar elliptic boundary value problems. Assuming a smooth exact solution, standard finite element methods for such problems provide optimal order approximation in the maximum norm for both the solution and its first derivatives. Thus, if finite elements based on piecewise polynomials of degree $d \geq 2$ are used, then the L^∞ norm of the error will converge to zero as $O(h^{d+1})$ as the meshsize h tends to zero, and the L^∞ norm of the gradient of the error will converge as $O(h^d)$. Simple approximation theory informs us that no better order of convergence can be achieved by any piecewise polynomial of degree d.

However, there is no obstruction to the error achieving higher order at isolated points. Indeed, under very general circumstances finite element errors can be shown to be of higher order when measured in Sobolev space norms of negative index. This indicates that the error is oscillatory, and thus must be smaller at some points than others. A point where the error achieves higher order than is possible globally is called a superconvergent point. For example, for a one-dimensional problem and continuous piecewise quadratic elements, the finite element solution is superconvergent at the mesh points and element midpoints, and its derivative is superconvergent at the two Gauss points in each element. Moreover, there can be no more superconvergent points for either the solution or its derivative, since an assumption to the contrary brings us into contradiction with approximation theory once again. These notes begin with an extensive study of such superconvergence in one dimension, including both the well-developed theory for merely continuous piecewise polynomials and more recent results (many due to the author) for smoother splines, in which case uniformity or symmetry restrictions are generally required on the mesh.

For higher-dimensional finite elements, superconvergence at readily identifiable points is a much more special phenomena and, except in the case of tensor product meshes and elements, the one-dimensional theory is of little guidance. However, some results are long known in the case of meshes with a high degree of uniformity. For example, on a uniform triangular mesh generated by three families of parallel lines, the element vertices and edge midpoints are superconvergent points when continuous piecewise quadratic elements are used. A major theme of these notes, and of the author's recent research in the area, is to relate such superconvergence behavior to the invariance of the mesh under the reflection through the superconvergent point. This approach leads to results for much more general meshes, though still requiring a high degree of symmetry. A second major theme is to localize the symmetry requirements to a neighborhood of the superconvergent points, with general meshes allowed far enough away from the points. The case of superconvergent behaviour of the L^2 projection is studied as well, since the techniques are similar to those used to study the finite element solution but analysis for the L^2 projection is somewhat simpler.

Other topics in the book include superconvergence by "trivial, or not so trivial" postprocesses, for example, the use of difference quotients on translation invariant meshes as superconvergent approximations of derivatives and the use of various averaging operators to achieve superconvergence; extensions to nonlinear problems; extensions to meshes which are smooth mappings of meshes with sufficient invariance properties; superconvergence results for boundary integral operators on one-dimensional boundary curves; and numerical studies to locate superconvergent points.

Superconvergence is presented in these notes as a fascinating and challenging area of mathematical analysis—the question of its significance in practical computation is not a major concern. The treatment is extensive, the references nearly exhaustive, the proofs complete, and the exposition clear and precise. The book will surely be appreciated by researchers wanting some guidance through the vast literature on superconvergence, and by readers who appreciate the intricate and subtle craft of technical finite element numerical analysis.

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2[35R30]—*Inverse problems in diffusion processes*, Heinz W. Engl and William Rundell (Editors), SIAM Proceedings Series, SIAM, Philadelphia, PA and GAMM, Regensburg, Germany, 1995, xii+232 pp., $25\frac{1}{2}$ cm, softcover, $58.00

The eleven papers in this book are based on some of the invited talks at a conference held in Austria during the summer of 1994.

Three papers deal with the inverse heat conduction problem: one by Beck on the function specification method, one by Eldén on a numerical method using Tikhonov regularization, and one by Murio, Liu, and Zheng on a mollification method for stabilizing the inverse problem.

Two papers deal with theoretical aspects of regularization: Seidman with general considerations in dealing with ill-posed problems, and Chavent with recent results on the regularization of nonlinear least squares problems.

Three papers deal with the determination of unknown coefficients in second-order parabolic equations. In particular, Isakov considers identifiability from lateral and final data; Lowe and Rundell consider identifiability using boundary fluxes from interior sources; and it first reviews the literature and then discusses methods based on transforming the inverse problem to one involving a nonlocal functional.

The paper by Kunisch is a survey of some recent work on numerical methods for estimating the coefficients of elliptic equations.

The paper by Vainikko uses projection discretization schemes with Tikhonov regularization to deal with an inverse problem in groundwater filtration.

The paper by Lorenzi deals with finding the characteristics of one-dimensional dispersive media governed by Maxwell's equations.

The book is a valuable guide to the current state of knowledge about inverse problems in diffusion processes.

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3[65-02, 45A55, 65D32, 65D30]—*Lattice methods for multiple integration*, by I. H. Sloan and S. Joe, Oxford University Press, New York, 1994, xii+239 pp., 24 cm, $69.95

When Milton Abramowitz said in his article [1] that a common experience of applied mathematicians was to have a scientist come to the office and say "I have an integral", he was primarily referring to one-dimensional integrals. In the early days of numerical computing, multiple-dimensional integrals (cubatures) were avoided if possible, or manipulated into one-dimensional problems. As computing power increased, however, more attention was directed to the numerical evaluation of practical integrals in several variables. But long before numerical libraries contained reliable cubature methods, statisticians and others were computing high-dimensional integrals using Monte Carlo techniques, and mathematicians were producing elegant numerical approximations, based on rules for integrating polynomials or trigonometric polynomials exactly. (For an interesting history of cubatures, from Maxwell's brick to adaptive techniques, see [2].) Lattice methods provide a link between elegant, if impractical, cubature methods, and the practical, if inaccurate, Monte Carlo methods of the statisticians. They are of interest in themselves, regardless of applications, and are the basis of an algorithm for integration over hypercubes of, in theory, any dimension. The preface of this book states that it is aimed not only at those who might be interested in lattice methods for their own sake, but also those who have practical integrals to compute. The book certainly fulfills the first claim, but it is not clear that it provides practical help to those with high-dimensional integrals to approximate.

Chapter 1 provides a nice introduction to cubature, and to the idea of trigonometric degree. The main topic of the book is introduced in Chapter 2, where lattice rules are defined, and the early history of these is discussed. (The history of lattice rules is scattered throughout the book, with recent results and historical summaries appearing in several chapters.) In Chapter 3 the concept of *rank* of a lattice rule is introduced, together with a canonical form for a general lattice rule, based on *invariants* of the rule. The original lattice rules, called *rank*-1 rules, are discussed briefly in Chapter 4, and higher rank rules are dealt with in Chapters 5–7. Since lattice rules are intended for periodic integrands, some modifications are required either to the function or to the formula if the function is not periodic. Methods for periodising the function are discussed in Chapter 2, while modifications of the rules are discussed in Chapter 8. Chapter 9 contains some miscellaneous topics which do not fit anywhere else in the scheme of the book. This takes us to page 164 in a book which, apart from the appendices, contains 215 pages. In the remaining 50 or so pages, practical methods for integration, and comparisons with existing methods, are dealt with. The only method described is based on rank-1 rules, and a sequence of embedded rules leading to a full rank rule. In addition, the rank-1 rules used are obtained by means of the restricted Korobov search. (It is worth noting that the only other lattice based software available, in the NAG scientific subroutine library, is based on rank-1 rules obtained in a similar manner.) Pseudo-code for the method is provided, and two appendices contain "recommended choices" for the rank-1 rules. It would perhaps have been more useful to have provided an ftp address for the reader to obtain the actual programs and tables electronically.

As an overview of lattice methods and their properties the book reads well. At times, however, the reader gets the feeling that the whole story is not yet known, and that results are obtained by a process of erosion. Chapter 9 is a prime example, where results on the numbers of rules with specific order, or invariants are given. The book provides an accessible entry point for anyone who wishes to understand the essentials of lattice rules, and armed with it, the interested reader will be more ready to deal with the sections on lattice rules in [3] and [4].

References

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2. R. Cools, *A survey of methods for constructing cubature formulae*, Numerical Integration, Recent Developments, Software and Applications (T. O. Espelid and A. Genz, editors), NATO ASI Series, C-357, Kluwer Academic Publishers, 1992. MR **94g:**65028
3. L. K. Hua and Y. Wang, *Applications of number theory to numerical analysis*, Springer-Verlag, Berlin, 1981. MR **83g:**10034
4. H. Niederrieter, *Random number generation and quasi-Monte Carlo methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 63, SIAM, Philadelphia, PA, 1992. MR **93h:**65008

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4[44-01, 42A38, 44A10, 44A15, 44A20, 33C10, 33C25]—*Integral transforms and their applications*, by L. Debnath, CRC Press, Boca Raton, FL, 1995, xvii+457 pp., 24 cm, $69.95

This book gives the standard integral transforms (Fourier, Laplace, Hankel, Mellin, Hilbert, Stieltjes, finite Fourier and Laplace, Z transforms and transforms with orthogonal polynomials (Legendre, Jacobi, Gegenbauer, Laguerre and Hermite)). In all chapters applications are given (Laplace transform applications are given in a separate chapter) for all kinds of boundary value problems, there is an Appendix with main properties of special functions that are used as kernels, and there are thirteen tables of integral transforms. Each chapter has worked examples, applications and exercises, and there is an extensive bibliography and a section with hints and answers to selected exercises.

The book was developed as a result from teaching advanced undergraduates and first-hear graduate students in mathematics, physics and engineering, and the author felt the need to provide lecture notes that were not too advanced for the beginner. This gives the book a quite recognizable place between other texts and reference books: the treatment of all transforms is rather to the point, without more advanced rigorous proofs. The many proofs given are usually formal without for example, discussing topics as the validity of interchanging the order of integration. This is not a point of criticism but to indicate the style of the book.

I expect that the book is a useful addition to the existing literature, of which many books are out of print, or indeed too simple or too advanced for those interested in applications of integral transforms. Those familiar with a modern setting of boundary value problems and integration theory may find the book useful when figuring out what is possible with certain types of transforms or differential equations. Comparing this book with my favorite one, "The Use of Integral Transforms" (1972, McGraw-Hill) by I. N. Sneddon, I should stay with the latter, but the present book gives many new references and I am not sure about the availability of Sneddon's book in the book stores.

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