ON SEARCHING FOR SOLUTIONS OF
THE DIOPHANTINE EQUATION $x^3 + y^3 + z^3 = n$

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Abstract. We propose a new search algorithm to solve the equation $x^3 + y^3 + z^3 = n$ for a fixed value of $n > 0$. By parametrizing $|x| = \min(|x|, |y|, |z|)$, this algorithm obtains $|y|$ and $|z|$ (if they exist) by solving a quadratic equation derived from divisors of $|x|^3 \pm n$. By using several efficient number-theoretic sieves, the new algorithm is much faster on average than previous straightforward algorithms. We performed a computer search for 51 values of $n$ below 1000 (except $n \equiv \pm 4 \pmod{9}$) for which no solution has previously been found. We found eight new integer solutions for $n = 75, 435, 444, 501, 600, 618, 912$, and $969$ in the range of $|x| \leq 2 \cdot 10^7$.

1. Introduction

Consider the Diophantine equation

$$x^3 + y^3 + z^3 = n,$$

where $n$ is a fixed positive integer and $x, y$ and $z$ can be any integers with minus signs allowed [4, 12, 15]. Note that there are no solutions of equation (1) when $n \equiv \pm 4 \pmod{9}$ because $a^3 \equiv 0, \pm 1 \pmod{9}$ for any integer $a$. There is no known general criterion for excluding any other values of $n$, although there are still many values of $n$ for which no solution has been found.

In finding all solutions for a range of values of $n$ with $\max(|x|, |y|, |z|) \leq U$, a straightforward two-dimensional algorithm [3, 8, 11] takes $O(U^2)$ steps. In [8], a computer search based on this algorithm in the range of $\max(|x|, |y|, |z|) \leq 2097151$ ($= 2^{21} - 1$), $0 < n < 1000$, was discussed. This range included the ones chosen in [3] and [11]. All 5418 solutions found were deposited into the UMT file of the American Mathematical Society. In particular, the search found solutions for 17 values of $n$ for which no solutions had been found before: $n = 39, 143, 180, 231, 312, 321, 367, 439, 462, 516, 542, 556, 600, 663, 754, 777$, and $870$. Recently, Koyama [9] extended a computer search to the range of $\max(|x|, |y|, |z|) \leq 3414387$, $0 < n < 1000$, on a CRAY-2 computer. He found other solutions for $n = 439$ as $(-489418, -2281057, 2322404)$ and for $n = 462$ as $(1612555, 2598019, -2790488)$ in differing ranges of [8] and [9]. Conn and Vaserstein [2] presented a search method by parametrizing another variable related to $(x, y, z)$ for a fixed value of $n$. They carried out a computer search in the range of $0 < n < 100$ on a Sun 4 and a Next workstation. Although they
missed some solutions, they found solutions for \( n = 39 \) and 84. In particular, a solution for \( n = 84 \) was found as \((-8\,241\,191, -41\,531\,726, 41\,639\,611\) beyond the range of [9]. Heath-Brown, Lioen and te Riele [6] presented a new algorithm based on the class number of \( Q(\sqrt{n}) \) for solving equation (1) with a fixed value of \( n \). Their algorithm takes \( O(c_0U \log U) \) steps to find all solutions in the range of \( \max(|x|, |y|, |z|) \leq U \), where the constant \( c_0 \) depends on \( n \). They did numerical experiments for \( n = 2, 3, 20, 30, 39, \) and 42 over an extended range on a CYBER 205 vector computer [6, 13]. According to recent private communications among Vaserstein, te Riele and Koyama, it appears that the solution \( (117\,367, 134\,476, -159\,380) \) for \( n = 39 \) was independently found by these three groups in 1991. In early 1995, Jagy [7] presented a search method by parametrizing \( r = x + y + z \) for a fixed value of \( n \). He found a solution for \( n = 478 \) as \((-1\,368\,722, -13\,434\,503, 13\,439\,237) \). With these recent results included, there are 51 values of \( n \) below 1000 (and \( \not\equiv \pm 4 \) mod 9) for which no solution has been found.

\[
\begin{align*}
  n &= 30, 33, 42, 52, 74, 75, 110, 114, 156, 165, \\
      &195, 290, 318, 366, 390, 420, 435, 444, 452, 501, \\
      &530, 534, 564, 579, 588, 600, 606, 609, 618, 627, \\
      &633, 732, 735, 758, 767, 786, 789, 795, 830, 834, \\
      &861, 894, 903, 906, 912, 921, 933, 948, 964, 969, \\
      &975.
\end{align*}
\]

In this paper, in order to find all solutions in the range of \( \min(|x|, |y|, |z|) \leq L \) for a fixed value of \( n \) in the above list, we propose a new search algorithm that takes \( O(cL^2) \) steps. The constant \( c \) depends on \( n \), and the computational complexity is much smaller than that of previous straightforward algorithms [3, 8, 11]. This improved efficiency is achieved by several number-theoretic sieves in the algorithm. We show the results of a computer search that used this algorithm.

\section{Outline of New Search Algorithm}

Without loss of generality, we may take 

\[ |x| \leq |y| \leq |z|. \]

The solutions are generally classified into the following three cases:

Case 0 : \( x \geq 0, y \geq 0, z \geq 0 \),

Case 1 : \( x > 0, y > 0, z < 0 \),

Case 2 : \( x \leq 0, y < 0, z > 0 \).

In case 0, the constraint \( 0 < x^3 + y^3 + z^3 < 1000 \) implies \( z \leq 9 \). Thus, it is easy to find all solutions for case 0, even if a three-dimensional exhaustive search is done, that is to say, \( x, y, z \) vary independently. In order to find all solutions for case 1 and case 2 over a range of values of \( n \), a two-dimensional exhaustive search with parameters \( y \) and \( z \) was done in [3, 8, 9, 11]. In order to find all solutions for case 1 and case 2 with a fixed value of \( n \), we propose a one-dimensional exhaustive search with one parameter \( x \). In case 1, we put \( X = x, Y = y, Z = -z \), and \( A = X^3 - n \), where \( X \) is assumed so that \( X^3 > n \). In case 2, we put \( X = -x, Y = -y, Z = z \),...
and \( A = X^3 + n \). Summarizing case 1 and case 2, we have

\[
(3) \quad Z^3 - Y^3 = A,
\]

where \( Z > Y > 0 \) and \( A > 0 \). Equation (3) can be rewritten as a product of two divisors

\[
(4) \quad (Z - Y)(Z^2 + ZY + Y^2) = A.
\]

Let \( C = Z - Y \) and \( D = Z^2 + ZY + Y^2 \). For given values of \( X \) and \( n \), we compute \( A \). By factorizing \( A \), we obtain candidates for the pair of divisors \( C \) and \( D \) such that \( A = CD \). By substituting \( Z = C + Y \) into \( D = Z^2 + ZY + Y^2 \), we get

\[
(5) \quad Y^2 + CY + \frac{C^2 - D}{3} = 0.
\]

Note that \((C^2 - D)/3\) is an integer. The value of \( Y \) (> 0) is obtained as one of the roots of equation (5) as

\[
(6) \quad Y = \frac{-C + \sqrt{Q}}{2}, \quad \text{where} \quad Q = \frac{4D - C^2}{3}.
\]

From \( Z = C + Y \), we have

\[
(7) \quad Z = \frac{C + \sqrt{Q}}{2}.
\]

Note that \( Q \) is a positive integer because \( C^2 = Z^2 - 2ZY + Y^2 < Z^2 + ZY + Y^2 = D \) and \( 4D \equiv C^2 \) (mod 3). If \( Q \) is a square, then \( Y \) and \( Z \), which are represented by equations (6) and (7), become integers because \( \sqrt{Q} \equiv C \) (mod 2).

3. Properties of sieves and their effect

To execute the above procedure, several sieves based on the following properties can be applied.

3.1. Congruence restriction between \( n \) and \( x \). If \( a = 1, 2, -3 \), then \( a^3 \equiv 1 \) (mod 7). If \( a = -1, -2, 3 \), then \( a^3 \equiv -1 \) (mod 7). Since \( a^3 \equiv 0, \pm 1 \) (mod 7) for any integer \( a \), we have \( Z^3 - Y^3 \equiv 0, \pm 1, \pm 2 \) (mod 7). Recall that

\[
Z^3 - Y^3 = X^3 - n = \begin{cases} x^3 - n & \text{for case 1,} \\ -x^3 + n & \text{for case 2.} \end{cases}
\]

Therefore, if \( n \equiv \pm 3 \) (mod 7), then \( x^3 \equiv 0 \) (mod 7). If \( n \equiv 2, 3 \) (mod 7), then \( x^3 \equiv -1 \) (mod 7). If \( n \equiv -2, -3 \) (mod 7), then \( x^3 \equiv 1 \) (mod 7). Thus, for given \( n \), the value of \( x \) is restricted as follows:

**Property 1.**

- If \( n \equiv 2 \) (mod 7), then \( x \equiv 0, 1, 2, -3 \) (mod 7).
- If \( n \equiv -2 \) (mod 7), then \( x \equiv 0, -1, -2, 3 \) (mod 7).
- If \( n \equiv 3 \) (mod 7), then \( x \equiv 1, 2, -3 \) (mod 7).
- If \( n \equiv -3 \) (mod 7), then \( x \equiv 1, -2, 3 \) (mod 7).

If \( n \equiv \pm 2 \) (mod 7), then the passing ratio for \( X \) in this sieve is \( 4/7 \). If \( n \equiv \pm 3 \) (mod 7), then the passing ratio for \( X \) in this sieve is \( 3/7 \). Among the 51 values of \( n \) in the list (2), there are 21 values of \( n \) satisfying \( n \equiv \pm 2 \) (mod 7) and 20 values of \( n \) satisfying \( n \equiv \pm 3 \) (mod 7).

Since \( a^3 \equiv 0, \pm 1 \) (mod 9) for any integer \( a \), we have \( Z^3 - Y^3 \equiv 0, \pm 1, \pm 2 \) (mod 9). It is well known that if \( n \equiv \pm 4 \) (mod 9), there is no solution. Note that for
Table 1 shows that about the relationship of factors of \( A = X^3 \pm n \) if and only if \( X^3 \equiv \mp n \pmod{p} \). Thus, for given \( n \), the factors of \( A \) are restricted as follows.

Property 2.  
- If \( n \equiv 2 \pmod{9} \), then \( x \equiv 0, 1 \pmod{3} \).
- If \( n \equiv -2 \pmod{9} \), then \( x \equiv 0, -1 \pmod{3} \).
- If \( n \equiv 3 \pmod{9} \), then \( x \equiv 1 \pmod{3} \).
- If \( n \equiv -3 \pmod{9} \), then \( x \equiv -1 \pmod{3} \).

If \( n \equiv \pm 2 \pmod{9} \), then the passing ratio for \( X \) in this sieve is 2/3. If \( n \equiv \pm 3 \pmod{9} \), then the passing ratio for \( X \) in this sieve is 1/3. Among the 51 values of \( n \) in the list (2), there are eight values of \( n \) satisfying \( n \equiv \pm 2 \pmod{9} \) and 41 values of \( n \) satisfying \( n \equiv \pm 3 \pmod{9} \). We have proven that no other values of modulus for \( n \) except 7 and 9 have the sieve effect of excluding some values of \( x \) for a solution [14].

### 3.2. Factor restriction of \( A \) based on cubic residuacity

A prime \( p \) is a factor of \( A = X^3 \pm n \) if and only if \( X^3 \equiv \mp n \pmod{p} \). Thus, for given \( n \), the factors of \( A \) are restricted as follows.

Property 3. Let \( p \) be a prime. If \( n \) is a cubic nonresidue modulo \( p \), then \( A = X^3 \pm n \) does not have the factor \( p \). When \( p \equiv 2 \pmod{3} \), all values of \( n \) are cubic residues modulo \( p \). When \( p \equiv 1 \pmod{3} \), \( n \) is a cubic residue modulo \( p \) if and only if \( n^{\frac{p-1}{3}} \equiv 1, 0 \pmod{3} \).

In advance, for fixed \( n \), we can easily pick primes \( p \) satisfying cubic residuacity (i.e., there is a solution \( X \) for \( X^3 \equiv \pm n \pmod{p} \)) from all primes below a certain limit. Let \( W_m \) be the set of primes satisfying \( p \equiv 2 \pmod{3} \) and \( p \leq m \). Let \( V_m(n) \) be the set of primes satisfying \( p \equiv 1 \pmod{3} \), \( n^{\frac{p-1}{3}} \equiv 1, 0 \pmod{3} \), and \( p \leq m \). Let \( P_m(n) \) be the set of primes satisfying \( p \equiv 1 \pmod{3} \), \( n \equiv \pm 1 \pmod{3} \), and \( p \leq m \). Let \( P_m(n) \) be the set of the union of \( W_m \) and \( V_m(n) \) that includes the prime 3. Note that \( |P_m(n)| = |W_m| + |V_m(n)| + 1 \), where \( |\cdot| \) means the cardinality of a set.

For example, there are 348,513 primes below 5,000,000, giving us \( |W_{5,000,000}| = 174,322 \). Table 1 shows \( |V_m(n)| \) and \( |P_m(n)| \) for several values of \( n \) and \( m = 5,000,000 \). From Table 1, we can observe that \( |P_m(n)| \) is about 66.7% of the number of all primes \((= 348,513)\). Using these prechosen primes, factoring based on trial and division can be more efficiently carried out.

**Table 1. Number of primes satisfying cubic residuacity below \( m = 5,000,000 \)**

<table>
<thead>
<tr>
<th>( n )</th>
<th>30</th>
<th>33</th>
<th>42</th>
<th>52</th>
<th>74</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>V_m(n)</td>
<td>)</td>
<td>58,145</td>
<td>58,079</td>
<td>57,912</td>
<td>58,097</td>
</tr>
<tr>
<td>(</td>
<td>P_m(n)</td>
<td>)</td>
<td>232,468</td>
<td>232,402</td>
<td>232,235</td>
<td>232,420</td>
</tr>
</tbody>
</table>

### 3.3. Factor restriction between \( A \) and \( C \)

We obtain the following theorem about the relationship of factors of \( A \) and \( C \). Hereafter, we denote \( p^e || N \) if \( p^e | N \) and \( p^{e+1} \nmid N \) for integer \( N \) and prime \( p \).

**Theorem 1.** Let \( p \) be a prime with \( p \equiv 2 \pmod{3} \). If \( p^e | A \ (e \geq 1) \), and \( p^f | C \ (f \geq 0) \), then \( e = f + 2g \) and \( f \geq g \), where \( g \) is a nonnegative integer.
Proof. Let $\omega = \frac{-1 + \sqrt[3]{3}}{2}$. A prime satisfying $p \equiv 2 \pmod{3}$ is a prime element in $\mathbb{Z}[\omega]$. Note that $A = Z^3 - Y^3 = (Z - Y)(Z - \omega Y)(Z - \omega^2 Y)$, where $C = Z - Y$ and $D = (Z - \omega Y)(Z - \omega^2 Y)$. Assume that $Z - \omega Y = p^a \cdot D_1$ and $Z - \omega^2 Y = p^b \cdot D_2$, where $p \nmid D_1$, $p \nmid D_2$, $a \geq 0$ and $b \geq 0$. For any integers $k$, $Y$ and $Z$, we have

$$k|(Z - \omega Y) \iff [k |Z$ and $k |Y] \iff k|(Z - \omega^2 Y).$$

Putting $k = p^a$ and $k = p^b$ into the above relation, we have $a = b$, which is denoted by $g$. Thus, $p^{2g} \| D$, which implies $e = f + 2g$. Furthermore, $p^g|(Z - Y)$, that is, $p^g|C$. Thus, $f \geq g$. $$\square$$

As a result of this theorem, divisor $C$ is restricted as:

**Property 4.** Let $p$ be a prime with $p \equiv 2 \pmod{3}$. Assume that $p^e||A$, where $e \geq 1$. Then $p^h|C$ and $p^f||C$, where

$$(8) \quad h = \begin{cases} \left[ \frac{e}{2} \right] + (1 - (\frac{e}{2}) \pmod{2}) & \text{if } e \text{ is odd,} \\ \left[ \frac{e}{2} \right] + (\left[ \frac{e}{2} \right] \pmod{2}) & \text{if } e \text{ is even,} \end{cases}$$

$h \leq f \leq e$ and $f - h$ is even.

For example, if $e = 1, 3$, then $p|C$. If $e = 5, 7, 9$, then $p^3|C$. If $e = 2, 4, 6$, then $p^2|C$. If $e = 8, 10, 12$, then $p^4|C$. If $e = 3$, then either $f = 1$ or $f = 3$. Property 4 is effective in determining the candidates for divisor $C$ from the combination of prime factors of $A$. Note that, even if a prime factor $p$ of $A$ with $p \equiv 1 \pmod{3}$ is found, we cannot determine whether it is a factor of $C$ or not. For the prime factor 3, we obtain the following theorem.

**Theorem 2.** Assume that $3^e||A$, $3^f||C$ and $3^g||D$. Then $e = f + g$ and $f \geq \left[ \frac{e}{2} \right]$. Moreover, if $f > 0$, then $e \geq 2$, $f \geq 1$ and $g \geq 1$.

**Proof.** Let $\omega = \frac{-1 + \sqrt[3]{3}}{2}$ and $\pi = 1 - \omega$. For $Z, Y \in \mathbb{Z}$, if $\pi^a||(Z - \omega Y)$ and $\pi^b||(Z - \omega^2 Y)$, then $a = b$, which is denoted by $g$. Note that for $N \in \mathbb{Z}$, if $\pi|N$, then $k$ is even. Since $3 = -\pi^2$, we have $3|\pi$, we have $3|N \iff \pi^k||N$ for $N \in \mathbb{Z}$. If $\pi^2||Y$, then $g = \min(2f, 2f + 1)$ because $Z - \omega Y = C + \pi Y$ and $\pi^2||C$. Thus, we have $2f \geq g$ and $f \geq \left[ \frac{e}{2} \right]$.

Since $C^2 \equiv D \pmod{3}$, we have $3|C \iff 3|D$. $$\square$$

Note that if $3|A$, then $3^2|A, 3|C$ and $3|D$. By means of Theorem 2, divisor $C$ is restricted as:

**Property 5.** If $3^e||A$ and $e \geq 1$, then $3^h|C$, where $h = \left[ \frac{e}{2} \right]$.

For example, if $e = 2, 3$, then $h = 1$. If $e = 4, 5, 6$, then $h = 2$. Note that from Property 2, if $n \equiv \pm 2, \pm 3 \pmod{9}$, then $3 \nmid A$. Among the 51 values of $n$ in the list (2), there are two values of $n$ satisfying $n \equiv \pm 1 \pmod{9}$ for which $A$ may have a factor of 3.

3.4. **Size restriction of $C$.** Since $C^2 < D = A/C$, we have $C < A^{1/3}$. When $X \gg n$ such that $n < 1000$, $X > 100000$, we have $A = X^3 \pm n \approx X^3$ and a weak upper bound of $C$ is obtained as $C < X$. Furthermore, since $Z < 2^{1/3}Y$ and $Z > 2^{1/3}X$ if $X \gg n$, a stricter upper bound of $C$ is evaluated in a term of $X$ as:

$$C \approx \frac{X^3}{Y^2 + YZ + Z^2} < \frac{X^3}{Z^2(1 + 2^{-1/3} + 2^{-2/3})} < \frac{X}{1 + 2^{1/3} + 2^{2/3}} \approx 0.2599X.$$ 

This inequality implies the following property.
Property 6. $C < 0.26X$.

The combination of Properties 4, 5 and 6 is effective in finding prime factors of $A$, more exactly, prime factors of $C$. At the beginning of trial division factoring, an upper bound of searched primes is put as $B = \lceil 0.26X \rceil$. After prime factors $p_k^e$ of $A$ satisfying $p_k = 3$ or $p_k \equiv 2 \pmod{3}$ are found, the upper bound of primes for trial division factoring is dynamically reduced to $B = \left\lfloor \frac{0.26X}{\prod_k p_k^e} \right\rfloor$. The final upper bound $B$ depends on the distribution of prime factors of pseudo-random values of $A$.

3.5. Congruence restriction between $A$ and $C$. If $C \not\equiv 0 \pmod{3}$, then $D \equiv 1 \pmod{3}$. If $C \equiv 0 \pmod{2}$, then $D \equiv 1 \pmod{2}$. Thus, the following congruences of $A$ and $C$ for a particular modulus hold.

Property 7. $C \equiv A \pmod{6}$, that is, $C \equiv A \pmod{2}$ and $C \equiv A \pmod{3}$.

The relationship $C \equiv A \pmod{6}$ is effective in checking the appropriateness of pairs of $C$ and $D$. Furthermore, by combining Properties 4, 5, 6 and 7, a kernel divisor of $C$, which is denoted by $H$, can be computed and has a congruence relationship with $A$ as shown in the following theorem.

Theorem 3. Let $p_1 = 3$. Let $p_k$ ($k \geq 2$) be a prime satisfying $p_k \equiv 2 \pmod{3}$, $p_k < p_{k+1}$ and $p_k < \lfloor 0.26X \rfloor$. Assume that $p_k^e \parallel A$, $e_k \geq 0$ ($k = 1, 2, 3, \ldots$). Let $H$ be defined as

$$H = \prod_{k=1}^{e} p_k^{h_k},$$

where $\ell$ is the maximum integer satisfying $H < \lfloor 0.26X \rfloor$, and

$$h_k = \begin{cases} \left\lfloor \frac{e_k}{3} \right\rfloor & \text{if } 3^{e_k} \parallel A, \\ \left\lfloor \frac{e_k}{3} \right\rfloor + (1 - (\left\lfloor \frac{e_k}{3} \right\rfloor \mod 2)) & \text{if } p_k^e \parallel A, \; k \geq 2 \text{ and } e_k \text{ is odd,} \\ \left\lfloor \frac{e_k}{3} \right\rfloor + (\left\lfloor \frac{e_k}{3} \right\rfloor \mod 2) & \text{if } p_k^e \parallel A, \; k \geq 2 \text{ and } e_k \text{ is even.} \end{cases}$$

Then, $H \mid C$ and $H \equiv A \pmod{6}$.

Proof. It is clear that $H \mid C$ because of Properties 4 and 5. Since $H \equiv C \pmod{6}$ and $C \equiv A \pmod{6}$, we have $H \equiv A \pmod{6}$.

If $p_k \nmid A$ for all primes $p_k \in W_m$, $m = \lfloor 0.26X \rfloor$, then $H = 1$. In Theorem 3, $H$ is generally defined and discussed; however, when $2 \mid A$, the congruence $H \equiv A \pmod{2}$ always holds. When $3 \mid A$, the congruence $H \equiv A \pmod{3}$ always holds. When the factor 3 is excluded from $A$ and $H$, the following property can be used as a sieve before checking each candidate of $C$.

Property 8. Let $H = 3^b H'$, $3 \nmid H'$, $A = 3^b A'$ and $3 \nmid A'$. Then $H' \equiv A' \pmod{3}$.

In this sieve, two cases such that $\{H' \equiv 1 \pmod{3} \text{ and } A' \equiv 2 \pmod{3}\}$ and $\{H' \equiv 2 \pmod{3} \text{ and } A' \equiv 1 \pmod{3}\}$ are rejected, and two other cases such that $\{H' \equiv A' \equiv 1 \pmod{3}\}$ and $\{H' \equiv A' \equiv 2 \pmod{3}\}$ are accepted. From an extensive computer experiment, we can observe that the passing ratio for $X$ to satisfy $H' \equiv A' \pmod{3}$ is about 50%. Note that, even if $H = 1$, the passing ratio for $X$ to satisfy $H' \equiv A' \pmod{3}$ is also about 50%.

In our search algorithm, the first trial division factoring is carried out for the prime 3 and primes $\in W_B$, then congruence $H' \equiv A' \pmod{3}$ is checked. If the
where $x_n \equiv 9$, we have

If $n \equiv 1 \pmod{3}$, then $a^3 - 3a + 2 \equiv (a - 1)^2(a + 2) \equiv 0 \pmod{27}$. Thus, when $n \equiv 3 \pmod{9}$, we have $n \equiv x^3 + y^3 + z^3 \equiv (3x - 2) + (3y - 2) + (3z - 2) \equiv 3(x + y + z) - 6 \pmod{27}$, which implies $x + y + z \equiv 2 + \frac{6}{3} \pmod{9}$. On the other hand, if $n \equiv -3 \pmod{9}$, then $x \equiv y \equiv z \equiv -1 \pmod{3}$. If $a \equiv -1 \pmod{3}$, then $a^3 - 3a - 2 \equiv (a + 1)^2(a - 2) \equiv 0 \pmod{27}$. Thus, when $n \equiv -3 \pmod{9}$, we have $n \equiv x^3 + y^3 + z^3 \equiv (3x + 2) + (3y + 2) + (3z + 2) \equiv 3(x + y + z) + 6 \pmod{27}$, which implies $x + y + z \equiv -2 + \frac{6}{3} \pmod{9}$. These congruences imply the following property.

**Property 9.** If $n \equiv \pm 3 \pmod{9}$, then

$$C \equiv \begin{cases} X - k \pmod{9} & \text{for case 1,} \\ X + k \pmod{9} & \text{for case 2,} \end{cases}$$

where

$$k \equiv \begin{cases} 2 + \frac{n}{3} \pmod{9} & \text{if } n \equiv 3 \pmod{9}, \\ -2 + \frac{n}{3} \pmod{9} & \text{if } n \equiv -3 \pmod{9}. \end{cases}$$

If $n \equiv \pm 3 \pmod{9}$, then this sieve modulo 9 can be used in addition to the sieve modulo 6. There are 41 values of $n$ satisfying $n \equiv \pm 3 \pmod{9}$ in the list (2). They include the case for $n = 30$, which is the smallest in the list (2) and said in [4, Probl. D5] to be the most interesting.

3.6. **Congruence restriction between $C$ and $n$.** The value of $C$ is more restrictive for special values of $n$. We can extend the result that was analyzed for $n = 30$ in [13]. If $n \equiv 3 \pmod{9}$, then $x \equiv y \equiv z \equiv 1 \pmod{3}$. If $a \equiv 1 \pmod{3}$, then $a^3 - 3a + 2 \equiv (a - 1)^2(a + 2) \equiv 0 \pmod{27}$. Thus, when $n \equiv 3 \pmod{9}$, we have $n \equiv x^3 + y^3 + z^3 \equiv (3x - 2) + (3y - 2) + (3z - 2) \equiv 3(x + y + z) - 6 \pmod{27}$, which implies $x + y + z \equiv 2 + \frac{6}{3} \pmod{9}$. On the other hand, if $n \equiv -3 \pmod{9}$, then $x \equiv y \equiv z \equiv -1 \pmod{3}$. If $a \equiv -1 \pmod{3}$, then $a^3 - 3a - 2 \equiv (a + 1)^2(a - 2) \equiv 0 \pmod{27}$. Thus, when $n \equiv -3 \pmod{9}$, we have $n \equiv x^3 + y^3 + z^3 \equiv (3x + 2) + (3y + 2) + (3z + 2) \equiv 3(x + y + z) + 6 \pmod{27}$, which implies $x + y + z \equiv -2 + \frac{6}{3} \pmod{9}$. These congruences imply the following property.

**Property 9.** If $n \equiv \pm 3 \pmod{9}$, then

$$C \equiv \begin{cases} X - k \pmod{9} & \text{for case 1,} \\ X + k \pmod{9} & \text{for case 2,} \end{cases}$$

where

$$k \equiv \begin{cases} 2 + \frac{n}{3} \pmod{9} & \text{if } n \equiv 3 \pmod{9}, \\ -2 + \frac{n}{3} \pmod{9} & \text{if } n \equiv -3 \pmod{9}. \end{cases}$$

If $n \equiv \pm 3 \pmod{9}$, then this sieve modulo 9 can be used in addition to the sieve modulo 6. There are 41 values of $n$ satisfying $n \equiv \pm 3 \pmod{9}$ in the list (2). They include the case for $n = 30$, which is the smallest in the list (2) and said in [4, Probl. D5] to be the most interesting.

3.7. **Congruence restriction of $C$ based on quadratic residuacity.** If an integer $b$ is a quadratic nonresidue modulo $p$ for some prime $p$, then $b$ is not a square. This relationship of quadratic residuacity can be applied for choosing an appropriate value of $C$. An application of several primes, say $p = 5, 7$, seems to be practically effective. Recall $Q = (4D - C^2)/3$ is a square if there is a solution for equation (1). When $p = 5$, pairs of $(A, C)$ modulo 5 such that $(-2, 1), (-1, 2), (1, -2)$ and $(2, -1)$ imply the quadratic nonresidue condition $Q^{\frac{p+1}{2}} = Q^2 \equiv 1 \pmod{5}$. Thus, the value of $C$ is restricted by the value of $A$ modulo 5 as follows.

**Property 10.**

- If $A \equiv 1 \pmod{5}$, then $C \equiv \pm 1, 2 \pmod{5}$.
- If $A \equiv -1 \pmod{5}$, then $C \equiv \pm 1, -2 \pmod{5}$.
- If $A \equiv 2 \pmod{5}$, then $C \equiv 1, \pm 2 \pmod{5}$.
- If $A \equiv -2 \pmod{5}$, then $C \equiv -1, \pm 2 \pmod{5}$.

The characteristic that $A \equiv 0 \pmod{5}$ implies $C \equiv 0 \pmod{5}$ is common to Property 4. If $A \not\equiv 0 \pmod{5}$, then the passing ratio for $C$ in this sieve is $3/5$.

A similar restriction is obtained for another prime, $p = 7$. Recall $A \not\equiv \pm 3 \pmod{7}$. When $A \equiv \pm 2 \pmod{7}$, the value of $Q$ is always a quadratic residue modulo 7. Pairs of $(A, C)$ modulo 7 such that $(-1, 1), (-1, 2), (1, 3), (-1, -3), (1, -2)$, and $(1, -1)$ imply the quadratic nonresidue condition $Q^{\frac{p+1}{2}} = Q^3 \equiv 1 \pmod{7}$. Thus, the value of $C$ is restricted by the value of $A$ modulo 7 as follows.
step 4: Compute Check

Find prime factors

step 5: Let W_m and V_m(n) be the sets of primes satisfying

\[ W_m = \{ p_i \mid p_i \equiv 2 \pmod{3}, \ p_i \leq m \}, \]

\[ V_m(n) = \{ p_i \mid p_i \equiv 1 \pmod{3}, \ n^{(p_i-1)/3} \pmod{p_i} = \{0,1\}, \ p_i \leq m \}. \]

Collect primes \( p_i \in W_m \) and \( p_i \in V_m(n) \), where \( m = [0.26L] \).

step 2: Put \( X = S \).

step 3: Check \( x \) by the values of \( n \pmod{7} \) and \( n \pmod{9} \) by using Properties 1 and 2.

If \( x \) is not appropriate as a solution then go to step 11 endif.

step 4: Compute \( A = X^3 \pm n \)

(\( A \) is a representative of \( A_1 = X^3 - n \) and \( A_2 = X^3 + n \)).

step 5: Let \( B = \lfloor 0.26 X \rfloor \), \( H = 1 \) and \( F = 1 \).

If \( 3^e \| A \) (\( e \geq 1 \)) then put \( H = 3^h \), \( B = \lfloor B/3^h \rfloor \), \( F = 3^{e-h} \) endif.

step 6: Find prime factors \( p_i \in W_B \) of \( A \) by a revised trial division:

Do while \( p_i \leq B \)

if \( p_i^{e_i} \| A \) (\( e_i \geq 1 \))

then if \( p_i^{h_i} < B \)

then put \( H = H \cdot p_i^{h_i} \), \( B = \lfloor B/p_i^{h_i} \rfloor \), \( F = F \cdot p_i^{e_i-h_i} \)

else go to step 11 endif

endif
endo.

step 7: Let \( H' = H/3^h \) (\( h \geq 0 \)) and \( A' = A/3^e \) (\( e \geq 0 \)).

If \( H' \neq A' \pmod{3} \) then go to step 11 endif.

step 8: Find prime factors \( p_i \in V_B(n) \) of \( A \) by a trial division:

Do while \( p_i < B \)

if \( p_i^{e_i} \| A \) (\( e_i \geq 1 \)) then put \( F = F \cdot p_i^{e_i} \) endif
endo.

step 9: By using the information of the factors \( H \) and \( F \) of \( A \), choose divisor \( C_j \) as \( C_j = HF_j \) satisfying Properties 6, 7, 9, 10, and 11, where \( F_j \) is the \( j \)th element among combinations of factors of \( F \).

Compute another divisor \( D_j = A/C_j \) from each \( C_j \).
1 and case 2. When $X$ and $X$ show that this value of $X$ is not a solution for case 1. This value of $X$ may be a solution for case 2, and it follows that $A \equiv X^3 + n = 784730401134188690880$. Note that $|0.26 \times 19895059| = 5172715$. We apply trial division factoring of step 6 with primes $p_i$ satisfying $p_i \equiv 2 \pmod{3}$ and $p_i \leq 5172715$. After knowing that $A$ has the factor $2^6$, the upper bound of primes for the trial and division is reduced to $\left\lfloor \frac{0.26X}{2^6} \right\rfloor = 1293178$. Moreover, after knowing that $A$ has the factor 5, the upper bound is reduced to $\left\lfloor \frac{0.26X}{5} \right\rfloor = 258635$. After finding that $A$ has the factor 169553, step 6 ends with $\left\lfloor \frac{0.26X}{169553} \right\rfloor = 1$. Thus, we have $F = 2^4$ and $H = H' = 2^2 \cdot 5 \cdot 169553 = 3391060$, which holds $H' = A' (= A) \equiv 1 \pmod{3}$. Since the reduced upper bound becomes one, we do not need the trial division factoring of step 8 with primes $p_i$ satisfying $p_i \equiv 1 \pmod{3}$ and $501^{(p_i-1)/3} \equiv 0, 1 \pmod{p_i}$. Note that, although $A$ has factors 181 and 6073 below 5172715, they are not included into the factors of $F$. Thus, the candidates for divisor $C$ satisfying the exponent restriction and $A \equiv C \equiv 4 \pmod{6}$ are $\{H, H \cdot 2^2, H \cdot 2^4\}$. Among these candidates, only 3391060$(= H)$ satisfies $C < 0.26X$. For $C = 3391060$, we have $Q = (4D - C^2)/3 = 3092437844334864$, which is a square of 55609692. Thus, we can compute $Y = 26109316$ and $Z = 29500376$. Finally, we obtain the solution for $n = 501$ as $(-19895059, -26109316, 29500376)$. \[ \text{Numerical Example. When } n = 501, \text{ we found a new solution for case 2. We mention the values of the intermediate variables in the algorithm.} \]
5. Computer search and its results

By using the search algorithm mentioned in §4, we performed a computer search for solutions of equation (1) for the 51 values of \( n \) below 1000 in the list (2). The range of the search was determined as follows. The ratio \( Z/X \) is maximal when \( Z - Y = 1 \) and \( X \gg n \), which imply

\[
X \approx (Z^2 + ZY + Y^2)^{1/3} \approx (3Z^2)^{1/3} = 3^{1/3}Z^{2/3} \approx 1.442Z^{2/3}.
\]

The ratio \( Z/X \) is minimal when \( X \approx 2^{-1/3}Z \approx 0.7937Z \). As a result, the range of \( X \) is represented in terms of \( Z \) as

\[
1.442Z^{2/3} < X < 0.7937Z.
\]

In [9], a search for all solutions in the range of \( \max(|x|,|y|,|z|) = Z \leq 3414387 \) was done. That is to say, a complete search for all solutions in the range of \( X \leq \lfloor 3^{1/3} \cdot 3414387^{2/3} \rfloor = 32702 \) and a partial search for solutions in the range of \( 32702 < X \leq \lfloor 2^{-1/3} \cdot 3414387 \rfloor = 2710000 \); a search for solutions in the range of \( 2710000 < X \) was not done.

Our new search algorithm parametrizes a positive integer \( X \) that is in the range of \( S \leq X \leq L \), where \( \min(|x|,|y|,|z|) = X \). To keep a continuous and exhaustive search going, we put \( S = 32702 \). Taking into account our computer’s power, we put \( L = 2 \cdot 10^7 \). The CPU-time on a DEC Alpha Server 2100 computer (4 processors, 190 MHz) was about 4 months.

We found eight new integer solutions for \( n = 75, 435, 444, 501, 600, 618, 912, \) and 969 as shown in Table 2. Note that the solution \((x', y', z')\) for \( n = 600 \) is derived from the solution \((x, y, z)\) for \( n = 75 \) because \( 600 = 75 \cdot 2^3 \) and \((x', y', z') = (2x, 2y, 2z)\). Since our search algorithm is deterministic and exhaustive, we can also confirm that there is no solution for 43 values of \( n \) below 1000 exempting the above eight values of \( n \) in the range of \( |x| \leq 2 \cdot 10^7 \).

Quite recently, a referee informed us of the related work [1, 5, 10]. Bremner [1, 5] presented a search method by parametrizing \( m = y + z \) and \( x \) to find solutions for a fixed value of \( n \). It appears that he and we independently found solutions for \( n = 75 \) (and \( n = 600 \)). By using Bremner’s search method, Lukes [10] found a new solution for \( n = 110 \) as \((109938919, 16540290630, -16540291649)\) and another solution for \( n = 435 \) as \((-981038126, -509795654285, 509795655496)\). These solutions were found beyond the range of our search. As a result, there are 42 values of \( n \) below 1000 (exempting \( n \equiv \pm 4 \pmod{9} \)) for which no solutions have been found.

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SOLUTIONS OF \( x^3 + y^3 + z^3 = n \)

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