COMPUTATION OF RELATIVE CLASS NUMBERS
OF CM-FIELDS

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Abstract. It was well known that it is easy to compute relative class numbers
of abelian CM-fields by using generalized Bernoulli numbers (see Theorem
4.17 in Introduction to cyclotomic fields by L. C. Washington, Grad. Texts
in Math., vol. 83, Springer-Verlag, 1982). Here, we provide a technique for
computing the relative class number of any CM-field.

1. Statement of the results

Proposition 1. Let $n \geq 1$ be an integer and $\alpha > 1$ be real. Set $P_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} x^k$,

\begin{equation}
f_n(s) = \Gamma^n(s) A^{-2s} \left( \frac{1}{2s-1} + \frac{1}{2s-2} \right)
\end{equation}

and

\begin{equation}
K_n(A) = \frac{A^2}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} f_n(s) ds.
\end{equation}

Then, it holds

\begin{equation}
0 \leq K_n(A) \leq 2P_n(nA^{2/n}) e^{-nA^{2/n}} \leq 2n \exp(-A^{2/n}).
\end{equation}

Theorem 2. Let $N$ be a totally imaginary number field of degree $2n$ which is a
quadratic extension of a totally real number field $N^+$ of degree $n$, i.e. $N$ is a CM-
field. Let $w_N$ be the number of roots of unity in $N$, $Q_N \in \{1, 2\}$ be the Hasse unit
index of $N$, and $d_N$, $\zeta_N$ and $d_{N^+}$, $\zeta_{N^+}$ be the absolute values of the discriminants
and the Dedekind zeta functions of $N$ and $N^+$, respectively. Let $\chi_{N/N^+}$ be the
quadratic character associated with the quadratic extension $N/N^+$ and let $\phi_k$
be the coefficients of the Dirichlet series $(\zeta_N/\zeta_{N^+})(s) = L(s, \chi_{N/N^+}) = \sum_{k \geq 1} \phi_k k^{-s}$,

$\Re(s) > 1$. Set $A_{N/N^+} = \sqrt{d_N/n^2d_{N^+}}$.

We have

\begin{equation}
h_N^- = \frac{Q_N w_N}{(2\pi)^n} \sqrt{\frac{d_N}{d_{N^+}}} \sum_{k \geq 1} \frac{\phi_k}{k} K_n(k/A_{N/N^+}),
\end{equation}

and according to (3) this series (4) is absolutely convergent. Moreover, set

\begin{equation}
B(N) \overset{def}{=} A_{N/N^+} \left( \frac{\lambda}{n} \log A_{N/N^+} \right)^{n/2}.
\end{equation}

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Then, if \( \lambda > 1 \) and \( n \) are given, then the limit of \(|h^{-\lambda}_{N}-h^{-\lambda}_{N}(M)|\) as \( A_{N}/N^{+} \) approaches infinity is equal to 0, where \( h^{-\lambda}_{N}(M) \) is the approximation of the relative class number obtained by disregarding in the series occurring in (4) the indices \( k > M \geq B(N) \).

For example, if \( N \) of degree \( m = 2n \) is the narrow Hilbert class field of a real quadratic number field \( L \) of discriminant \( d_{L} \), we have

\[
B(N) = \left( \frac{\lambda}{4\pi} \right)^{m/4} \frac{m}{d_{L}^{m/8}} \log^{m/4}(d_{L}/\pi^2).
\]

The following Proposition 3 explains how we compute the numerical values of the function \( A \mapsto K_{n}(A) \) according to its series expansion:

**Proposition 3.** Take \( A > 0 \). It holds

\[
K_{n}(A) = 1 + \pi^{n/2}A + 2A^{2} \sum_{m \geq 0} \text{Res}_{s=-m}(f_n).
\]

This series is absolutely convergent and for any integer \( M \geq 0 \) we have

\[
\left| 2A^{2} \sum_{m > M} \text{Res}_{s=-m}(f_n) \right| \leq \frac{\pi^{n/2}A^{2M+3}}{(M+1)(M!/2)^{n}}.
\]

Finally, the following Proposition 4 explains how to compute recursively the values of the residues \( \text{Res}_{s=-m}(f_n) \) occurring in (6):

**Proposition 4.** We have \(^{1}\)

\[
\text{Res}_{s=-m}(f_n) = -(-1)^{nm} A^{2m} \frac{(m!)}{(m+1)!} \sum_{i=-n}^{-1} 2^{-1-i} h_{i}(m)((2m + 1)^i + (2m + 2)^i)
\]

where the \( h_{i}(m) \)'s are computed recursively from the \( h_{i}(0) \)'s by using

\[
h_{i}(m+1) = \sum_{j=-n}^{i} h_{j}(m) \frac{b_{i-j}}{(m+1)^{i-j}} \quad \text{and} \quad \sum_{j=-n}^{-1} h_{j}(0)s^j + O(1) = \Gamma^{n}(s)A^{-2s},
\]

where \( b_{k} = C_{n+k-1}^{m-1} = ((k+n-1)!/k!(n-1)!). \) Thus, if

\[
\Gamma^{n}(s+1) = \sum_{i=0}^{n-1} h_{i}s^i + O(s^n),
\]

then

\[
h_{j-n}(0) = \sum_{i=0}^{j} \frac{(-2\log A)^{i}}{i!} h_{j-i} \quad (0 \leq j \leq n-1).
\]

For proving these results, obvious questions of convergence of series and integrals, and questions of inversions of integrals and summations will not be gone into.

\(^{1}\)Note the misprint in the formula given in [Lou 2].
2. Introduction

Prior to the method we have developed here, the only general method for computing the relative class number of any CM-field was that developed by T. Shintani (see [Oka 1] and [Oka 2] for examples of actual relative class number computations using Shintani’s ideas). However, his method requires the knowledge of a great deal of information on the maximal totally real subfield \( \mathbf{N}^+ \). In particular, it requires the knowledge of a system of fundamental units of the group of totally positive units of \( \mathbf{N}^+ \). However, what makes the concept of CM-field an attractive one is that the relative class number formula

\begin{equation}
(12) \quad h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \frac{\text{Res}_{s=1}(\zeta_{\mathbf{N}})}{\text{Res}_{s=1}(\zeta_{\mathbf{N}^+})} = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n} \left( \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \right) L(1, \chi_{\mathbf{N}/\mathbf{N}^+})
\end{equation}

enables us to get lower bounds on relative class numbers and solve class number and class group problems for CM-fields precisely because (12) does not involve any regulator (see [Lou-Oka] and [LOO]). Thus, the reader may possibly feel dissatisfied that he should have to know beforehand a good grasp of the unit group of \( \mathbf{N}^+ \) before he can compute \( h_{\mathbf{N}}^- \), whereas (12) gives an expression for \( h_{\mathbf{N}}^- \) which does not involve units. The reader may now possibly feel satisfied that this paper shows how using (12) he indeed gets an efficient method for computing \( h_{\mathbf{N}}^- \) provided that he only knows how to compute the decomposition of any rational prime into a product of prime ideals of \( \mathbf{N} \). The key point of our method is to establish the holomorphic continuation of \( s \mapsto (\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = L(s, \chi_{\mathbf{N}/\mathbf{N}^+}) \) in the same way Riemann did in the case of the Riemann zeta function (by using Mellin transformation) and to evaluate the resulting series at \( s = 1 \) (see section 4).

Finally, we note that the results of this paper are better than those of [Lou 3]. Indeed, \( B(\mathbf{N}) \) in (5) is \( n^{n/2} \)-fold better than the one we gave in [Lou 3]. Moreover, our proof of (3) (in section 3) is more satisfactory and elegant than the one we gave in [Lou 3].

3. Proof of Proposition 1

We use:

**Lemma 5.** Let \( \alpha > 1 \) be real. We have

\[
\int_{\alpha-i\infty}^{\alpha+i\infty} u^{s} \frac{ds}{2s-1} = \begin{cases} 0 & \text{if } u < 1, \\ i\pi \sqrt{u} & \text{if } u > 1; \end{cases} \quad \text{and} \quad \int_{\alpha-i\infty}^{\alpha+i\infty} u^{s} \frac{ds}{2s-2} = \begin{cases} 0 & \text{if } u < 1, \\ i\pi u & \text{if } u > 1. \end{cases}
\]

Now, using

\[
\Gamma^n(s) = \int \int e^{-\text{Tr}(y)} y^s \frac{dy}{y}
\]

where the multiple integral ranges over \( (y_1, \cdots, y_n) \in (\mathbb{R}^+_+)^n \) and where we set \( y = y_1 y_2 \cdots y_n \) and \( \text{Tr}(y) = y_1 + y_2 + \cdots + y_n \), leads to

\[
K_n(A) = \frac{A^2}{i\pi} \int \int e^{-\text{Tr}(y)} \left( \int_{\alpha-i\infty}^{\alpha+i\infty} \left( \frac{1}{2s-1} + \frac{1}{2s-2} \right) \frac{dy}{y} \right).
\]
Using Lemma 5 yields  
\[ K_n(A) = A^2 \int_{y \geq A^2} \left( \sqrt{y/A^2} + (y/A^2) \right) e^{-\text{Tr}(y)} \frac{dy}{y} \leq 2 \int_{y \geq A^2} e^{-\text{Tr}(y)} dy. \]

For example, we get \( K_1(A) \leq 2e^{-A^2} \). Now, using the arithmetic-geometric mean inequality yields that \( \{(y_1, \cdots, y_n) ; y \geq A^2\} \) is included in \( \{(y_1, \cdots, y_n) ; \text{Tr}(y) \geq nA^{2/n}\} \), which yields  
\[ K_n(A) \leq 2 \int_{\text{Tr}(y) \geq nA^{2/n}} e^{-\text{Tr}(y)} dy. \]

Then, the following easily proved Lemma 6 provides us with the desired result.

**Lemma 6.** Set \( P_n(x) = \sum_{k=0}^{n-1} x^k/k! \). Then  
\[ P_n(\alpha)e^{-\alpha} = \int_{(y_1, \cdots, y_n) \in \mathbb{R}^n_+} e^{-\text{Tr}(y)} dy \leq n \int_{\alpha/n}^{+\infty} e^{-y} dy = ne^{-\alpha/n}. \]

**Proof.** Use  
\[ \{(y_1, \cdots, y_n) \in \mathbb{R}^n_+ , \text{Tr}(y) \geq \alpha \} \subseteq \bigcup_{i=1}^{n} \{(y_1, \cdots, y_n), y_i \geq \frac{\alpha}{n} \text{ and } y_j \geq 0 \text{ for } j \neq i \} \]. □

4. **Proof of Theorem 2**

Let \( K \) be a number field of degree \( n = r_1 + 2r_2 \), where \( r_1 \) is the number of real places of \( K \) and \( r_2 \) the number of complex places of \( K \). Let \( \zeta_K \) and \( \text{Reg}_K \) be the Dedekind zeta function and regulator of \( K \). We set  
\[ A_K = 2^{-r_2}d_K^{1/2}\pi^{-\left(r_1 + 2r_2\right)/2}, \]

\[ \lambda_K = \frac{2^{r_1}h_K \text{Reg}_K}{w_K} \text{ where } w_K \text{ is the number of roots of unity in } K, \]

\[ F_K(s) = A_K^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s). \]

Hence, \( F_K \) a simple pole at \( s = 1 \) with residue \( \lambda_K \), and \( F_K(s) = F_K(1 - s) \).

From now on, we let \( N \) be a CM-field of degree \( 2n \), i.e. \( N \) is a totally imaginary number field of degree \( 2n \) which is a quadratic extension of a totally real number field \( N^+ \) of degree \( n \). Define the \( \phi_k \)'s by :

\[ \Phi_{N/N^+}(s) = \frac{\zeta_N}{\zeta_{N^+}}(s) = \sum_{k \geq 1} \phi_k k^{-s} \quad (\Re(s) > 1). \]

Then, \( (\zeta_N/\zeta_{N^+})(s) = L\left(s, \chi_{N/N^+}\right) \) yields  
\[ \phi_k = \sum_{N_{N^+/I}=k} \chi_{N/N^+}(I) \]

where \( I \) ranges over the integral ideals of \( N^+ \) of norm \( k \). Now,  
\[ \Phi_{N/N^+} = \zeta_N/\zeta_{N^+} \quad \text{and} \quad \Psi_{N/N^+} = F_N/F_{N^+} \]
are entire and \( \Psi_{N/N^+}(s) = \Psi_{N/N^+}(1 - s) \). Notice that

\[
\Psi_{N/N^+}(1) = \frac{\lambda_N}{\lambda_{N^+}} = \frac{h_{N^-}}{Q_N w_N}
\]

where \( Q_N \in \{1, 2\} \) is the Hasse unit index of \( N \) (see [Wa, Th. 4.16]). Since

\[
\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s + 1}{2}\right),
\]

using (13) for \( N \) and \( N^+ \) leads to

\[
\Psi_{N/N^+}(s) = c_{N/N^+} A_{N/N^+}^n \left(\frac{s + 1}{2}\right) \Phi_{N/N^+}(s)
\]

where

\[
c_{N/N^+} = 1/(4\pi)^{n/2} \quad \text{and} \quad A_{N/N^+} = \sqrt{d_N / \pi^d d_{N^+}}.
\]

Note that

\[
c_{N/N^+} A_{N/N^+} = \frac{1}{(2\pi)^n} \sqrt{d_N / d_{N^+}}.
\]

Set

\[
\hat{\Psi}_{N/N^+}(x) = \frac{1}{2i\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \Psi_{N/N^+}(s)x^{-s} ds \quad (\alpha > 1),
\]

i.e., \( \hat{\Psi}_{N/N^+} \) is the Mellin transform of the function \( \Psi_{N/N^+} \). Using (18) and (16) yields

\[
\hat{\Psi}_{N/N^+}(x) = \frac{1}{2i\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \Phi_{N/N^+}(kx/A_{N/N^+}) \quad (x > 0),
\]

with

\[
H_n(x) = \frac{1}{2i\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \Gamma_n\left(\frac{s + 1}{2}\right) x^{-s} ds
\]

\[
= \frac{1}{i\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \Gamma_n(S)x^{1-2S} dS \quad (x > 0 \text{ and } \alpha > 0).
\]

Now, we move the integral (18) to the line \( \Re(s) = 1 - \alpha \). Since \( \Psi_{N/N^+} \) is entire, we do not pick up any residue. Then, we use the functional equation

\[
\Psi_{N/N^+}(s) = \Psi_{N/N^+}(1 - s)
\]

satisfied by \( \Psi_{N/N^+} \) to come back to the line \( \Re(s) = \alpha \). We get

\[
x \Psi_{N/N^+}(x) = \hat{\Psi}_{N/N^+}(1/x) \quad (x > 0).
\]

Mellin’s inversion formula and (21) yield

\[
\Psi_{N/N^+}(s) = \int_1^\infty \hat{\Psi}_{N/N^+}(x)x^{-s} \frac{dx}{x} = \int_1^\infty \hat{\Psi}_{N/N^+}(x)\left\{x^{s-1} + x^{-s}\right\} dx.
\]
Moreover, yield the second derivative of $g$, and we have

$$
\Psi_{N/N^+}(s) = c_{N/N^+} \sum_{k \geq 1} \phi_k \int_1^\infty \left( \frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \left( \frac{kx}{A_{N/N^+}} \right)^{1-2S} \Gamma_n(S) \left\{ x^{s-1} + x^{-s} \right\} dS \right) dx
$$

and the following yields (4):

$$
h_N^{-} = Q_{NwN} \Psi_{N/N^+}(1) = Q_{NwN} \Psi_{N/N^+}(0)
$$

$$
= Q_{NwNC_{N/N^+}} \sum_{k \geq 1} \phi_k \frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma_n(S) (k/A_{N/N^+})^{1-2S} \left( \frac{1}{2S-2} + \frac{1}{2S-1} \right) dS
$$

$$
= Q_{NwNC_{N/N^+}A_{N/N^+}} \sum_{k \geq 1} \phi_k K_n(k/A_{N/N^+})
$$

$$
= \frac{Q_{NwN}}{(2\pi)^n} \sqrt{\frac{d_N}{d_{N^+}}} \sum_{k \geq 1} \phi_k K_n(k/A_{N/N^+}).
$$

Now, we prove the assertion below (5). To start with we quote some elementary facts we will need.

1) We have

$$
P_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} x^k \leq \sum_{k=0}^{n-1} \frac{1}{k!} x^{n-1} \leq e x^{n-1} \quad (x \geq 1).
$$

2) The derivative of

$$
g(x) = x^{2n-2} e^{-nx^{2/n}}
$$

is

$$
g'(x) = \frac{1}{n} \left( (2n-2) - 2nx^{2/n} \right) x^{n-2} e^{-nx^{2/n}}
$$

and we have $g'(x) \leq 0$ if $x \geq 1$ and

$$
|g'(x)| \leq 2xe^{-nx^{2/n}} \quad (x \geq 1).
$$

Moreover,

$$
g''(x) = \frac{1}{n^2} \left( 4n^2 x^4/n - (6n^2 - 4n)x^{2/n} + (2n^2 - 6n + 4) \right) x^{-2/n} e^{-nx^{2/n}},
$$

the second derivative of $g$, satisfies $g''(x) \geq 0$ if $x \geq 2^{n/2}$. Note that (3) and (24) yield

$$
K_n(x) \leq 2e n^{-1} g(x) \quad (x \geq 1).
$$
3) If $g(x) \geq 0$, $g'(x) \leq 0$ and $g''(x) \geq 0$ on $[\alpha, +\infty]$, then $\alpha \leq a \leq b$ implies

(27) \[ 0 \leq g(a) - g(b) \leq (a - b)g'(a). \]

4) If $A((n+1)/2)^{n/2} \geq 1$, then the derivative of

(28) \[ h(x) = x \log^n (ex)e^{-n(x/A)^{2/n}} \]

is

\[ h'(x) = (n - (2(x/A)^{2/n} - 1) \log(ex)) \log^{n-1}(ex)e^{-n(x/A)^{2/n}} \]

and we have $h'(x) \leq 0$ provided that $x \geq A((n+1)/2)^{n/2}$ and $x \geq 1$, hence provided that $x \geq A((n+1)/2)^{n/2}$ if $A((n + 1)/2)^{n/2} \geq 1$.

Now, we set $A = A_{N/N^+}$, $S_n(k) = \sum_{i=1}^{k} \frac{d_n(i)}{i}$ where $d_n(i)$ is the number of ways of writing $i$ as an ordered product of $n$ positive integers, and

\[ R_M = \sum_{k > M} \frac{\phi_k}{k} K_n(k/A). \]

We want an upper bound on $R_M$. We note that (14) yields $|\phi_k| \leq d_n(k)$. Moreover,

\[ S_n(k) = \sum_{i=1}^{k} \frac{d_n(i)}{i} \leq \left( \sum_{i=1}^{k} \frac{1}{i} \right)^n \leq \log^n (ek). \]

Thus, we have

\[ |R_M| \leq \sum_{k > M} \frac{d_n(k)}{k} K_n(k/A) \leq 2en^{n-1} \sum_{k > M} (S_n(k) - S_n(k - 1))g(k/A) \quad \text{(if $M \geq A$)} \]

(by using (26))

\[ \leq 2en^{n-1} \sum_{k > M} S_n(k)(g(k/A) - g((k + 1)/A)) \]

\[ \leq \frac{2en^{n-1}}{A} \sum_{k > M} S_n(k)g'(k/A) \quad \text{(if $M \geq 2^{n/2}A$)} \]

(by using (27))

\[ \leq \frac{4en^{n-1}}{A^2} \sum_{k > M} k \log^n (ek)e^{-n(k/A)^{2/n}} \]

(by using (25))

\[ = \frac{4en^{n-1}}{A^2} \sum_{k > M} h(k) \leq \frac{4en^{n-1}}{A^2} \int_{M}^{\infty} h(x)dx \quad \text{(if $M \geq \left( \frac{n + 1}{2} \right)^{n/2} A \geq 1$)} \]

(by using (28)).
Now, we set $B = (eA)^{2/n}$ and we change the variable by setting $x = Ay^{n/2}$. We get

$$|R_M| \leq 2e(n^2/2)^n \int_{(M/A)^{2/n}}^{\infty} y^n \log^n(By)e^{-ny} \frac{dy}{y}.$$  

Since $H(y) = y^{n+1} \log^n(By)e^{-ny}$ decreases on $[(M/A)^{2/n}, +\infty]$ if $M \geq (2n+2)^n A \geq e^{(n/2)-1}$ (since its derivative $H'(y) = ((n+1-ny) \log(By) + n)y^n \log^{n-1}(By)e^{-ny}$ satisfies $H'(y) \leq 0$ if $y \geq (2n+2)/n$ and $B^{2n+2} \geq e$), we get

$$|R_M| \leq 2e(n^2/2)^n \int_{(M/A)^{2/n}}^{\infty} H(y) \frac{dy}{y^2} \leq 2e(n^2/2)^n H((M/A)^{2/n}) \int_{(M/A)^{2/n}}^{\infty} \frac{dy}{y^2} = 2e(n^2/2)^n H((M/A)^{2/n})/(M/A)^{2/n},$$

i.e., if $M \geq (2n+2)^n A \geq e^{(n/2)-1}$, then we have the following explicit upper bound:

$$|R_M| \leq 2e \left( \frac{n^2}{2} G((M/A)^{2/n}) \right)^n$$

where $G(y) = y \log(By)e^{-y}$ and $B = (eA)^{2/n}$.

Now, we choose $M \approx B(N) = A \left( \frac{\lambda}{n} \log A \right)^{n/2}$ and note that $G \left( \frac{\lambda}{n} \log A \right) = O_n \left( A^{-\lambda/n} \log^2 A \right)$ yields the desired result:

$$|h_{-N}^\infty - h_{-N}^{-\infty}(M)| = \frac{Q_{N^\infty}}{2n \pi^{n/2}} A |R_M| = O_n \left( \frac{\log A}{A^{\lambda-1}} \right).$$

5. PROOF OF PROPOSITION 3

Let $M \geq 0$ be a given integer. Shifting the integral (2) to the left to the line $\Re(s) = -M - \frac{1}{2}$, we pick a residue at $s = 1$, a residue at $s = 1/2$, and a residue $\text{Res}_{s=-m}(f_n)$ at each nonpositive integer $-m \leq 0$. Hence, by using $\Gamma(1/2) = \sqrt{\pi}$ we get

$$K_n(A) = 1 + \pi^{n/2} A + 2A^2 \sum_{m=0}^{M} \text{Res}_{s=-m}(f_n) + \frac{A^2}{i\pi} \int_{-M - \frac{1}{2} - i\infty}^{-M - \frac{1}{2} + i\infty} f_n(s)ds.$$

Now, it is well known that for any nonnegative integer $l \geq 0$ we have

$$\left| \Gamma \left( \frac{2l+1}{2} + it \right) \right|^2 = \frac{\pi}{\text{ch}(\pi t)} \prod_{k=0}^{l-1} \left| \frac{2k+1}{2} + it \right|^2,$$

where $\text{ch}(x) = (e^x + e^{-x})/2$. Hence, using the functional equation $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ leads to

$$\left| \Gamma \left( -\frac{2l+1}{2} + it \right) \right|^2 = \frac{\pi}{\text{ch}(\pi t)} \prod_{k=0}^{l} \left| \frac{2k+1}{2} + it \right|^{-2},$$

and
If we set
\[ f_n(-M - \frac{1}{2} + it) \]
\[ \leq \left( \frac{\pi}{\text{ch}(\pi t)} \right)^{n/2} \frac{A^{2M+1}}{\prod_{k=0}^{M} |\frac{2k+1}{2} + it|^{n/2}} \left( \frac{1}{|2M + 2 + it|} + \frac{1}{|2M + 3 + it|} \right) \]
\[ \leq \frac{1}{(\text{ch}(\pi t))^{n/2}} \frac{\pi^{n/2} A^{2M+1}}{2 (M! / n)^{n/2}} \frac{2}{M + 2}. \]

Set
\[ c_n = \int_{-\infty}^{+\infty} \frac{dt}{(\text{ch}(\pi t))^{n/2}} = \frac{2}{\pi} \int_{0}^{1} \frac{\frac{2}{u + u^{-1}}}{u} \frac{du}{u} \]
(note that the sequence \((c_n)_{n \geq 0}\) decreases, that \(c_1 = \frac{4\sqrt{2}}{\pi} \int_{0}^{1} \frac{dv}{\sqrt{v^2 + 1}} \leq 4\sqrt{2}/\pi \) and \(c_2 = 1\). Then,
\[ \left| \frac{A^2}{i\pi} \int_{-M - \frac{1}{2} + i\infty}^{-M + \frac{1}{2} + i\infty} f_n(s) ds \right| \leq \frac{c_n}{\pi} \frac{\pi^{n/2} A^{2M+1}}{2 (M+1)(M! / n)^{n/2}}. \]

Note that the greater the value of \(n\), the faster the series (6) converges.

6. Proof of Proposition 4

We have
\[ \text{Res}_{s=-m}(f_n) = -A^{2m} \text{Res}_{s=0} \left( s \mapsto \Gamma^n(-m + s) A^{-2s} \left( \frac{1}{2m + 1 - 2s} + \frac{1}{2m + 2 - 2s} \right) \right). \]

If we set
\[ \Gamma^n(-m + s) A^{-2s} = \sum_{i=-n}^{-1} a_i(m) s^i + O(1), \]
then we get
\[ \text{Res}_{s=-m}(f_n) = -A^{2m} \sum_{i=-n}^{-1} a_i(m) 2^{-1-i} ((2m + 1)^i + (2m + 2)^i). \]

Now, \(\Gamma(s) = \frac{1}{s} \Gamma(s + 1)\) yields
\[ \sum_{i=-n}^{-1} a_i(m + 1) s^i + O(1) = (-1)^n (m + 1 - s)^{-n} \left( \sum_{j=-n}^{-1} a_j(m) s^j + O(1) \right) \]
and
\[ (m + 1 - s)^{-n} = \frac{1}{(m + 1)^n} \sum_{k=0}^{n-1} C_{k+n-1}^{n-1} \frac{s^k}{(m + 1)^k} + O(s^n) \]
yields
\[ a_i(m + 1) = \frac{(-1)^n}{(m + 1)^n} \sum_{j=-n}^{i} \frac{a_j(m)}{(m + 1)^{j-i}} C_{i-j+n-1}^{n-1}. \]

Thus, in order to simplify the recursion relation (36), we define
\[ h_i(m) = (-1)^nm!a_i(m). \]
Then, using (35) yields (8), and using (34) and (36) yields (9). Note that (10) makes it easy to compute the numerical values of the $h_i$’s by using Maple, for example.

7. Examples of relative class numbers computations

In order to use (4) to compute relative class numbers, it remains to explain how we compute the $\phi_k$’s. Since

$$\phi_k = \sum_{N^{N+}_+/Q(I) = k} \chi_{N^{N^+}/Q}(I)$$

(see (14)), then $k \mapsto \phi_k$ is multiplicative and we only have to explain how we compute the $\phi_p^m$ where $p$ is prime and $m \geq 1$. We will only explain this when $N$ is normal over $\mathbb{Q}$. In that case, let $e$ and $f$ be the inertia and residual degrees of $p$ in $N^+$. Set $g = n/(ef)$. Then in $\mathbb{N}^+$ we have $(p) = (P_1 \cdots P_g)^e$ and

$$\chi_{N^{N^+}/Q}(P_1) = \cdots = \chi_{N^{N^+}/Q}(P_g),$$

and we let $\epsilon_p$ be the common value of these $g$ symbols. Now, $N_{N^+}/\mathbb{Q}(I) = p^m$ if and only if $I = \prod_{i=1}^g P_i^{e_i}$ with $\sum_{i=1}^g e_i = m$. Set

$$C_j = \frac{i!}{j!(i-j)!}.$$ 

Since the equation $\sum_{i=1}^g e_i = K$ has $C_{g-1}^{g-1}$ solutions in nonnegative integers $e_i$, we easily get

$$\phi_{p^m} = \begin{cases} 
0 & \text{if } f \text{ does not divide } m, \\
\epsilon_p^k C_{K+g-1}^{g-1} & \text{if } f \text{ divides } m \text{ and } m = kf.
\end{cases}$$

This formula (37) makes it easy to compute the $\phi_{p^m}$. We refer the reader to [Lou 1], [Lou 2], [Lou 3], [Lou-Oka] and [LOO] for actual computations of relative class numbers.

REFERENCES


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