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COMPUTATION OF RELATIVE CLASS NUMBERS OF CM-FIELDS

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Abstract. It was well known that it is easy to compute relative class numbers of abelian CM-fields by using generalized Bernoulli numbers (see Theorem 4.17 in *Introduction to cyclotomic fields* by L. C. Washington, Grad. Texts in Math., vol. 83, Springer-Verlag, 1982). Here, we provide a technique for computing the relative class number of any CM-field.

1. Statement of the results

Proposition 1. *Let $n \geq 1$ be an integer and $\alpha > 1$ be real. Set $P_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!}x^k$,*

$$f_n(s) = \Gamma^n(s) A^{-2s} \left(\frac{1}{2s-1} + \frac{1}{2s-2} \right) \tag{1}$$

and

$$K_n(A) = \frac{A^2}{i\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} f_n(s) ds. \tag{2}$$

Then, it holds

$$0 \leq K_n(A) \leq 2P_n(nA^{2/n})e^{-nA^{2/n}} \leq 2n \exp(-A^{2/n}). \tag{3}$$

Theorem 2. *Let $\mathbf{N}$ be a totally imaginary number field of degree $2n$ which is a quadratic extension of a totally real number field $\mathbf{N}^+$ of degree n, i.e. $\mathbf{N}$ is a* CM-*field. Let $w_{\mathbf{N}}$ be the number of roots of unity in $\mathbf{N}$, $Q_{\mathbf{N}} \in \{1, 2\}$ be the Hasse unit index of $\mathbf{N}$, and $d_{\mathbf{N}}$, $\zeta_{\mathbf{N}}$ and $d_{\mathbf{N}^+}$, $\zeta_{\mathbf{N}^+}$ be the absolute values of the discriminants and the Dedekind zeta functions of $\mathbf{N}$ and $\mathbf{N}^+$, respectively. Let $\chi_{\mathbf{N}/\mathbf{N}^+}$ be the quadratic character assocciated with the quadratic extension $\mathbf{N}/\mathbf{N}^+$ and let ϕ_k be the coefficients of the Dirichlet series $(\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = L(s, \chi_{\mathbf{N}/\mathbf{N}^+}) = \sum_{k \geq 1} \phi_k k^{-s}$, $\Re(s) > 1$. Set $A_{\mathbf{N}/\mathbf{N}^+} = \sqrt{d_{\mathbf{N}}/\pi^n d_{\mathbf{N}^+}}$.*

We have

$$h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \sum_{k \geq 1} \frac{\phi_k}{k} K_n(k/A_{\mathbf{N}/\mathbf{N}^+}), \tag{4}$$

and according to (3) this series (4) is absolutely convergent. Moreover, set

$$B(\mathbf{N}) \stackrel{def}{=} A_{\mathbf{N}/\mathbf{N}^+} \left(\frac{\lambda}{n} \log A_{\mathbf{N}/\mathbf{N}^+} \right)^{n/2}. \tag{5}$$

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Then, if $\lambda > 1$ *and* n *are given, then the limit of* $|h_{\mathbf{N}}^- - h_{\mathbf{N}}^-(M)|$ *as* $A_{\mathbf{N}/\mathbf{N}^+}$ *approaches infinity is equal to* 0*, where* $h_{\mathbf{N}}^-(M)$ *is the approximation of the relative class number obtained by disregarding in the series occurring in (4) the indices* $k > M \geq B(\mathbf{N})$.

For example, if $\mathbf{N}$ of degree $m = 2n$ is the narrow Hilbert class field of a real quadratic number field $\mathbf{L}$ of discriminant $d_{\mathbf{L}}$, we have

$$B(\mathbf{N}) = \left(\frac{\lambda}{4\pi}\right)^{m/4} d_{\mathbf{L}}^{m/8} \log^{m/4}(d_{\mathbf{L}}/\pi^2).$$

The following Proposition 3 explains how we compute the numerical values of the function $A \mapsto K_n(A)$ according to its series expansion:

Proposition 3. *Take* $A > 0$*. It holds*

$$K_n(A) = 1 + \pi^{n/2} A + 2A^2 \sum_{m \geq 0} \mathrm{Res}_{s=-m}(f_n). \tag{6}$$

This series is absolutely convergent and for any integer $M \geq 0$ *we have*

$$\left| 2A^2 \sum_{m > M} \mathrm{Res}_{s=-m}(f_n) \right| \leq \frac{\pi^{n/2} A^{2M+3}}{(M+1)(M!/2)^n}. \tag{7}$$

Finally, the following Proposition 4 explains how to compute recursively the values of the residues $\mathrm{Res}_{s=-m}(f_n)$ occurring in (6):

Proposition 4. *We have* [1]

$$\mathrm{Res}_{s=-m}(f_n) = -(-1)^{nm} \frac{A^{2m}}{(m!)^n} \sum_{i=-n}^{-1} 2^{-1-i} h_i(m)((2m+1)^i + (2m+2)^i) \tag{8}$$

where the $h_i(m)$*'s are computed recursively from the* $h_i(0)$*'s by using*

(9)

$$h_i(m+1) = \sum_{j=-n}^{i} h_j(m) \frac{b_{i-j}}{(m+1)^{i-j}} \quad \text{and} \quad \sum_{j=-n}^{-1} h_j(0) s^j + O(1) = \Gamma^n(s) A^{-2s},$$

where $b_k = C_{n+k-1}^{n-1} = ((k+n-1)!/k!(n-1)!)$*. Thus, if*

$$\Gamma^n(s+1) = \sum_{i=0}^{n-1} h_i s^i \ + O(s^n), \tag{10}$$

then

$$h_{j-n}(0) = \sum_{i=0}^{j} \frac{(-2\log A)^i}{i!} h_{j-i} \qquad (0 \leq j \leq n-1). \tag{11}$$

For proving these results, obvious questions of convergence of series and integrals, and questions of inversions of integrals and summations will not be gone into.

[1] Note the misprint in the formula given in [Lou 2].

2. Introduction

Prior to the method we have developed here, the only general method for computing the relative class number of any CM-field was that developed by T. Shintani (see [Oka 1] and [Oka 2] for examples of actual relative class number computations using Shintani's ideas). However, his method requires the knowledge of a great deal of information on the maximal totally real subfield $\mathbf{N}^+$. In particular, it requires the knowledge of a system of fundamental units of the group of totally positive units of $\mathbf{N}^+$. However, what makes the concept of CM-field an attractive one is that the relative class number formula

$$h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n}\sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}}\frac{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})}{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+})} = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n}\left(\sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}}\right) L(1,\chi_{\mathbf{N}/\mathbf{N}^+}) \tag{12}$$

enables us to get lower bounds on relative class numbers and solve class number and class group problems for CM-fields precisely because (12) does not involve any regulator (see [Lou-Oka] and [LOO]). Thus, the reader may possibly feel dissatisfied that he should have to know beforehand a good grasp of the unit group of $\mathbf{N}^+$ before he can compute $h_{\mathbf{N}}^-$, whereas (12) gives an expression for $h_{\mathbf{N}}^-$ which does not involve units. The reader may now possibly feel satisfied that this paper shows how using (12) he indeed gets an efficient method for computing $h_{\mathbf{N}}^-$ provided that he only knows how to compute the decomposition of any rational prime into a product of prime ideals of $\mathbf{N}$. The key point of our method is to establish the holomorphic continuation of $s \mapsto (\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = L(s,\chi_{\mathbf{N}/\mathbf{N}^+})$ in the same way Riemann did in the case of the Riemann zeta function (by using Mellin transformation) and to evaluate the resulting series at $s=1$ (see section 4).

Finally, we note that the results of this paper are better than those of [Lou 3]. Indeed, $B(\mathbf{N})$ in (5) is $n^{n/2}$-fold better than the one we gave in [Lou 3]. Moreover, our proof of (3) (in section 3) is more satisfactory and elegant than the one we gave in [Lou 3].

3. Proof of Proposition 1

We use:

Lemma 5. *Let $\alpha > 1$ be real. We have*

$$\int_{\alpha-i\infty}^{\alpha+i\infty} u^s \frac{ds}{2s-1} = \begin{cases} 0 & \text{if } u<1, \\ i\pi\sqrt{u} & \text{if } u>1; \end{cases} \quad \text{and} \quad \int_{\alpha-i\infty}^{\alpha+i\infty} u^s \frac{ds}{2s-2} = \begin{cases} 0 & \text{if } u<1, \\ i\pi u & \text{if } u>1. \end{cases}$$

Now, using

$$\Gamma^n(s) = \int\int e^{-\mathrm{Tr}(y)} y^s \frac{dy}{y}$$

where the multiple integral ranges over $(y_1,\cdots,y_n) \in (\mathbf{R}_+^*)^n$ and where we set $y = y_1 y_2 \cdots y_n$ and $\mathrm{Tr}(y) = y_1+y_2+\cdots+y_n$, leads to

$$K_n(A) = \frac{A^2}{i\pi}\int\int e^{-\mathrm{Tr}(y)}\left(\int_{\alpha-i\infty}^{\alpha+i\infty}(y/A^2)\left(\frac{1}{2s-1}+\frac{1}{2s-2}\right)\right)\frac{dy}{y}.$$

Using Lemma 5 yields

$$K_n(A) = A^2 \int\int_{y \geq A^2} \big(\sqrt{y/A^2} + (y/A^2)\big) e^{-\mathrm{Tr}(y)} \frac{dy}{y} \leq 2 \int\int_{y \geq A^2} e^{-\mathrm{Tr}(y)} dy.$$

For example, we get $K_1(A) \leq 2e^{-A^2}$. Now, using the arithmetic-geometric mean inequality yields that $\{(y_1, \cdots, y_n);\ y \geq A^2\}$ is included in $\{(y_1, \cdots, y_n);\ \mathrm{Tr}(y) \geq nA^{2/n}\}$, which yields

$$K_n(A) \leq 2 \int\int_{\mathrm{Tr}(y) \geq nA^{2/n}} e^{-\mathrm{Tr}(y)} dy.$$

Then, the following easily proved Lemma 6 provides us with the desired result. We finally notice that we get a shorter and more satisfactory proof of [Lou 3, Proposition 1]:

Lemma 6. *Set* $P_n(x) = \sum_{k=0}^{n-1} x^k/k!$. *Then*

$$P_n(\alpha)e^{-\alpha} = \int\int_{\substack{(y_1,\cdots,y_n)\in\mathbf{R}_+^* \\ \mathrm{Tr}(y)\geq\alpha}} e^{-\mathrm{Tr}(y)} dy \leq n \int_{\alpha/n}^{+\infty} e^{-y} dy = ne^{-\alpha/n}.$$

Proof. Use

$$\begin{aligned} &\{(y_1, \cdots, y_n) \in \mathbf{R}_+^*,\ \mathrm{Tr}(y) \geq \alpha\} \\ &\quad \subseteq \bigcup_{i=1}^{n} \left\{(y_1, \cdots, y_n),\ y_i \geq \frac{\alpha}{n} \text{ and } y_j \geq 0 \text{ for } j \neq i\right\}. \quad \square \end{aligned}$$

4. Proof of Theorem 2

Let $\mathbf{K}$ be a number field of degree $n = r_1 + 2r_2$, where r_1 is the number of real places of $\mathbf{K}$ and r_2 the number of complex places of $\mathbf{K}$. Let $\zeta_{\mathbf{K}}$ and $\mathrm{Reg}_{\mathbf{K}}$ be the Dedekind zeta function and regulator of $\mathbf{K}$. We set

$$\begin{aligned} &A_{\mathbf{K}} = 2^{-r_2} d_{\mathbf{K}}^{1/2} \pi^{-\big(r_1+2r_2\big)/2}, \\ &\lambda_{\mathbf{K}} = \frac{2^{r_1} h_{\mathbf{K}} \mathrm{Reg}_{\mathbf{K}}}{w_{\mathbf{K}}} \quad \text{where } w_{\mathbf{K}} \text{ is the number of roots of unity in } \mathbf{K}, \\ &F_{\mathbf{K}}(s) = A_{\mathbf{K}}^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_{\mathbf{K}}(s). \end{aligned} \tag{13}$$

Hence, $F_{\mathbf{K}}$ a a simple pole at $s = 1$ with residue $\lambda_{\mathbf{K}}$, and $F_{\mathbf{K}}(s) = F_{\mathbf{K}}(1-s)$.

From now on, we let $\mathbf{N}$ be a CM-field of degree $2n$, i.e. $\mathbf{N}$ is a totally imaginary number field of degree $2n$ which is a quadratic extension of a totally real number field $\mathbf{N}^+$ of degree n. Define the ϕ_k's by :

$$\Phi_{\mathbf{N}/\mathbf{N}^+}(s) = \frac{\zeta_{\mathbf{N}}}{\zeta_{\mathbf{N}^+}}(s) = \sum_{k\geq 1} \phi_k k^{-s} \qquad (\Re(s) > 1).$$

Then, $(\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = L\big(s, \chi_{\mathbf{N}/\mathbf{N}^+}\big)$ yields

$$\phi_k = \sum_{N_{\mathbf{N}^+/\mathbf{Q}}(\mathbf{I})=k} \chi_{\mathbf{N}/\mathbf{N}^+}(\mathbf{I}) \tag{14}$$

where $\mathbf{I}$ ranges over the integral ideals of $\mathbf{N}^+$ of norm k. Now,

$$\Phi_{\mathbf{N}/\mathbf{N}^+} = \zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+} \quad \text{and} \quad \Psi_{\mathbf{N}/\mathbf{N}^+} = F_{\mathbf{N}}/F_{\mathbf{N}^+}$$

are entire and $\Psi_{\mathbf{N}/\mathbf{N}^+}(s) = \Psi_{\mathbf{N}/\mathbf{N}^+}(1-s)$. Notice that

$$\Psi_{\mathbf{N}/\mathbf{N}^+}(1) = \frac{\lambda_{\mathbf{N}}}{\lambda_{\mathbf{N}^+}} = \frac{h_{\mathbf{N}}^-}{Q_{\mathbf{N}} w_{\mathbf{N}}} \tag{15}$$

where $Q_{\mathbf{N}} \in \{1, 2\}$ is the Hasse unit index of $\mathbf{N}$ (see [Wa, Th. 4.16]). Since

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right),$$

using (13) for $\mathbf{N}$ and $\mathbf{N}^+$ leads to

$$\Psi_{\mathbf{N}/\mathbf{N}^+}(s) = c_{\mathbf{N}/\mathbf{N}^+} A_{\mathbf{N}/\mathbf{N}^+}^s \Gamma^n\left(\frac{s+1}{2}\right) \Phi_{\mathbf{N}/\mathbf{N}^+}(s) \tag{16}$$

where

$$c_{\mathbf{N}/\mathbf{N}^+} = 1/(4\pi)^{n/2} \quad \text{and} \quad A_{\mathbf{N}/\mathbf{N}^+} = \sqrt{d_{\mathbf{N}}/\pi^n d_{\mathbf{N}^+}}.$$

Note that

$$c_{\mathbf{N}/\mathbf{N}^+} A_{\mathbf{N}/\mathbf{N}^+} = \frac{1}{(2\pi)^n} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}}. \tag{17}$$

Set

$$\hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) = \frac{1}{2i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Psi_{\mathbf{N}/\mathbf{N}^+}(s) x^{-s} ds \qquad (\alpha > 1), \tag{18}$$

i.e., $\hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}$ is the Mellin transform of the function $\Psi_{\mathbf{N}/\mathbf{N}^+}$. Using (18) and (16) yields

$$\hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) = c_{\mathbf{N}/\mathbf{N}^+} \sum_{k \geq 1} \phi_k H_n\left(kx/A_{\mathbf{N}/\mathbf{N}^+}\right) \qquad (x > 0), \tag{19}$$

with

$$\begin{aligned} H_n(x) &= \frac{1}{2i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma^n\left(\frac{s+1}{2}\right) x^{-s} ds \\ &= \frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma^n(S) x^{1-2S}\, dS \qquad (x > 0 \text{ and } \alpha > 0). \end{aligned} \tag{20}$$

Now, we move the integral (18) to the line $\Re(s) = 1 - \alpha$. Since $\Psi_{\mathbf{N}/\mathbf{N}^+}$ is entire, we do not pick up any residue. Then, we use the functional equation

$$\Psi_{\mathbf{N}/\mathbf{N}^+}(s) = \Psi_{\mathbf{N}/\mathbf{N}^+}(1-s)$$

satisfied by $\Psi_{\mathbf{N}/\mathbf{N}^+}$ to come back to the line $\Re(s) = \alpha$. We get

$$x\hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) = \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(1/x) \qquad (x > 0). \tag{21}$$

Mellin's inversion formula and (21) yield

$$\Psi_{\mathbf{N}/\mathbf{N}^+}(s) = \int_0^\infty \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) x^s \frac{dx}{x} = \int_1^\infty \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) \left\{x^{s-1} + x^{-s}\right\} dx. \tag{22}$$

By using (22), (19) and (20) we thus get

$$\begin{aligned}
&\Psi_{\mathbf{N}/\mathbf{N}^+}(s)\\
&= c_{\mathbf{N}/\mathbf{N}^+}\sum_{k\geq 1}\phi_k\int_1^\infty\left(\frac{1}{i\pi}\int_{\alpha-i\infty}^{\alpha+i\infty}\left(\frac{kx}{A_{\mathbf{N}/\mathbf{N}^+}}\right)^{1-2S}\Gamma^n(S)\left\{x^{s-1}+x^{-s}\right\}dS\right)dx\\
&= c_{\mathbf{N}/\mathbf{N}^+}\sum_{k\geq 1}\phi_k\frac{1}{i\pi}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma^n(S)\left(\int_1^\infty\left(\frac{kx}{A_{\mathbf{N}/\mathbf{N}^+}}\right)^{1-2S}\left\{x^{s-1}+x^{-s}\right\}dx\right)dS\\
&= c_{\mathbf{N}/\mathbf{N}^+}\sum_{k\geq 1}\phi_k\frac{1}{i\pi}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma^n(S)\left(k/A_{\mathbf{N}/\mathbf{N}^+}\right)^{1-2S}\left(\frac{1}{2S-s-1}+\frac{1}{2S+s-2}\right)dS,
\end{aligned}$$

and the following yields (4):

$$\begin{aligned}
h_{\mathbf{N}}^- &= Q_{\mathbf{N}}w_{\mathbf{N}}\Psi_{\mathbf{N}/\mathbf{N}^+}(1) = Q_{\mathbf{N}}w_{\mathbf{N}}\Psi_{\mathbf{N}/\mathbf{N}^+}(0)\\
&= Q_{\mathbf{N}}w_{\mathbf{N}}c_{\mathbf{N}/\mathbf{N}^+}\sum_{k\geq 1}\phi_k\frac{1}{i\pi}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma^n(S)\big(k/A_{\mathbf{N}/\mathbf{N}^+}\big)^{1-2S}\left(\frac{1}{2S-2}+\frac{1}{2S-1}\right)dS\\
&= Q_{\mathbf{N}}w_{\mathbf{N}}c_{\mathbf{N}/\mathbf{N}^+}A_{\mathbf{N}/\mathbf{N}^+}\sum_{k\geq 1}\frac{\phi_k}{k}K_n\left(k/A_{\mathbf{N}/\mathbf{N}^+}\right)\\
&= \frac{Q_{\mathbf{N}}w_{\mathbf{N}}}{(2\pi)^n}\sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}}\sum_{k\geq 1}\frac{\phi_k}{k}K_n\big(k/A_{\mathbf{N}/\mathbf{N}^+}\big).
\end{aligned}$$

Now, we prove the assertion below (5).

To start with we quote some elementary facts we will need.

1) We have

$$P_n(x)=\sum_{k=0}^{n-1}\frac{1}{k!}x^k \;\leq \sum_{k=0}^{n-1}\frac{1}{k!}x^{n-1}\;\leq ex^{n-1}\qquad (x\geq 1). \tag{23}$$

2) The derivative of

$$g(x)=x^{\frac{2n-2}{n}}e^{-nx^{2/n}} \tag{24}$$

is

$$g'(x)=\frac{1}{n}\left((2n-2)-2nx^{2/n}\right)x^{\frac{n-2}{n}}e^{-nx^{2/n}}$$

and we have $g'(x)\leq 0$ if $x\geq 1$ and

$$|g'(x)|\leq 2xe^{-nx^{2/n}}\qquad (x\geq 1). \tag{25}$$

Moreover,

$$g''(x)=\frac{1}{n^2}\left(4n^2x^{4/n}-(6n^2-4n)x^{2/n}+(2n^2-6n+4)\right)x^{-2/n}e^{-nx^{2/n}},$$

the second derivative of g, satisfies $g''(x)\geq 0$ if $x\geq 2^{n/2}$. Note that (3) and (24) yield

$$K_n(x)\leq 2en^{n-1}g(x)\qquad (x\geq 1). \tag{26}$$

3) If $g(x) \geq 0$, $g'(x) \leq 0$ and $g''(x) \geq 0$ on $[\alpha, +\infty, [$, then $\alpha \leq a \leq b$ implies

$$0 \leq g(a) - g(b) \leq (a-b)g'(a). \tag{27}$$

4) If $A\big((n+1)/2\big)^{n/2} \geq 1$, then the derivative of

$$h(x) = x \log^n(ex) e^{-n(x/A)^{2/n}} \tag{28}$$

is

$$h'(x) = \big(n - (2(x/A)^{2/n} - 1)\log(ex)\big) \log^{n-1}(ex) e^{-n(x/A)^{2/n}}$$

and we have $h'(x) \leq 0$ provided that $x \geq A\big((n+1)/2\big)^{n/2}$ and $x \geq 1$, hence provided that $x \geq A\big((n+1)/2\big)^{n/2}$ if $A\big((n+1)/2\big)^{n/2} \geq 1$.

Now, we set $A = A_{\mathbf{N}/\mathbf{N}^+}$, $S_n(k) = \sum_{i=1}^k \frac{d_n(i)}{i}$ where $d_n(i)$ is the number of ways of writing i as an ordered product of n positive integers, and

$$R_M = \sum_{k>M} \frac{\phi_k}{k} K_n(k/A).$$

We want an upper bound on R_M. We note that (14) yields $|\phi_k| \leq d_n(k)$. Moreover,

$$S_n(k) = \sum_{i=1}^k \frac{d_n(i)}{i} \leq \left(\sum_{i=1}^k \frac{1}{i}\right)^n \leq \log^n(ek).$$

Thus, we have

$$\begin{aligned}
|R_M| &\leq \sum_{k>M} \frac{d_n(k)}{k} K_n(k/A) \\
&\leq 2en^{n-1} \sum_{k>M} \big(S_n(k) - S_n(k-1)\big) g(k/A) \quad (\text{if } M \geq A) \\
&\text{(by using (26))} \\
&\leq 2en^{n-1} \sum_{k>M} S_n(k) \big(g(k/A) - g((k+1)/A)\big) \\
&\leq \frac{2en^{n-1}}{A} \sum_{k>M} S_n(k) g'(k/A) \quad (\text{if } M \geq 2^{n/2}A) \\
&\text{(by using (27))} \\
&\leq \frac{4en^{n-1}}{A^2} \sum_{k>M} k \log^n(ek) e^{-n(k/A)^{2/n}} \\
&\text{(by using (25))} \\
&= \frac{4en^{n-1}}{A^2} \sum_{k>M} h(k) \leq \frac{4en^{n-1}}{A^2} \int_M^\infty h(x)dx \quad \left(\text{ if } M \geq \left(\frac{n+1}{2}\right)^{n/2} A \geq 1\right) \\
&\text{(by using (28))}.
\end{aligned}$$

Now, we set $B = (eA)^{2/n}$ and we change the variable by setting $x = Ay^{n/2}$. We get

$$|R_M| \leq 2e(n^2/2)^n \int_{(M/A)^{2/n}}^{\infty} y^n \log^n(By)e^{-ny}\frac{dy}{y}.$$

Since $H(y) = y^{n+1}\log^n(By)e^{-ny}$ decreases on $[(M/A)^{2/n}, +\infty[$ if $M \geq \left(\frac{2n+2}{n}\right)^{n/2} A \geq e^{(n/2)-1}$ (since its derivative

$$H'(y) = \big((n+1-ny)\log(By) + n\big)y^n \log^{n-1}(By)e^{-ny}$$

satisfies $H'(y) \leq 0$ if $y \geq (2n+2)/n$ and $B\frac{2n+2}{2} \geq e$), we get

$$\begin{aligned} |R_M| &\leq 2e(n^2/2)^n \int_{(M/A)^{2/n}}^{\infty} H(y)\frac{dy}{y^2} \\ &\leq 2e(n^2/2)^n H((M/A)^{2/n}) \int_{(M/A)^{2/n}}^{\infty} \frac{dy}{y^2} \\ &= 2e(n^2/2)^n H((M/A)^{2/n})/(M/A)^{2/n}, \end{aligned}$$

i.e., if $M \geq \left(\frac{2n+2}{n}\right)^{n/2} A \geq e^{(n/2)-1}$, then we have the following explicit upper bound :

(29)

$$|R_M| \leq 2e\left(\frac{n^2}{2}G((M/A)^{2/n})\right)^n \quad \text{where } G(y) = y\log(By)e^{-y} \text{ and } B = (eA)^{2/n}.$$

Now, we choose $M \approx B(\mathbf{N}) = A\left(\frac{\lambda}{n}\log A\right)^{n/2}$ and note that

$$G(\frac{\lambda}{n}\log A) = O_n(A^{-\lambda/n}\log^2 A)$$

yields the desired result :

$$|h_{\mathbf{N}}^- - h_{\mathbf{N}}^-(M)| = \frac{Q_{\mathbf{N}}w_{\mathbf{N}}}{2^n\pi^{n/2}}A|R_M| = O_n\left(\frac{\log^{2n} A}{A^{\lambda-1}}\right). \tag{30}$$

5. Proof of Proposition 3

Let $M \geq 0$ be a given integer. Shifting the integral (2) to the left to the line $\Re(s) = -M - \frac{1}{2}$, we pick a residue at $s = 1$, a residue at $s = 1/2$, and a residue $\mathrm{Res}_{s=-m}(f_n)$ at each nonpositive integer $-m \leq 0$. Hence, by using $\Gamma(1/2) = \sqrt{\pi}$ we get

$$K_n(A) = 1 + \pi^{n/2}A + 2A^2\sum_{m=0}^{M}\mathrm{Res}_{s=-m}(f_n) + \frac{A^2}{i\pi}\int_{-M-\frac{1}{2}-i\infty}^{-M-\frac{1}{2}+i\infty} f_n(s)ds. \tag{31}$$

Now, it is well known that for any nonnegative integer $l \geq 0$ we have

$$\left|\Gamma\left(\frac{2l+1}{2}+it\right)\right|^2 = \frac{\pi}{\mathrm{ch}(\pi t)}\prod_{k=0}^{l-1}\left|\frac{2k+1}{2}+it\right|^2,$$

where $\mathrm{ch}(x) = (e^x + e^{-x})/2$. Hence, using the functional equation $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ leads to

$$\left|\Gamma\left(-\frac{2l+1}{2}+it\right)\right|^2 = \frac{\pi}{\mathrm{ch}(\pi t)}\prod_{k=0}^{l}\left|\frac{2k+1}{2}+it\right|^{-2}, \tag{32}$$

and

$$\begin{aligned}&\left|f_n(-M-\frac{1}{2}+it)\right|\\&\quad\le\left(\frac{\pi}{\mathrm{ch}(\pi t)}\right)^{n/2}\frac{A^{2M+1}}{\left(\prod_{k=0}^{M}\left|\frac{2k+1}{2}+it\right|\right)^n}\left(\frac{1}{|2M+2+it|}+\frac{1}{|2M+3+it|}\right)\\&\quad\le\frac{1}{(\mathrm{ch}(\pi t))^{n/2}}\frac{\pi^{n/2}A^{2M+1}}{(M!/2)^n}\frac{2}{2M+2}.\end{aligned}$$

Set

$$c_n=\int_{-\infty}^{+\infty}\frac{dt}{(\mathrm{ch}(\pi t))^{n/2}}=\frac{2}{\pi}\int_0^1\left(\frac{2}{u+u^{-1}}\right)^{n/2}\frac{du}{u}$$

(note that the sequence $(c_n)_{n\ge0}$ decreases, that $c_1=\frac{4\sqrt{2}}{\pi}\int_0^1\frac{dv}{\sqrt{v^4+1}}\le4\sqrt{2}/\pi$ and $c_2=1$). Then,

$$\left|\frac{A^2}{i\pi}\int_{-M-\frac{1}{2}-i\infty}^{-M-\frac{1}{2}+i\infty}f_n(s)ds\right|\le\frac{c_n}{\pi}\frac{\pi^{n/2}A^{2M+3}}{(M+1)(M!/2)^n}.\tag{33}$$

Note that the greater the value of n, the faster the series (6) converges.

6. Proof of Proposition 4

We have

$$\begin{aligned}&\mathrm{Res}_{s=-m}(f_n)\\&=-A^{2m}\,\mathrm{Res}_{s=0}\left(s\mapsto\Gamma^n(-m+s)A^{-2s}\left(\frac{1}{2m+1-2s}+\frac{1}{2m+2-2s}\right)\right).\end{aligned}$$

If we set

$$\Gamma^n(-m+s)A^{-2s}=\sum_{i=-n}^{-1}a_i(m)s^i+O(1),\tag{34}$$

then we get

$$\mathrm{Res}_{s=-m}(f_n)=-A^{2m}\sum_{i=-n}^{-1}a_i(m)2^{-1-i}\left((2m+1)^i+(2m+2)^i\right).\tag{35}$$

Now, $\Gamma(s)=\frac{1}{s}\Gamma(s+1)$ yields

$$\sum_{i=-n}^{-1}a_i(m+1)s^i+O(1)=(-1)^n(m+1-s)^{-n}\left(\sum_{j=-n}^{-1}a_j(m)s^j+O(1)\right)$$

and

$$(m+1-s)^{-n}=\frac{1}{(m+1)^n}\sum_{k=0}^{n-1}C_{k+n-1}^{n-1}\frac{s^k}{(m+1)^k}+O(s^n)$$

yields

$$a_i(m+1)=\frac{(-1)^n}{(m+1)^n}\sum_{j=-n}^{i}\frac{a_j(m)}{(m+1)^{i-j}}C_{i-j+n-1}^{n-1}.\tag{36}$$

Thus, in order to simplify the recursion relation (36), we define

$$h_i(m)=(-1)^{nm}(m!)^na_i(m).$$

Then, using (35) yields (8), and using (34) and (36) yields (9). Note that (10) makes it easy to compute the numerical values of the h_i's by using Maple, for example.

7. Examples of relative class numbers computations

In order to use (4) to compute relative class numbers, it remains to explain how we compute the ϕ_k's. Since

$$\phi_k = \sum_{N_{\mathbf{N}^+/\mathbf{Q}}(\mathbf{I})=k} \chi_{\mathbf{N}/\mathbf{N}^+}(\mathbf{I})$$

(see (14)), then $k \mapsto \phi_k$ is multiplicative and we only have to explain how we compute the ϕ_{p^m} where p is prime and $m \geq 1$. We will only explain this when $\mathbf{N}$ is normal over $\mathbb{Q}$. In that case, let e and f be the inertia and residual degrees of p in $\mathbf{N}_+$. Set $g = n/(ef)$. Then in $\mathbf{N}^+$ we have $(p) = (\mathcal{P}_1 \cdots \mathcal{P}_g)^e$ and

$$\chi_{\mathbf{N}/\mathbf{N}^+}(\mathcal{P}_1) = \cdots = \chi_{\mathbf{N}/\mathbf{N}^+}(\mathcal{P}_g),$$

and we let ϵ_p be the common value of these g symbols. Now, $N_{\mathbf{N}^+/\mathbf{Q}}(\mathbf{I}) = p^m$ if and only if $\mathbf{I} = \prod_{i=1}^g \mathcal{P}_i^{e_i}$ with $f \sum_{i=1}^g e_i = m$. Set

$$C_i^j = \frac{i!}{j!(i-j)!}.$$

Since the equation $\sum_{i=1}^g e_i = K$ has C_{K+g-1}^{g-1} solutions in nonnegative integers e_i, we easily get

$$\phi_{p^m} = \begin{cases} 0 & \text{if } f \text{ does not divide } m, \\ \epsilon_p^k C_{k+g-1}^{g-1} & \text{if } f \text{ divides } m \text{ and } m = kf. \end{cases} \tag{37}$$

This formula (37) makes it easy to compute the ϕ_{p^m}. We refer the reader to [Lou 1], [Lou 2], [Lou 3], [Lou-Oka] and [LOO] for actual computations of relative class numbers.

References

[Lou 1] S. Louboutin, *Calcul des nombres de classes relatifs: application aux corps octiques quaternioniques à multiplication complexe,* C. R. Acad. Sci. Paris **317** (1993), 643-646. MR **94j:**11111

[Lou 2] S. Louboutin, *Calcul des nombres de classes relatifs de certains corps de classes de Hilbert,* C. R. Acad. Sci. Paris. **319** (1994), 321-325. MR **95g:**11111

[Lou 3] S. Louboutin, *Calcul du nombre de classes des corps de nombres,* Pacific J. Math. **171** (1995), 455–467. MR **97a:**11176

[Lou-Oka] S. Louboutin and R. Okazaki, *The class number one problem for some non-abelian normal* CM*-fields of* 2*-power degrees,* preprint Univ. Caen 1996, to be submitted.

[LOO] S. Louboutin, R. Okazaki and M. Olivier, *The class number one problem for some non-abelian normal* CM*-fields,* to appear in Trans. Amer. Math. Soc. CMP 96:12

[Oka 1] R. Okazaki, *On evaluation of L-functions over real quadratic fields,* J. Math. Kyoto Univ. **31** (1991), 1125-1153. MR **93b:**11154

[Oka 2] R. Okazaki, *An elementary proof for a theorem of Thomas and Vasquez,* J. Nb. Th. **55** (1995), 197-208. MR **96m:**11099

[Shin] T. Shintani, *On evaluation of zeta functions of totally real algebraic number fields at non-positive integers,* J. Fac. Sci. Univ. Tokyo **23** (1976), 393-417. MR **55:**266

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields,* Grad. Texts Math. **83**, Springer-Verlag. MR **85g:**11001

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