

ON DIVISIBILITY OF THE CLASS NUMBER h^+ OF THE REAL CYCLOTOMIC FIELDS OF PRIME DEGREE l

STANISLAV JAKUBEC

ABSTRACT. In this paper, criteria of divisibility of the class number h^+ of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$ of a prime conductor p and of a prime degree l by primes q the order modulo l of which is $\frac{l-1}{2}$, are given. A corollary of these criteria is the possibility to make a computational proof that a given q does not divide h^+ for any p (conductor) such that both $\frac{p-1}{2}, \frac{p-3}{4}$ are primes. Note that on the basis of Schinzel's hypothesis there are infinitely many such primes p .

INTRODUCTION

Let l, p be primes such that $p = 2l + 1$. To consider divisibility of the class number h^+ of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$ by primes q it is suitable to sort primes q according to their order modulo l . The simplest case is the case when the order of q modulo l is $l - 1$, i.e. when q is a primitive root modulo l . In this case the problem is completely solved, because it is proved that q does not divide h^+ . The proof for $q = 2$ can be found in [1] and for $q > 2$ in [4]. According to complexity, the further case is the case when the order of q modulo l is $\frac{l-1}{2}$, hence when q generates the group of quadratic residues modulo l .

In this case we have:

- 1) $q = 2$. If $l \equiv 3 \pmod{4}$, then 2 does not divide h^+ . (For the proof see [2].)
- 2) $q = 3$. The prime 3 does not divide h^+ . (For the proof see [5].)
- 3) $q = 5$. If $l \equiv 3 \pmod{4}$ then 5 does not divide h^+ . (For the proof see [6].)

The divisibility of h^+ by a general prime q under the assumption $p \equiv -1 \pmod{q}$, $p \not\equiv -1 \pmod{q^3}$ was considered in the papers [7], [8].

The aim of this paper is to derive criteria for divisibility of h^+ by a prime q without any restriction imposed on $p \pmod{q}$. As an application of derived criteria we shall prove Theorem 7.

Theorem 7. *Let q be prime, $q \leq 23$. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, and let the order of the prime q modulo l be $l - 1$ or $\frac{l-1}{2}$. The prime q does not divide h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$.*

Note that if $l = 2l_1 + 1$, where l_1 is a prime, then each $q \not\equiv 0, \pm 1 \pmod{l}$ satisfies the conditions of Theorem 5.

This implies the following Corollary.

Received by the editor March 16, 1995 and, in revised form, April 12, 1996.
1991 *Mathematics Subject Classification.* Primary 11R29.

Corollary. *Let l_1, l, p be primes such that $l = 2l_1 + 1, p = 2l + 1$. The prime q does not divide h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$, for $q \leq 23$.*

* * *

Let q be an odd prime. Define the numbers $A_0, A_1, A_2, \dots, A_{q-1}$ as follows:

$$A_0 = 0, \quad A_j = \sum_{i=1}^j \frac{1}{i}, \quad \text{for } j = 1, 2, \dots, q - 1.$$

Let s be a rational q -integer. Put $A_s = A_j$ for an integer $j, 0 \leq j < q, s \equiv j \pmod{q}$.

Let m, n be natural numbers, $m \equiv 1 \pmod{2}, (m, n) = 1$. Associate to the number n the permutation $\phi_{m,n}$ of the numbers $1, 2, \dots, \frac{m-1}{2}$ as follows:

$$\phi_{m,n}(x) \equiv \pm nx \pmod{m}, \quad \text{for } x = 1, 2, \dots, \frac{m-1}{2}.$$

Further, associate to the number n the quadratic form $Q_{m,n}(X_1, X_2, \dots, X_{\frac{m-1}{2}})$,

$$Q_{m,n}(X_1, X_2, \dots, X_{\frac{m-1}{2}}) = X_1^2 + X_2^2 + \dots + X_{\frac{m-1}{2}}^2 - \sum_{i=1}^{\frac{m-1}{2}} X_i X_{\phi_{m,n}(i)}.$$

The following theorem holds

Theorem 1. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1, l \equiv 3 \pmod{4}, p \equiv -m \pmod{q}, m \equiv 1 \pmod{2}, m > 0$, and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then for each divisor $n, (n, q) = 1$, of the number $p + m$, the following congruence holds:*

(i)

$$\frac{p+m}{2q} \frac{n^{q-1} - 1}{q} \equiv Q_{m,n}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{\frac{-t}{m}}) \pmod{q}.$$

(ii) *If $nq \mid \frac{p+m}{q}$, then*

$$\frac{p+m}{2q^2} \equiv -Q_{m,qn}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{\frac{-t}{m}}) \pmod{q},$$

where $t = \frac{m-1}{2}$.

Proof. To prove this theorem, the following assertion from [4] will be used:

Proposition 1. *Let l, p, q be primes, $p \equiv 1 \pmod{l}, q \neq 2; q \neq l; q < p$. Let K be a subfield of the field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$, $[K : \mathbf{Q}] = l$ and let h_K be the class number of the field K . If $q \mid h_K$, then $q \mid N_{\mathbf{Q}(\zeta_l)/\mathbf{Q}}(\omega)$, where*

$$\omega = b_1 \sum_{i \equiv 1 \pmod{q}} \chi(i) + b_2 \sum_{i \equiv 2 \pmod{q}} \chi(i) + \dots + b_{q-1} \sum_{i \equiv q-1 \pmod{q}} \chi(i),$$

with the sums all taken with $1 \leq i \leq p - 1$, with $\chi(x)$ a Dirichlet character modulo p of order l , and b_j defined by the expressions

$$\frac{p}{q} \left(\frac{(\zeta_p - 1)^q}{\zeta_p^q - 1} - 1 \right) \equiv b_1 \zeta_p + b_2 \zeta_p^2 + \dots + b_{p-1} \zeta_p^{p-1} \pmod{q}.$$

The following lemma will determine the coefficients b_1, b_2, \dots, b_{q-1} .

Lemma 1. *Let $p \equiv z \pmod{q}$. Then*

$$b_i = A_{\frac{-i}{z}}, \text{ for } i = 1, 2, \dots, q - 1.$$

Proof. We note that the b_j can be determined explicitly by multiplying the above expression through by $\zeta_p^q - 1$: In fact we get (taking $b_0 = 0$ and each $b_k = b_{k \pmod{p}}$)

$$\begin{aligned} \frac{1}{p} \sum_{j=0}^{p-1} (b_{j-q} - b_j) \zeta_p^j &\equiv \left(\frac{(\zeta_p - 1)^q - (\zeta_p^q - 1)}{q} \right) = \sum_{i=1}^{q-1} \frac{1}{q} \binom{q}{i} (-1)^{q-i} \zeta_p^i \\ &\equiv \sum_{i=1}^{q-1} \frac{\zeta_p^i}{i} \pmod{q}, \end{aligned}$$

since

$$\frac{1}{q} \binom{q}{i} = \frac{1}{i} \frac{(q-1)(q-2)\dots(q-i+1)}{(i-1)!} \equiv \frac{(-1)^{i-1}}{i} \pmod{q}.$$

Comparing coefficients we see that $b_{j-q} - b_j \equiv b_{-q} - b_0 + p\delta_j \pmod{q}$, where $\delta_j = \frac{1}{j}$ if $1 \leq j \leq q - 1$ and $\delta_j = 0$ otherwise. Adding these congruences together for $j = 0, -q, -2q, \dots, -(n-1)q$ and noting that $b_0 = 0$, we obtain $b_{-nq} \equiv nb_{-q} + \frac{p}{\binom{p}{p}} + \frac{p}{\binom{2p}{q}} + \dots + \frac{p}{\binom{mp}{q}} \pmod{q}$, where $(m+1)p \geq nq > mp$ and $(jp)_q$ is the least positive residue of $jp \pmod{q}$, by an easy induction. Taking $n = p$ gives that $0 = b_0 \equiv pb_{-q} + 1 + \frac{1}{2} + \dots + \frac{1}{q-1} \equiv pb_{-q} \pmod{q}$ (since $\frac{1}{j} + \frac{1}{q-j} \equiv 0 \pmod{q}$ for each j), and thus $b_{-q} \equiv 0 \pmod{q}$. Therefore, if $1 \leq j \leq p - 1$ we write $j = (m+1)p - nq$, so that

$$b_j = b_{-nq} \equiv 1 + \frac{1}{2} + \dots + \frac{1}{m} \equiv 1 + \frac{1}{2} + \dots + \frac{1}{(-j/p)_q} \pmod{q}.$$

Lemma 1 is proved. □

Let $p \equiv z \pmod{q}$. By Proposition 1 we have

$$\omega = \sum_{i=1}^{p-1} A_{\frac{-i}{z}} \chi(i).$$

Denote

$$\tau = \sum_{0 < i < \frac{p}{2}} A_{\frac{-i}{z}} \chi(i).$$

It is easy to see that $\omega = 2\tau$.

Since the order of q modulo l is $\frac{l-1}{2}$, according to [10], Theorem 2.13, we have that q is splitting to two divisors in $\mathbf{Q}(\zeta_l)$. Because $l \equiv 3 \pmod{4}$, it holds that $\left(\frac{-1}{l}\right) = -1$, hence if $q | \mathbf{N}_{\mathbf{Q}(\zeta_l)/\mathbf{Q}}(\omega)$, then q divides $\tau\bar{\tau}$.

The following formula holds

$$(1) \quad \tau\bar{\tau} = \sum_{i,j < \frac{p}{2}} A_{\frac{-i}{z}} A_{\frac{-j}{z}} \chi(ij^{-1}) = d_0 + d_1\zeta_l + d_2\zeta_l^2 + \dots + d_{l-1}\zeta_l^{l-1}.$$

Then $q | \tau\bar{\tau}$ if and only if

$$d_0 \equiv d_1 \equiv \dots \equiv d_{l-1} \pmod{q}.$$

Let $p \equiv -m \pmod{q}$, $m > 0$, $m \equiv 1 \pmod{2}$. Hence $b_i = A_{\frac{i}{m}}$. Denote by r such a number that $r < l$, $g^r \equiv \pm n \pmod{p}$. Let $\chi(ij^{-1}) = \zeta_l^r$. Then either

$\text{ind}(ij^{-1}) = r$ or $r + l$, therefore

$$(2) \quad ij^{-1} \equiv \pm n \pmod{p}, \quad i, j < \frac{p}{2}.$$

The following lemma determines the coefficient d_r of (1).

Lemma 2. *Let $p \equiv -m \pmod{q}$, $m > 0$, $m \equiv 1 \pmod{2}$, $g^r \equiv \pm n \pmod{p}$. For the coefficient d_r , $r < l$, the following holds:*

$$d_r = \sum_{0 < j < \frac{p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}} + \sum_{\frac{p}{n} < j < \frac{2p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+1} \\ + \sum_{\frac{2p}{n} < j < \frac{3p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+2} + \dots + \sum_{\frac{\frac{n-1}{2}p}{n} < j < \frac{p}{2}} A_{\frac{j}{m}} A_{\frac{jn}{m} + \frac{n-1}{2}},$$

for n odd,

$$d_r = \sum_{0 < j < \frac{p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}} + \sum_{\frac{p}{n} < j < \frac{2p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+1} \\ + \sum_{\frac{2p}{n} < j < \frac{3p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+2} + \dots + \sum_{\frac{(\frac{n}{2}-1)p}{n} < j < \frac{p}{2}} A_{\frac{j}{m}} A_{\frac{jn}{m} + \frac{n}{2}-1},$$

for n even.

Proof. By (2), $ij^{-1} \equiv \pm n \pmod{p}$, $i, j < \frac{p}{2}$. Therefore either $i \equiv nj \pmod{p}$ or $i \equiv p - nj \pmod{p}$. Let $nj < p$. From (1) we get the term $A_{\frac{j}{m}} A_{\frac{nj}{m}} \chi(ij^{-1})$ if $nj < \frac{p}{2}$ and $A_{\frac{j}{m}} A_{\frac{p-nj}{m}} \chi(ij^{-1})$ if $nj > \frac{p}{2}$. Clearly $\frac{p-nj}{m} \equiv -1 - \frac{nj}{m} \pmod{q}$. From $\frac{p-nj}{m} + \frac{nj}{m} \equiv -1 \pmod{q}$ we get $A_{\frac{nj}{m}} = A_{\frac{p-nj}{m}}$. If $p < nj < 2p$, then the coefficient of $\chi(ij^{-1})$ is $A_{\frac{j}{m}} A_{\frac{nj-p}{m}}$ and hence $A_{\frac{j}{m}} A_{\frac{nj}{m}+1}$. Repeating this procedure we obtain

$$d_r = \sum_{0 < j < \frac{p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}} + \sum_{\frac{p}{n} < j < \frac{2p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+1} + \sum_{\frac{2p}{n} < j < \frac{3p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+2} + \dots \quad \square$$

The following lemma determines the coefficient d_r , $g^r \equiv \pm n \pmod{p}$ in the special case when $n \mid \frac{p+m}{q}$. The reason why we restrict ourselves to such special coefficients is that in this case it is possible to give such criterion of divisibility h^+ that has a simple form (see Theorem 1). If n does not divide $\frac{p+m}{q}$, then things are more complicated and even in the most simple case when $n = 3$ and 3 does not divide $\frac{p+m}{q}$, the corresponding criteria have a more complicated form than Theorem 1 (see Theorem 2).

Lemma 3. *Let $p \equiv -m \pmod{q}$, $m > 0$, $m \equiv 1 \pmod{2}$, $g^r \equiv \pm n \pmod{p}$. For the coefficient d_r , $r < l$, $n \mid \frac{p+m}{q}$ the following holds:*

$$(3) \quad d_r \equiv \frac{p+m}{qn} \left(\sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} + \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} \right. \\ \left. + \dots + \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m} + \frac{n-3}{2}} + \frac{1}{2} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m} + \frac{n-1}{2}} \right) \\ - \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m} + \lceil \frac{ni}{m} \rceil} \pmod{q},$$

for $n \equiv 1 \pmod{2}$,

$$d_r \equiv \frac{p+m}{qn} \left(\sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} + \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} + \dots + \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n}{2}-1} \right) - \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m}+\lceil \frac{ni}{m} \rceil} \pmod{q},$$

for $n \equiv 0 \pmod{2}$.

Proof. The following congruences hold

$$\begin{aligned} \sum_{0 < j < \frac{p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}} &\equiv \frac{p+m}{qn} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} - \sum_{\lceil \frac{in}{m} \rceil=0}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m}}, \\ \sum_{\frac{p}{n} < j < \frac{2p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+1} &\equiv \frac{p+m}{qn} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} - \sum_{\lceil \frac{in}{m} \rceil=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m}+1}, \\ &\vdots \\ \sum_{\frac{p \frac{n-1}{2}}{n} < j < \frac{p}{2}} A_{\frac{j}{m}} A_{\frac{jn}{m}+\frac{n-1}{2}} &\equiv \frac{p+m}{2qn} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n-1}{2}} - \sum_{\lceil \frac{in}{m} \rceil=\frac{n-1}{2}}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m}+\frac{n-1}{2}}, \end{aligned}$$

for n odd.

And

$$\begin{aligned} \sum_{0 < j < \frac{p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}} &\equiv \frac{p+m}{qn} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} - \sum_{\lceil \frac{in}{m} \rceil=0}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m}}, \\ \sum_{\frac{p}{n} < j < \frac{2p}{n}} A_{\frac{j}{m}} A_{\frac{jn}{m}+1} &\equiv \frac{p+m}{qn} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} - \sum_{\lceil \frac{in}{m} \rceil=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m}+1}, \\ &\vdots \\ \sum_{\frac{p(\frac{n}{2}-1)}{n} < j < \frac{p}{2}} A_{\frac{j}{m}} A_{\frac{jn}{m}+\frac{n}{2}-1} &\equiv \frac{p+m}{qn} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n}{2}-1} - \sum_{\lceil \frac{in}{m} \rceil=\frac{n}{2}-1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m}+\frac{n}{2}-1}, \end{aligned}$$

for n even.

These congruences can be proved as follows. Let n be odd. If $s \equiv t \pmod{q}$, then $A_s \equiv A_t \pmod{q}$. On the basis of this fact it is enough to prove that for each $k = 1, 2, \dots, \frac{n-1}{2}$ the following holds: the set $\{j | \frac{kp}{n} < j < \frac{(k+1)p}{n}\} \cup \{-i | \lceil \frac{in}{m} \rceil = k, i \leq \frac{m-1}{2}\}$ gives $\frac{p+m}{qn}$ exemplars of the full residue system modulo q for $k =$

$1, 2, \dots, \frac{n-3}{2}$, and $\frac{p+m}{2qn}$ exemplars of the full residue system modulo q for $k = \frac{n-1}{2}$.

From $n|p+m$ we get that $(m, n) = 1$. Hence $\left[\frac{im}{m}\right] = k$, $k \neq 0$ if and only if

$$\frac{km}{n} < i < \frac{(k+1)m}{n}.$$

Denote $\frac{p+m}{nq} = v$, hence $m = nqv - p$. It implies

$$kqv - \frac{kp}{n} < i < (k+1)qv - \frac{(k+1)p}{n}.$$

Multiplying by -1 and adding $(k+1)qv$, we get

$$\frac{(k+1)p}{n} < -i + (k+1)qv < \frac{kp}{n} + qv.$$

Denote $i^* = -i + (k+1)qv$. Now we have

$$\frac{kp}{n} < i < \frac{(k+1)p}{n}; \quad \frac{(k+1)p}{n} < i^* < \frac{kp}{n} + qv.$$

This provides qv successive natural numbers, hence we have $v = \frac{p+m}{qn}$ exemplars of full residue systems modulo q . If $k = 0$, then the terms A_0 and A_{qv} will be missing. Since $A_0 = A_{qv} = 0$, the congruence will hold for $k = 0$ as well. For $k = \frac{n-1}{2}$, by the same method we get $\frac{p+m}{2qn}$ exemplars of the full residue system modulo q . Summing the congruences we get the required congruence. The same procedure applies for n even. Lemma 3 is proved. \square

In the formula for d_r , there is the sum

$$\sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m} + \left[\frac{ni}{m}\right]}.$$

We shall prove that

$$\sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-ni}{m} + \left[\frac{ni}{m}\right]} \equiv \sum_{i=1}^{\frac{m-1}{2}} X_i X_{\phi_{m,n}(i)} \pmod{q},$$

for $X_i = A_{\frac{-i}{m}}$, for $i = 1, 2, \dots, \frac{m-1}{2}$.

Clearly

$$\frac{-ni}{m} + \left[\frac{ni}{m}\right] \equiv \frac{-1}{m} \left(ni - m \left[\frac{ni}{m}\right] \right) \pmod{q}.$$

The number $ni - m \left[\frac{ni}{m}\right]$ is equal to the residuum ni modulo m . It follows that if $ni - m \left[\frac{ni}{m}\right] < \frac{m}{2}$, then $ni - m \left[\frac{ni}{m}\right] = \phi_{m,n}(i)$. If $ni - m \left[\frac{ni}{m}\right] > \frac{m}{2}$, then $ni - m \left[\frac{ni}{m}\right] = m - \phi_{m,n}(i)$.

Consider the numbers

$$A_{\frac{-\phi_{m,n}(i)}{m}} \text{ resp. } A_{\frac{-1}{m}(m-\phi_{m,n}(i))}.$$

Since

$$\frac{-1}{m} \phi_{m,n}(i) + \frac{-1}{m} (m - \phi_{m,n}(i)) = -1,$$

there holds

$$A_{-\frac{\phi_{m,n}(i)}{m}} \equiv A_{\frac{-1}{m}(m-\phi_{m,n}(i))} \pmod{q},$$

which implies the required relation.

Now we shall express the coefficient d_0 corresponding to the value $n = 1$. The substitution into (3) gives

$$d_0 = \frac{p+m}{2q} \sum_{i=1}^{q-1} A_i^2 - \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}}^2.$$

If $q|h^+$, then $d_0 \equiv d_r \pmod{q}$ and hence for $n \equiv 1 \pmod{2}$ there holds:

$$\begin{aligned} \frac{p+m}{qn} & \left(\sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} + \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} + \dots + \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n-3}{2}} \right. \\ & \left. + \frac{1}{2} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n-1}{2}} \right) \\ & - \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{-\frac{\phi_{m,n}(i)}{m}} \equiv \frac{p+m}{2q} \sum_{i=1}^{q-1} A_i^2 - \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}}^2 \pmod{q}. \end{aligned}$$

It is easy to prove that $\sum_{i=1}^{q-1} A_i^2 \equiv -2 \pmod{q}$. Therefore

$$\begin{aligned} \frac{p+m}{q} & \left(\frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} + \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} + \dots + \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n-3}{2}} \right. \\ & \left. + \frac{1}{2n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n-1}{2}} + 1 \right) \\ & \equiv -Q_{m,n}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{\frac{-t}{m}}) \pmod{q}, \end{aligned}$$

where $t = \frac{m-1}{2}$.

By [8] (proof of Theorem 1), the following holds:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} + \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} + \dots + \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n-3}{2}} \\ + \frac{1}{2n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n-1}{2}} + 1 \\ \equiv -\frac{1}{2} \frac{n^{q-1} - 1}{q} \pmod{q}. \end{aligned}$$

The congruence (i) is now proved for $n \equiv 1 \pmod{2}$. Analogically, the congruence (i) can be proved for $n \equiv 0 \pmod{2}$, on the basis of the congruence

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}} + \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+1} + \dots + \frac{1}{n} \sum_{i=1}^{q-1} A_{\frac{i}{m}} A_{\frac{ni}{m}+\frac{n}{2}-1} + 1 \\ \equiv -\frac{1}{2} \frac{n^{q-1} - 1}{q} \pmod{q}. \end{aligned}$$

Now we shall prove the congruence (ii). Substituting nq , where $nq \mid \frac{p+m}{q}$, instead of n into the formula for the computation d_r , we get for $n \equiv 1 \pmod{2}$ the following sum:

$$A_1(A_1 + A_2 + \cdots + A_{q-1}) + A_2(A_1 + A_2 + \cdots + A_{q-1}) \\ + \cdots + \frac{1}{2}A_{\frac{nq-1}{2}}(A_1 + A_2 + \cdots + A_{q-1}).$$

It is easy to see that $A_1 + A_2 + \cdots + A_{q-1} \equiv 1 \pmod{q}$, therefore

$$A_1(A_1 + A_2 + \cdots + A_{q-1}) + A_2(A_1 + A_2 + \cdots + A_{q-1}) \\ + \cdots + \frac{1}{2}A_{\frac{nq-1}{2}}(A_1 + A_2 + \cdots + A_{q-1}) \equiv \frac{n}{2} \pmod{q}.$$

Analogously for $n \equiv 0 \pmod{2}$ we get

$$A_1(A_1 + A_2 + \cdots + A_{q-1}) + A_2(A_1 + A_2 + \cdots + A_{q-1}) \\ + \cdots + A_{\frac{nq}{2}-1} \equiv \frac{n}{2} \pmod{q}.$$

Theorem 1 is proved. \square

We shall show 12 corollaries of Theorem 1.

Corollary 1. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -3 \pmod{q}$, $p \not\equiv -3 \pmod{q^3}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then $2^{q-1} \equiv 1 \pmod{q^2}$.*

Proof. By Theorem 1, (i) putting $n = 2$ we have

$$\frac{p+3}{2q} \frac{2^{q-1}-1}{q} \equiv Q_{3,2}(A_{\frac{-1}{3}}) \pmod{q}.$$

Clearly $Q_{3,2}(X_1) = 0$, hence

$$\frac{p+3}{2q} \frac{2^{q-1}-1}{q} \equiv 0 \pmod{q}.$$

If $\frac{p+3}{2q} \not\equiv 0 \pmod{q}$, then $\frac{2^{q-1}-1}{q} \equiv 0 \pmod{q}$. Suppose that $q \mid \frac{p+3}{q}$. By Theorem 1, (ii) we have

$$-\frac{p+3}{2q^2} \equiv Q_{3,q}(A_{\frac{-1}{3}}) \equiv 0 \pmod{q},$$

hence $p+3 \equiv 0 \pmod{q^3}$ —a contradiction. \square

Corollary 2. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -5 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$F_{q-\left(\frac{5}{q}\right)} \equiv 0 \pmod{q^2},$$

where F_n is the n th Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $0 \leq n$).

Moreover, if $p \not\equiv -5 \pmod{q^3}$, then $2^{q-1} \equiv 1 \pmod{q^2}$.

Proof. The number $p + 5$ has the divisors $n = 2, 4$. Therefore by Theorem 1 (i)

$$\frac{p + 5}{2q} \frac{2^{q-1} - 1}{q} \equiv Q_{5,2}(A_{\frac{-1}{5}}, A_{\frac{-2}{5}}) \pmod{q},$$

$$\frac{p + 5}{2q} \frac{4^{q-1} - 1}{q} \equiv Q_{5,4}(A_{\frac{-1}{5}}, A_{\frac{-2}{5}}) \pmod{q}.$$

Clearly

$$\phi_{5,2} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \phi_{5,4} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Hence

$$Q_{5,2}(X_1, X_2) = X_1^2 + X_2^2 - 2X_1X_2 = (X_1 - X_2)^2, Q_{5,4}(X_1, X_2) = 0.$$

It is easy to see that

$$(A_{\frac{-1}{5}} - A_{\frac{-2}{5}})^2 \equiv \left(\sum_{\frac{q}{5} < i < \frac{2q}{5}} \frac{1}{i} \right)^2 \pmod{q}.$$

Therefore

$$\frac{p + 5}{2q} \frac{2^{q-1} - 1}{q} \equiv \left(\sum_{\frac{q}{5} < i < \frac{2q}{5}} \frac{1}{i} \right)^2 \pmod{q},$$

$$\frac{p + 5}{2q} \frac{4^{q-1} - 1}{q} \equiv 0 \pmod{q}.$$

Because $\frac{2^{q-1}-1}{q} \equiv 0 \pmod{q}$ if and only if $\frac{4^{q-1}-1}{q} \equiv 0 \pmod{q}$, we get that if $q|h^+$, then

$$\sum_{\frac{q}{5} < i < \frac{2q}{5}} \frac{1}{i} \equiv 0 \pmod{q}.$$

By [11], for $q > 5$ there holds

$$\frac{2}{5} \sum_{\frac{q}{5} < i < \frac{2q}{5}} \frac{1}{i} \equiv \frac{1}{q} F_{q-(\frac{5}{q})} \pmod{q},$$

which proves the first assertion of Corollary 2.

If $\frac{2^{q-1}-1}{q} \not\equiv 0 \pmod{q}$, then $\frac{p+5}{2q^2} \equiv 0 \pmod{q}$. By (ii) we get $\frac{p+5}{2q^2} \equiv 0 \pmod{q}$ —a contradiction. \square

Remark. P.L. Montgomery [9] reports no solution of $F_{q-(\frac{5}{q})} \equiv 0 \pmod{q^2}$ with $q < 2^{32}$.

Corollary 3. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1, l \equiv 3 \pmod{4}, p \equiv -7 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$(*) \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right) \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right) \equiv 0 \pmod{q}.$$

Moreover, if $p \not\equiv -7 \pmod{q^3}$, then $2^{q-1} \equiv 3^{q-1} \equiv 1 \pmod{q^2}$.

Proof. The number $p+7$ has the divisors $n = 2, 3, 6$. By Theorem 1 (i) the following holds

$$\frac{p+7}{2q} \frac{2^{q-1}-1}{q} \equiv Q_{7,2}(A_{\frac{-1}{7}}, A_{\frac{-2}{7}}, A_{\frac{-3}{7}}) \pmod{q},$$

$$\frac{p+7}{2q} \frac{3^{q-1}-1}{q} \equiv Q_{7,3}(A_{\frac{-1}{7}}, A_{\frac{-2}{7}}, A_{\frac{-3}{7}}) \pmod{q},$$

$$\frac{p+7}{2q} \frac{6^{q-1}-1}{q} \equiv Q_{7,6}(A_{\frac{-1}{7}}, A_{\frac{-2}{7}}, A_{\frac{-3}{7}}) \pmod{q}.$$

Clearly

$$\phi_{7,2} = \phi_{7,3} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \phi_{7,6} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Hence

$$Q_{7,2}(X_1, X_2, X_3) = Q_{7,3}(X_1, X_2, X_3), \quad Q_{7,6}(X_1, X_2, X_3) = 0.$$

By rearrangement we get

$$\begin{aligned} & Q_{7,2}(A_{\frac{-1}{7}}, A_{\frac{-2}{7}}, A_{\frac{-3}{7}}) \\ & \equiv \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right) \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right) \pmod{q}, \end{aligned}$$

Therefore we have

$$\begin{aligned} & \frac{p+7}{2q} \frac{2^{q-1}-1}{q} \\ & \equiv \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right) \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right) \pmod{q}, \\ & \frac{p+7}{2q} \frac{3^{q-1}-1}{q} \\ & \equiv \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right) \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right) \pmod{q}, \\ & \frac{p+7}{2q} \frac{6^{q-1}-1}{q} \equiv 0 \pmod{q}. \end{aligned}$$

If

$$\left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \right) \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \right) \not\equiv 0 \pmod{q},$$

then $\frac{p+7}{2q} \not\equiv 0 \pmod{q}$, $\frac{6^{q-1}-1}{q} \equiv 0 \pmod{q}$ and $\frac{2^{q-1}-1}{q} \equiv \frac{3^{q-1}-1}{q} \pmod{q}$ and $\frac{2^{q-1}-1}{q} \not\equiv 0 \pmod{q}$. This easily yields a contradiction.
 If

$$\left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i}\right) \left(\sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i}\right) \equiv 0 \pmod{q},$$

and $\frac{p+7}{2q} \not\equiv 0 \pmod{q}$, then

$$2^{q-1} \equiv 3^{q-1} \equiv 1 \pmod{q^2}.$$

If $\frac{p+7}{2q} \equiv 0 \pmod{q}$, then by Theorem 1 (ii) $\frac{p+7}{2q^2} \equiv 0 \pmod{q}$ and therefore $p \equiv -7 \pmod{q^3}$ —a contradiction. \square

Corollary 4. *Let q be an odd prime, $q \equiv 2 \pmod{3}$. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -7 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$\sum_{\frac{q}{7} < i < \frac{2q}{7}} \frac{1}{i} \equiv \sum_{\frac{2q}{7} < i < \frac{3q}{7}} \frac{1}{i} \equiv 0 \pmod{q}.$$

Proof. The left side of the congruence (*) can be expressed as the norm of the field $\mathbf{Q}(\zeta_3)$ into \mathbf{Q} . If $q \equiv 2 \pmod{3}$, then q does not decompose in the field $\mathbf{Q}(\zeta_3)$, and it implies the assertion of Corollary 4. \square

By [3] there holds: For $1 \leq a \leq 6$, and any odd prime $q \neq 7$,

$$B_{q-1} \left(\frac{a}{7}\right) - B_{q-1} \equiv \frac{7}{2q}(U_q(7, a, b) - 1) \pmod{q},$$

where $b = 1, 2$ or 3 with $b \equiv \pm q \pmod{7}$, and U_n satisfies the recurrence relation

$$U_{n+3} = 7U_{n+2} - 14U_{n+1} + 7U_n.$$

The values of U_1, U_2, U_3 are given in the table below

$\pm a$	$\pm b$	U_1	U_2	U_3
2	1	1	2	5
3	2	2	7	26
1	3	2	6	19
3	1	1	2	6
1	2	3	11	41
2	3	2	5	13
a	a	1	3	10

From Corollary 4 and the just mentioned result we get:

Corollary 5. *Let q be an odd prime, $b \equiv \pm q \pmod{7}$ where $b = 1, 2$ or 3 and $q \equiv 2 \pmod{3}$. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -7 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$U_q(7, 1, b) \equiv U_q(7, 2, b) \equiv U_q(7, 3, b) \pmod{q^2}.$$

Corollary 6. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -9 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$\left(\sum_{\frac{q}{9} < i < \frac{2q}{9}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{2q}{9} < i < \frac{4q}{9}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{q}{9} < i < \frac{2q}{9}} \frac{1}{i}\right) \left(\sum_{\frac{2q}{9} < i < \frac{4q}{9}} \frac{1}{i}\right) \equiv 0 \pmod{q}.$$

Moreover, if $p \not\equiv -9 \pmod{q^3}$, then $2^{q-1} \equiv 1 \pmod{q^2}$.

Proof. The number $p + 9$ has the divisors $n = 2, 4, 8$, which follows from $p + 9 = 2l + 10 = 2(l + 5) = 2(4k + 3 + 5) = 8(k + 2)$. Therefore, we have

$$\phi_{9,2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \quad \phi_{9,4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}, \quad \phi_{9,8} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Hence

$$\begin{aligned} Q_{9,2}(X_1 X_2, X_3, X_4) &= Q_{9,4}(X_1 X_2, X_3, X_4) \\ &= X_1^2 + X_2^2 + X_4^2 - (X_1 X_2 + X_1 X_4 + X_2 X_4), \end{aligned}$$

and

$$Q_{9,8}(X_1 X_2, X_3, X_4) = 0.$$

By rearrangement we get

$$\begin{aligned} &Q_{9,2}(A_{\frac{-1}{9}}, A_{\frac{-2}{9}}, A_{\frac{-3}{9}}, A_{\frac{-4}{9}}) \\ &\equiv \left(\sum_{\frac{q}{9} < i < \frac{2q}{9}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{2q}{9} < i < \frac{4q}{9}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{q}{9} < i < \frac{2q}{9}} \frac{1}{i}\right) \left(\sum_{\frac{2q}{9} < i < \frac{4q}{9}} \frac{1}{i}\right) \pmod{q}. \end{aligned}$$

The rest of the proof is the same as in the case of Corollary 3. □

To prove the remaining corollaries, the following fact will be used.

1. If $n \equiv \pm 1 \pmod{m}$, then the permutation $\phi_{m,n}$ is identical and therefore $Q_{m,n}(X_1, X_2, \dots, X_{\frac{m-1}{2}}) = 0$.
2. If $n_1 n_2 \equiv \pm 1 \pmod{m}$, then the permutations $\phi_{m,n_1}, \phi_{m,n_2}$ are inverse and therefore

$$Q_{m,n_1}(X_1, X_2, \dots, X_{\frac{m-1}{2}}) = Q_{m,n_2}(X_1, X_2, \dots, X_{\frac{m-1}{2}}).$$

Corollary 7. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -13 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$\begin{aligned} &Q_{13,2}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \\ &\equiv Q_{13,3}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \equiv 0 \pmod{q}. \end{aligned}$$

Moreover, if $p \not\equiv -13 \pmod{q^3}$, then

$$2^{q-1} \equiv 3^{q-1} \equiv 1 \pmod{q^2}.$$

Proof. The number $p + 13$ has the divisors $n = 2, 3, 4, 6, 12$. By Theorem 1 (i) we have

$$\frac{p + 13}{2q} \frac{2^{q-1} - 1}{q} \equiv Q_{13,2}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \pmod{q},$$

$$\frac{p + 13}{2q} \frac{3^{q-1} - 1}{q} \equiv Q_{13,3}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \pmod{q},$$

$$\frac{p + 13}{2q} \frac{4^{q-1} - 1}{q} \equiv Q_{13,3}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \pmod{q},$$

$$\frac{p + 13}{2q} \frac{6^{q-1} - 1}{q} \equiv Q_{13,2}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \pmod{q},$$

$$\frac{p + 13}{2q} \frac{12^{q-1} - 1}{q} \equiv 0 \pmod{q}.$$

If either

$$Q_{13,2}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \not\equiv 0 \pmod{q}$$

or

$$Q_{13,3}(A_{\frac{-1}{13}}, A_{\frac{-2}{13}}, A_{\frac{-3}{13}}, A_{\frac{-4}{13}}, A_{\frac{-5}{13}}, A_{\frac{-6}{13}}) \not\equiv 0 \pmod{q},$$

then $\frac{p+13}{2q} \not\equiv 0 \pmod{q}$, hence $\frac{12^{q-1}-1}{q} \equiv 0 \pmod{q}$ and this yields a contradiction. \square

Corollary 8. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -17 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$\begin{aligned} & Q_{17,2}(A_{\frac{-1}{17}}, A_{\frac{-2}{17}}, A_{\frac{-3}{17}}, A_{\frac{-4}{17}}, A_{\frac{-5}{17}}, A_{\frac{-6}{17}}, A_{\frac{-7}{17}}, A_{\frac{-8}{17}}) \\ & \equiv Q_{17,4}(A_{\frac{-1}{17}}, A_{\frac{-2}{17}}, A_{\frac{-3}{17}}, A_{\frac{-4}{17}}, A_{\frac{-5}{17}}, A_{\frac{-6}{17}}, A_{\frac{-7}{17}}, A_{\frac{-8}{17}}) \equiv 0 \pmod{q}. \end{aligned}$$

Moreover, if $p \not\equiv -17 \pmod{q^2}$, then $2^{q-1} \equiv 1 \pmod{q^2}$.

Proof. The number $p + 17$ has the divisors $n = 2, 4, 8$. By Theorem 1 (i) we have

$$\frac{p + 17}{2q} \frac{2^{q-1} - 1}{q} \equiv Q_{17,2}(A_{\frac{-1}{17}}, A_{\frac{-2}{17}}, A_{\frac{-3}{17}}, A_{\frac{-4}{17}}, A_{\frac{-5}{17}}, A_{\frac{-6}{17}}, A_{\frac{-7}{17}}, A_{\frac{-8}{17}}) \pmod{q},$$

$$\frac{p + 17}{2q} \frac{4^{q-1} - 1}{q} \equiv Q_{17,4}(A_{\frac{-1}{17}}, A_{\frac{-2}{17}}, A_{\frac{-3}{17}}, A_{\frac{-4}{17}}, A_{\frac{-5}{17}}, A_{\frac{-6}{17}}, A_{\frac{-7}{17}}, A_{\frac{-8}{17}}) \pmod{q},$$

$$\frac{p + 17}{2q} \frac{8^{q-1} - 1}{q} \equiv Q_{17,2}(A_{\frac{-1}{17}}, A_{\frac{-2}{17}}, A_{\frac{-3}{17}}, A_{\frac{-4}{17}}, A_{\frac{-5}{17}}, A_{\frac{-6}{17}}, A_{\frac{-7}{17}}, A_{\frac{-8}{17}}) \pmod{q}.$$

If either $Q_{17,2} \not\equiv 0 \pmod{q}$ or $Q_{17,4} \not\equiv 0 \pmod{q}$, then $\frac{p+17}{2q} \not\equiv 0 \pmod{q}$ and $\frac{2^{q-1}-1}{q} \not\equiv 0 \pmod{q}$. The first and the third congruence imply that

$$\frac{2^{q-1}-1}{q} \equiv \frac{8^{q-1}-1}{q} \pmod{q},$$

therefore $\frac{2^{q-1}-1}{q} \equiv 0 \pmod{q}$ —a contradiction. \square

From now on, the function values of quadratic forms will be omitted, i.e., instead of $Q_{19,2}(\dots)$ we shall write $Q_{19,2}$.

Corollary 9. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -19 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then $Q_{19,2} \equiv 0 \pmod{q}$. If $Q_{19,3} \not\equiv 0 \pmod{q}$, then $2^{q-1} \equiv 1 \pmod{q^2}$. Moreover, if $p \not\equiv -19 \pmod{q^2}$, then $2^{q-1} \equiv 1 \pmod{q^2}$.*

Proof. The number $p + 19$ has the divisors $n = 2, 3, 6$. Hence

$$\frac{p+19}{2q} \frac{2^{q-1}-1}{q} \equiv Q_{19,2} \pmod{q},$$

$$\frac{p+19}{2q} \frac{3^{q-1}-1}{q} \equiv Q_{19,3} \pmod{q},$$

$$\frac{p+19}{2q} \frac{6^{q-1}-1}{q} \equiv Q_{19,3} \pmod{q}.$$

If $Q_{19,2} \not\equiv 0 \pmod{q}$, then $\frac{2^{q-1}-1}{q} \not\equiv 0 \pmod{q}$. The second and the third congruence imply that

$$\frac{3^{q-1}-1}{q} \equiv \frac{6^{q-1}-1}{q} \pmod{q},$$

which is not possible, because $\frac{2^{q-1}-1}{q} \not\equiv 0 \pmod{q}$. If $Q_{19,3} \not\equiv 0 \pmod{q}$, then

$$\frac{3^{q-1}-1}{q} \equiv \frac{6^{q-1}-1}{q} \pmod{q},$$

and it follows that $2^{q-1} \equiv 1 \pmod{q^2}$. \square

Corollary 10. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -25 \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then*

$$Q_{25,2} \equiv Q_{25,3} \equiv Q_{25,4} \equiv 0 \pmod{q}.$$

Moreover, if $p \not\equiv -25 \pmod{q^3}$, then $2^{q-1} \equiv 3^{q-1} \equiv 1 \pmod{q^2}$.

The proof is analogous as for $p \equiv -13 \pmod{q}$.

Corollary 11. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -m \pmod{q}$, $p \not\equiv -m \pmod{q^2}$, $m > 0$, $m \equiv 1 \pmod{2}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that there exist divisors n_1, n_2 of the number $p + m$ such that $n_1 n_2 \equiv \pm 1 \pmod{m}$ or $n_1 \equiv \pm n_2 \pmod{m}$. If $q|h^+$, then*

$$n_1^{q-1} \equiv n_2^{q-1} \pmod{q^2}.$$

Proof. Since $n_1 n_2 \equiv \pm 1 \pmod{m}$ or $n_1 \equiv \pm n_2 \pmod{m}$, we have

$$Q_{m,n_1}(X_1, X_2, \dots, X_{\frac{m-1}{2}}) \equiv Q_{m,n_2}(X_1, X_2, \dots, X_{\frac{m-1}{2}}) \pmod{q},$$

and hence

$$\frac{p+m}{2q} \frac{n_1^{q-1} - 1}{q} \equiv \frac{p+m}{2q} \frac{n_2^{q-1} - 1}{q} \pmod{q}.$$

The Corollary now follows from $\frac{p+m}{2q} \not\equiv 0 \pmod{q}$. □

Corollary 12. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1, l \equiv 3 \pmod{4}, p \equiv -m \pmod{q}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then for arbitrary n_1, n_2 such that $n_1 n_2 | p + m, (n_1 n_2, q) = 1$, the following congruence holds.*

$$\begin{aligned} & Q_{m,n_1 n_2}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{-\frac{t}{m}}) \\ & \equiv Q_{m,n_1}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{-\frac{t}{m}}) + Q_{m,n_2}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{-\frac{t}{m}}) \pmod{q}, \end{aligned}$$

where $t = \frac{m-1}{2}$.

Proof. Since $\frac{(n_1 n_2)^{q-1} - 1}{q} \equiv \frac{n_1^{q-1} - 1}{q} + \frac{n_2^{q-1} - 1}{q} \pmod{q}$, the preceding congruence implies Theorem 1 (i). □

The following example shows the possibility of applying the congruence of Corollary 12 in order to find out the divisibility of the class number h^+ of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$.

Example 1. Let $p \equiv -11 \pmod{43}$. If $p \not\equiv \pm 2 \pmod{11}$, then 43 does not divide the class number h^+ . If $p \equiv \pm 2 \pmod{11}$ and $43 | h^+$, then

$$p + 11 = 2 \cdot 43^s \cdot p_1^{s_1} p_2^{s_2} \dots p_n^{s_n},$$

where $p_i \equiv \pm 1 \pmod{11}$, for $i = 1, 2, \dots, n$.

Proof. Let $43^s | p + 11$ and 43^{s+1} does not divide $p + 11$, where $1 \leq s$. Put $n_1 = \frac{p+11}{2 \cdot 43^s}, n_2 = 2$. Then it holds:

$$\begin{aligned} & Q_{11,2n_1}(A_{39}, A_{35}, A_{31}, A_{27}, A_{23}) \\ & \equiv Q_{11,n_1}(A_{39}, A_{35}, A_{31}, A_{27}, A_{23}) + Q_{11,2}(A_{39}, A_{35}, A_{31}, A_{27}, A_{23}) \pmod{43}. \end{aligned}$$

In the following we shall write quadratic forms without arguments. Because $43 \equiv -1 \pmod{11}$ we have $2n_1 = \frac{p+11}{43^s} \equiv \pm p \pmod{11}$. Because $Q_{m,n} = Q_{m,-n}$, it is enough to consider the cases $p \equiv 1, 2, 3, 4, 5 \pmod{11}$.

1) $p \equiv 1 \pmod{11}$, then $Q_{11,1} = 0 \equiv Q_{11,\frac{1}{2}} + Q_{11,2} \pmod{43}$. From $Q_{11,\frac{1}{2}} = Q_{11,2}$ we have $Q_{11,2} \equiv 0 \pmod{43}$.

2) $p \equiv 2 \pmod{11}$, then $Q_{11,2} \equiv Q_{11,1} + Q_{11,2} \pmod{43}$, hence in this case we do not have any information, as $Q_{11,1} = 0$.

3) $p \equiv 3 \pmod{11}$, hence $Q_{11,3} \equiv Q_{11,\frac{3}{2}} + Q_{11,2} \pmod{43}$, $\frac{3}{2} \equiv 7 \pmod{11}$, $3 \cdot 7 \equiv -1 \pmod{11}$ therefore $Q_{11,\frac{3}{2}} = Q_{11,3}$ and we get that $Q_{11,2} \equiv 0 \pmod{43}$.

4) $p \equiv 4 \pmod{11}$, then analogically as in the preceding cases we get the congruence $Q_{11,3} \equiv 2Q_{11,2} \pmod{43}$.

5) $p \equiv 5 \pmod{11}$, then we get $Q_{11,3} \equiv 0 \pmod{43}$.

By substituting $A_{39}, A_{35}, A_{31}, A_{27}, A_{23}$ we have $Q_{11,2}(A_{39}, A_{35}, A_{31}, A_{27}, A_{23}) \equiv Q_{11,2}(9, 33, 15, 20, 10) \equiv 11 \pmod{43}$ and $Q_{11,3}(9, 33, 15, 20, 10) \equiv 39 \pmod{43}$.

Hence $Q_{11,2} \not\equiv 0 \pmod{43}$, $Q_{11,3} \not\equiv 0 \pmod{43}$, and $Q_{11,3} \not\equiv 2Q_{11,2} \pmod{43}$. By this we proved that if $p \not\equiv \pm 2 \pmod{11}$, then 43 does not divide h^+ .

The preceding calculations show that if $p+11$ had another divisor than 2 different from $\pm 1 \pmod{11}$, then 43 would not divide h^+ . Therefore $p+11$ must have the above mentioned form. \square

Throughout the rest of the paper, we shall consider the divisibility of h^+ by the concrete primes $q = 7, 11, 13, 17, 19, 23$. Theorem 1 and its corollaries would not sufficiently solve this task. The reason is that for some m (e.g. $m = 11$), only one suitable divisor of $p+m$ is known, namely $n = 2$.

In what follows, B_j resp. $B_j(X)$ will denote a Bernoulli number resp. a Bernoulli polynomial.

Theorem 2. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -m \pmod{q}$, for $m = 1, 3, 5, \dots, 2q - 3$, $m \equiv 0, 2 \pmod{3}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then the following holds:*

I. $m \equiv 0 \pmod{3}$.

(i) if $q \equiv 1 \pmod{3}$, then

$$\frac{p+m}{2q} \frac{3^{q-1} - 1}{q} + \frac{1}{9} B_{q-2} \left(\frac{1}{3} \right) \equiv C_m \pmod{q}.$$

(ii) if $q \equiv 2 \pmod{3}$, $m + 2 < q$, then

$$\frac{p+m}{2q} \frac{3^{q-1} - 1}{q} + \frac{2}{9} B_{q-2} \left(\frac{1}{3} \right) \equiv C_m \pmod{q}.$$

(iii) if $q \equiv 2 \pmod{3}$ and $m + 2 \geq q$, then

$$\frac{p+m}{2q} \frac{3^{q-1} - 1}{q} - \frac{1}{9} B_{q-2} \left(\frac{1}{3} \right) \equiv C_m \pmod{q}.$$

II. $m \equiv 2 \pmod{3}$

(i) if $q \equiv 2 \pmod{3}$, then

$$\frac{p+m}{2q} \frac{3^{q-1} - 1}{q} + \frac{1}{9} B_{q-2} \left(\frac{1}{3} \right) \equiv C_m \pmod{q}.$$

(ii) if $q \equiv 1 \pmod{3}$, $m + 2 < q$, then

$$\frac{p+m}{2q} \frac{3^{q-1} - 1}{q} + \frac{2}{9} B_{q-2} \left(\frac{1}{3} \right) \equiv C_m \pmod{q}.$$

(iii) if $q \equiv 1 \pmod{3}$, $m + 2 \geq q$, then

$$\frac{p+m}{2q} \frac{3^{q-1} - 1}{q} - \frac{1}{9} B_{q-2} \left(\frac{1}{3} \right) \equiv C_m \pmod{q},$$

where

$$C_m = \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}}^2 - \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-3i}{m}+1} + \sum_{\substack{i=1 \\ 3i \not\equiv m \pmod{q}}}^{k-1} \frac{1}{\frac{-3i}{m} + 1} A_{\frac{-i}{m}},$$

and $k \equiv \frac{m+2}{3} \pmod{q}$, $0 \leq k < q$.

Proof. By Lemma 2, for the coefficient d_r , where $g^r \equiv \pm 3 \pmod{p}$, we get

$$d_r = \sum_{0 < i < \frac{p}{3}} A_{\frac{i}{m}} A_{\frac{3i}{m}} + \sum_{\frac{p}{3} < i < \frac{p}{2}} A_{\frac{i}{m}} A_{\frac{3i}{m}+1}.$$

Then we proceed similarly as in the proof of Lemma 3. The corresponding congruence will be obtained from the fact that $q|h^+$ implies $d_0 \equiv d_r \pmod{q}$, using the following results of [8]. □

Theorem 3. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1, l \equiv 3 \pmod{4}, p \equiv -m \pmod{q}$, for $m = 1, 3, 5, \dots, 2q - 3, m \equiv 0, 2 \pmod{3}$ and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then the following holds:*

(i) $m \equiv 0 \pmod{3}, q \equiv 1 \pmod{3}$, then

$$\frac{p + m + 4q}{2q} \frac{3^{q-1} - 1}{q} \equiv Q_{m+4q,3}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{\frac{-t}{m}}) \pmod{q},$$

where $t = \frac{4q+m-1}{2}$.

(ii) $m \equiv 0 \pmod{3}, q \equiv 2 \pmod{3}$, then

$$\frac{p + m + 2q}{2q} \frac{3^{q-1} - 1}{q} \equiv Q_{m+2q,3}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{\frac{-t}{m}}) \pmod{q},$$

where $t = \frac{2q+m-1}{2}$.

(iii) $m \equiv 2 \pmod{3}, q \equiv 1 \pmod{3}$, then

$$\frac{p + m + 2q}{2q} \frac{3^{q-1} - 1}{q} \equiv Q_{m+2q,3}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{\frac{-t}{m}}) \pmod{q},$$

where $t = \frac{2q+m-1}{2}$.

(iv) $m \equiv 2 \pmod{3}, q \equiv 2 \pmod{3}$, then

$$\frac{p + m + 4q}{2q} \frac{3^{q-1} - 1}{q} \equiv Q_{m+4q,3}(A_{\frac{-1}{m}}, A_{\frac{-2}{m}}, \dots, A_{\frac{-t}{m}}) \pmod{q},$$

where $t = \frac{4q+m-1}{2}$.

Proof. (i) If $m \equiv 0 \pmod{3}$ and $q \equiv 1 \pmod{3}$, then because $p \equiv 2 \pmod{3}$ we have $p + m + 4q \equiv 0 \pmod{3}$ and the assertion (i) follows from Theorem 1 (i). Further we proceed analogously. □

Lemma 2 of [8]. *Let n, k be integers such that $nk \not\equiv 0 \pmod{q}$. Then*

$$\sum_{\substack{i=1 \\ ni \not\equiv -k \pmod{q}}}^{q-1} \frac{A_i}{ni + k} \equiv \frac{1}{n} B_{q-2} \left(\frac{k}{n} \right) \pmod{q}.$$

Lemma 3 of [8]. *Let n be an odd number. Then*

$$\begin{aligned} \sum_{i=1}^{q-1} A_i A_{ni} &\equiv \frac{-1}{n^2} (n-2) B_{q-2} \left(\frac{1}{n} \right) + \frac{-1}{n^2} (n-4) B_{q-2} \left(\frac{2}{n} \right) \\ &+ \dots + \frac{-1}{n^2} B_{q-2} \left(\frac{\frac{n-1}{2}}{n} \right) - 2 - \frac{n^{q-1} - 1}{q} \pmod{q}. \end{aligned}$$

By Lemma 2 of [8] we get

$$\begin{aligned} \sum_{i=1}^{q-1} A_i A_{ni+1} &\equiv \sum_{i=1}^{q-1} A_i A_{ni} + \frac{1}{n} B_{q-2} \left(\frac{1}{n} \right) \pmod{q}, \\ \sum_{i=1}^{q-1} A_i A_{ni+2} &\equiv \sum_{i=1}^{q-1} A_i A_{ni} + \frac{1}{n} B_{q-2} \left(\frac{1}{n} \right) + \frac{1}{n} B_{q-2} \left(\frac{2}{n} \right) \pmod{q}, \\ &\vdots \\ \sum_{i=1}^{q-1} A_i A_{ni+\frac{n-1}{2}} &\equiv \sum_{i=1}^{q-1} A_i A_{ni} + \frac{1}{n} B_{q-2} \left(\frac{1}{n} \right) + \frac{1}{n} B_{q-2} \left(\frac{2}{n} \right) \\ &\quad + \cdots + \frac{1}{n} B_{q-2} \left(\frac{\frac{n-1}{2}}{n} \right) \pmod{q}. \end{aligned}$$

Theorem 4. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1, l \equiv 3 \pmod{4}, p \equiv -m \pmod{q}$, for $m = 1, 3, 5, \dots, 2q - 3, m \equiv 3 \pmod{4}$, and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then the following holds:*

(i) *if $\frac{m+3}{2} < q$, then*

$$\frac{p+m}{2q} \frac{4^{q-1} - 1}{q} - \frac{1}{8} B_{q-2} \left(\frac{1}{4} \right) \equiv C_m \pmod{q}.$$

(ii) *if $\frac{m+3}{2} \geq q$, then*

$$\frac{p+m}{2q} \frac{3^{q-1} - 1}{q} + \frac{1}{8} B_{q-2} \left(\frac{1}{4} \right) \equiv C_m \pmod{q},$$

where

$$C_m = \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}}^2 - \sum_{i=1}^{\frac{m-1}{2}} A_{\frac{-i}{m}} A_{\frac{-4i}{m}+1} + \sum_{\substack{i=1 \\ 4i \not\equiv m \pmod{q}}}^{k-1} \frac{1}{\frac{-4i}{m} + 1} A_{\frac{-i}{m}},$$

and $k \equiv \frac{m+3}{4} \pmod{q}, 0 \leq k < q$.

Proof. Analogous to the proof of Theorem 2. □

To prove that q does not divide h^+ for $p \equiv -1 \pmod{q}$, the following Theorem 5 will be necessary.

Let j be an integer, $0 < j < 2q, j \equiv 0 \pmod{2}$. Define the sums

$$S_j = \sum_{i=1}^{\frac{q-1}{2}} A_i \sum_{\substack{k=1 \\ 2ji \not\equiv -k \pmod{q}}}^{j-1} \frac{1}{2ji+k} - \sum_{i=\frac{q+1}{2}}^{q-1} A_i \sum_{\substack{k=1 \\ 2ji \not\equiv -k \pmod{q}}}^{j-1} \frac{1}{2ji+k}.$$

Theorem 5. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1, l \equiv 3 \pmod{4}, p \equiv -1 \pmod{q}$, and let the order of the prime q modulo l be $\frac{l-1}{2}$. Suppose that for each j such that $S_j \equiv 0 \pmod{q}$ there exists $n, (n, 2q) = 1, n|p+1$ such that $S_{j^*} \not\equiv 0 \pmod{q}$, where $j^* \equiv nj \pmod{2q}$. Then q does not divide h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$.*

Proof. Let $2^v|p+1$ and let 2^{v+1} not divide $p+1$. Let n be a divisor of $p+1$, $(n, 2q) = 1$. Denote $M = 2^{v+1}n$. We shall compute the coefficient d_r , $r < l$ in (2), where $g^r \equiv \pm M \pmod{p}$. By Lemma 2 we have

$$d_r = \sum_{0 < i < \frac{p}{M}} A_i A_{Mi} + \sum_{\frac{p}{M} < i < \frac{2p}{M}} A_i A_{Mi+1} + \dots$$

It implies that

$$d_r \equiv S + \left(\frac{p+1}{qN} - \frac{1}{2} \right) \sum_{k=0}^{\frac{M}{2}-1} \sum_{i=1}^{q-1} A_i A_{Mi+k} \pmod{q},$$

where

$$\begin{aligned} S = & \sum_{i=1}^{\frac{q-1}{2}} A_i A_{Mi} + \sum_{i=\frac{q+1}{2}}^{q-1} A_i A_{Mi+1} + \sum_{i=1}^{\frac{q-1}{2}} A_i A_{Mi+2} + \sum_{i=\frac{q+1}{2}}^{q-1} A_i A_{Mi+3} \\ & + \dots + \sum_{i=1}^{\frac{q-1}{2}} A_i A_{Mi+\frac{M}{2}-2} + \sum_{i=\frac{q+1}{2}}^{q-1} A_i A_{Mi+\frac{M}{2}-1}. \end{aligned}$$

Therefore

$$S = \sum_{k=0}^{\frac{M}{4}-1} \sum_{i=1}^{q-1} A_i A_{Mi+2k} + \sum_{i=\frac{q+1}{2}}^{q-1} A_i \sum_{\substack{k=1 \\ Mi \not\equiv -k \pmod{q}}}^{\frac{M}{2}-1} \frac{1}{Mi+k}.$$

By Lemma 2 of [8] and Lemma 3 of [8] we get

$$\begin{aligned} \sum_{k=0}^{\frac{M}{4}-1} \sum_{i=1}^{q-1} A_i A_{Mi+2k} & \equiv -\frac{M}{4} \left(2 + \frac{M^{q-1}-1}{q} \right) \\ & - \frac{1}{2M} \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2}}}^{\frac{M}{2}-1} B_{q-2} \left(\frac{k}{M} \right) \pmod{q}. \end{aligned}$$

If $q|h^+$ then $d_r \equiv d_0 \pmod{q}$, hence

$$S + \frac{p+1}{q} \left(1 + \frac{1}{M} \sum_{k=0}^{\frac{M}{2}-1} \sum_{i=1}^{q-1} A_i A_{Mi+k} \right) - \frac{1}{2} \sum_{k=0}^{\frac{M}{2}-1} \sum_{i=1}^{q-1} A_i A_{Mi+k} \equiv -\frac{p+1}{q} \pmod{q}.$$

By [8] we have

$$1 + \frac{1}{M} \sum_{k=0}^{\frac{M}{2}-1} \sum_{i=1}^{q-1} A_i A_{Mi+k} \equiv -\frac{1}{2} \frac{M^{q-1}-1}{q} \pmod{q}.$$

The congruence

$$-\frac{p+1}{2q} \frac{M^{q-1}-1}{q} + \frac{M}{2} \left(\frac{1}{2} \frac{M^{q-1}-1}{q} + 1 \right) + S \equiv 0 \pmod{q}.$$

follows.

Substituting for S we get

$$-\frac{p+1}{2q} \frac{M^{q-1}-1}{q} + \sum_{i=\frac{q+1}{2}}^{q-1} A_i \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2} \\ Mi \not\equiv -k \pmod{q}}}^{\frac{M}{2}-1} \frac{1}{Mi+k} \equiv \frac{1}{2M} \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2}}}^{\frac{M}{2}-1} B_{q-2} \left(\frac{k}{M} \right).$$

By Theorem 1, $q|h^+$ implies that

$$\frac{p+1}{2q} \frac{M^{q-1}-1}{q} \equiv 0 \pmod{q}.$$

Therefore

$$\sum_{i=\frac{q+1}{2}}^{q-1} A_i \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2} \\ Mi \not\equiv -k \pmod{q}}}^{\frac{M}{2}-1} \frac{1}{Mi+k} \equiv \frac{1}{2M} \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2}}}^{\frac{M}{2}-1} B_{q-2} \left(\frac{k}{M} \right) \pmod{q}.$$

By Lemma 2 of [8] and Lemma 3 of [8] we get

$$(4) \quad \sum_{i=\frac{q+1}{2}}^{q-1} A_i \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2} \\ Mi \not\equiv -k \pmod{q}}}^{\frac{M}{2}-1} \frac{1}{Mi+k} \equiv \sum_{i=1}^{q-1} A_i \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2} \\ Mi \not\equiv -k \pmod{q}}}^{\frac{M}{2}-1} \frac{1}{Mi+k} \pmod{q}.$$

Clearly

$$\sum_{\substack{k=1 \\ k \equiv 1 \pmod{2} \\ Mi \not\equiv -k \pmod{q}}}^{2q-1} \frac{1}{Mi+k} \equiv 0 \pmod{q}.$$

Therefore the congruence (4) can be rewritten as follows

$$\sum_{i=\frac{q+1}{2}}^{q-1} A_i \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2} \\ 2ji \not\equiv -k \pmod{q}}}^{j-1} \frac{1}{2ji+k} - \sum_{i=1}^{\frac{q-1}{2}} A_i \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2} \\ 2ji \not\equiv -k \pmod{q}}}^{j-1} \frac{1}{2ji+k} \equiv 0 \pmod{q},$$

where $j \equiv 2^v n \pmod{2q}$.

Let $2^v | p+1$ and let 2^{v+1} not divide $p+1$. If p runs through all primes of the form $2l+1$, then the numbers $2^v \pmod{2q}$ run through the set $\{j | j = 2, 4, 6, \dots, 2q-2\}$. If $S_j \not\equiv 0 \pmod{q}$ for all $j = 2, 4, 6, \dots, 2q-2$, then q does not divide h^+ . Let $S_j \equiv 0 \pmod{q}$ for some j . For this j there exists the corresponding coefficient d_r , $r < l$, where $g^r \equiv \pm 2^{v+1} \pmod{p}$. Consider the coefficient $d_{r'}$, $r' < l$, where $g^{r'} \equiv \pm 2^{v+1} n \pmod{q}$, $n|p+1$, $(n, 2q) = 1$. If $q|h^+$, then $d_r \equiv d_{r'} \equiv d_0 \pmod{q}$. Hence $S_{j^*} \equiv 0 \pmod{q}$, where $j^* \equiv nj \pmod{2q}$. Theorem 5 is proved. \square

Theorem 6. *Let q be an odd prime. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, $p \equiv -1 \pmod{q}$, the order of the prime q modulo l be $\frac{l-1}{2}$ and let the congruence $2^{q-1} \equiv 3^{q-1} \equiv 1 \pmod{q^2}$ not hold.*

Suppose that q divides h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. Then for each k , $(k, q) = 1$, the following congruence holds:

$$k \frac{k^{q-1} - 1}{q} \equiv Q_{1+2kq, \frac{p}{2q}}(A_{-1}, A_{-2}, \dots, A_{-t}) \pmod{q},$$

where $t = kq$.

Proof. By Theorem 1 (i) put $n = \frac{p+1+2kq}{2q} = \frac{p+1}{2q} + k$. If $q|h^+$, then similarly as in the proof of Corollary 1 we get $\frac{p+1}{2q} \equiv 0 \pmod{q^2}$ and hence $n \equiv k \pmod{q^2}$. Clearly $n = \frac{p+1+2kq}{2q} \equiv \frac{p}{2q} \pmod{1+2kq}$ and Theorem 6 is proved. \square

Theorem 7. Let q be prime, $q \leq 23$. Let l, p be primes such that $p = 2l + 1$, $l \equiv 3 \pmod{4}$, and let the order of the prime q modulo l be $l - 1$ or $\frac{l-1}{2}$. The prime q does not divide h^+ , the class number of the real cyclotomic field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$.

Proof. If the order of q modulo l is $l - 1$, then q does not divide h^+ by [1] and [3]. Suppose that the order of q modulo l is $\frac{l-1}{2}$. For $q = 2, 3, 5$, Theorem 7 was proved in the papers [2],[5],[6].

Now we shall prove that q does not divide h^+ for $q = 7, 11, 13, 17, 19, 23$.

Let $p \equiv -1 \pmod{q}$. By a computation we get that $S_j \equiv 0 \pmod{q}$ if and only if either $j = q - 1$ or $j = q + 1$. Since $3|p + 1$, by Theorem 5 we get that q does not divide h^+ . On the basis of the Remark after Corollary 2, the case $m = 5$ need not be considered.

I. Case $q = 7$

By the assumption of Theorem 1, we have that the order of q modulo l is $\frac{l-1}{2}$. Therefore

$$1 = \left(\frac{7}{l}\right) = -\left(\frac{l}{7}\right).$$

Since $l \equiv 3, 5, 6 \pmod{7}$, then $p = 2l + 1 \equiv 4, 6 \pmod{7}$. Therefore $m = 1, 3$, i.e. either $p \equiv -1 \pmod{7}$ or $p \equiv -3 \pmod{7}$.

For $p \equiv -3 \pmod{7}$ by Corollary 1 we get

$$\frac{p + 3}{14} \frac{2^6 - 1}{7} \equiv 0 \pmod{7}.$$

By Theorem 2, I,(i) we have

$$\frac{p + 3}{14} \frac{3^6 - 1}{7} + \frac{1}{9} B_5 \left(\frac{1}{3}\right) \equiv C_3 \pmod{7}.$$

By computation,

$$\frac{3^6 - 1}{7} \equiv 6 \pmod{7}, C_3 \equiv 6 \pmod{7}, B_5 \left(\frac{1}{3}\right) \equiv 6 \pmod{7}.$$

Hence

$$6 \frac{p + 3}{14} + \frac{6}{9} \equiv 6 \pmod{7},$$

which is a contradiction

$$\frac{p + 3}{14} \frac{2^6 - 1}{7} \equiv 0 \pmod{7}.$$

II. Case $q = 11$

Analogously for $q = 7$ we get $m = 1, 5, 7, 9, 17$.

1. $m = 7, p \equiv -7 \pmod{11}$.

By Corollary 3, if $11|h^+$, then

$$\left(\sum_{\frac{11}{7} < i < \frac{22}{7}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{22}{7} < i < \frac{33}{7}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{11}{7} < i < \frac{22}{7}} \frac{1}{i}\right) \left(\sum_{\frac{22}{7} < i < \frac{33}{7}} \frac{1}{i}\right) \equiv 0 \pmod{11}.$$

By computation, we get that this sum is $10^2 + 3^2 + 3 \cdot 10 \equiv 7 \pmod{11}$, therefore 11 does not divide h^+ .

2. $m = 9, p \equiv -9 \pmod{11}$.

By Corollary 6, we have

$$\begin{aligned} &\left(\sum_{\frac{11}{9} < i < \frac{22}{9}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{22}{9} < i < \frac{44}{9}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{11}{9} < i < \frac{22}{9}} \frac{1}{i}\right) \left(\sum_{\frac{22}{9} < i < \frac{44}{9}} \frac{1}{i}\right) \\ &\equiv 6^2 + 7^2 + 6 \cdot 7 \equiv 6 \pmod{11}. \end{aligned}$$

Therefore 11 does not divide h^+ .

3. $m = 17, p \equiv -17 \pmod{11}$.

By Corollary 8, it is enough to prove that

$$Q_{17,4}(A_{\frac{-1}{17}}, A_{\frac{-2}{17}}, A_{\frac{-3}{17}}, A_{\frac{-4}{17}}, A_{\frac{-5}{17}}, A_{\frac{-6}{17}}, A_{\frac{-7}{17}}, A_{\frac{-8}{17}}) \not\equiv 0 \pmod{11}.$$

By computation we have

$$Q_{17,4}(A_{\frac{-1}{17}}, A_{\frac{-2}{17}}, A_{\frac{-3}{17}}, A_{\frac{-4}{17}}, A_{\frac{-5}{17}}, A_{\frac{-6}{17}}, A_{\frac{-7}{17}}, A_{\frac{-8}{17}}) \equiv 3 \pmod{11},$$

therefore 11 does not divide h^+ .

III. Case $q = 13$

In this case we have $m = 1, 5, 7, 17, 19, 23$.

1. $m = 7, p \equiv -7 \pmod{13}$. By Corollary 3,

$$\begin{aligned} &\left(\sum_{\frac{13}{7} < i < \frac{26}{7}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{26}{7} < i < \frac{39}{7}} \frac{1}{i}\right)^2 + \left(\sum_{\frac{13}{7} < i < \frac{26}{7}} \frac{1}{i}\right) \left(\sum_{\frac{26}{7} < i < \frac{39}{7}} \frac{1}{i}\right) \\ &\equiv 3^2 + 5^2 + 3 \cdot 5 \equiv 10 \pmod{13}, \end{aligned}$$

therefore 13 does not divide h^+ .

2. $m = 17, p \equiv -17 \pmod{13}$.

By computation, using Corollary 8, we get that

$$A_1 = 1, A_2 = 8, A_3 = 4, A_4 = 1, A_5 = 9, A_6 = 7, A_7 = 9, A_8 = 1, A_9 = 4, A_{10} = 8, A_{11} = 1, A_{12} = 0.$$

For the permutation $\phi_{17,2}$ we have

$$\phi_{17,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 5 & 3 & 1 \end{pmatrix},$$

hence

$$Q_{17,2}(X_1, X_2, \dots, X_8) = X_1^2 + X_2^2 + \dots + X_8^2 - (X_1X_2 + X_2X_4 + \dots + X_8X_1).$$

By computation modulo 13 we get

$$A_{\frac{-1}{17}} = A_3 = 4, A_{\frac{-2}{17}} = A_6 = 7, A_{\frac{-3}{17}} = A_9 = 4, A_{\frac{-4}{17}} = A_{12} = 0, A_{\frac{-5}{17}} = A_2 = 8, A_{\frac{-6}{17}} = A_5 = 9, A_{\frac{-7}{17}} = A_8 = 1, A_{\frac{-8}{17}} = A_{11} = 1.$$

Hence

$$Q_{17,2}(4, 7, 4, 0, 8, 9, 1, 1) \equiv 11 \pmod{13},$$

therefore 13 does not divide h^+ .

3. $m = 19, p \equiv -19 \pmod{13}$.

By Corollary 9 we have that

$$Q_{19,2}(8, 1, 7, 1, 8, 0, 1, 4, 9) \equiv 6 \pmod{13},$$

therefore 13 does not divide h^+ .

4. $m = 23, p \equiv -23 \pmod{13}$.

By Theorem 1 (i), putting $n = 2$, we get

$$\frac{p + 23}{26} \frac{2^{12} - 1}{13} \equiv Q_{23,2}(A_{\frac{-1}{23}}, \dots, A_{\frac{-11}{23}}) \pmod{13}.$$

By computation we have

$$\frac{p + 23}{26} \equiv 1 \pmod{13}.$$

Further we proceed using Theorem 2, III, (iii). The congruence (iii) can be rewritten as

$$\frac{p + m}{2q} \frac{3^{q-1} - 1}{q} \equiv -\frac{1}{9} B_{q-2} \left(\frac{1}{3} \right) + \frac{3^{q-1} - 1}{q} - A_{\frac{-1}{3}} + \sum_{i=1}^{\frac{q-4}{3}} \frac{1}{1+i} A_{\frac{i}{3}} \pmod{q}.$$

By substitution $m = 23, q = 13$ and by computation we get $\frac{3^{12}-1}{13} \equiv 8 \pmod{13}$, $B_{11}(\frac{1}{3}) \equiv 7 \pmod{13}$, $A_{\frac{-1}{3}} = A_4 = 1$, $\sum_{i=1}^3 \frac{1}{1+i} A_{\frac{i}{3}} \equiv 2 \pmod{13}$.

This implies the congruence

$$8 \frac{p + 23}{26} \equiv 1 \pmod{13},$$

which is a contradiction with the congruence

$$\frac{p + 23}{26} \equiv 1 \pmod{13}.$$

The case III, $q = 13$ is solved.

IV. Case $q = 17$

By computation we get that the corresponding values of m are $m = 1, 3, 7, 15, 25, 29, 31$.

1. $m = 3, p \equiv -3 \pmod{17}$.

By Theorem 1 (i) and Theorem 2 I.(ii), the following congruences hold:

$$\frac{p + 3}{34} \frac{2^{16} - 1}{17} \equiv 0 \pmod{17},$$

$$\frac{p + 3}{34} \frac{3^{16} - 1}{17} + \frac{2}{9} B_{15} \left(\frac{1}{3} \right) \equiv C_3 \pmod{17},$$

where

$$C_3 = A_{11}^2 + \sum_{i=2}^{12} \frac{1}{1-i} A_{\frac{-1}{3}}.$$

By computation we get that $C_3 = 5$, $B_{15}(\frac{1}{3}) \equiv 8 \pmod{17}$. Therefore

$$10 \frac{p+3}{34} \equiv 7 \pmod{17},$$

$$\frac{p+3}{34} \frac{2^{16}-1}{17} \equiv 0 \pmod{17},$$

—a contradiction.

2. $m = 7$, $p \equiv -7 \pmod{17}$.

By Corollary 3 it is enough to prove that

$$\left(\sum_{\frac{17}{7} < i < \frac{34}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{34}{7} < i < \frac{51}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{17}{7} < i < \frac{34}{7}} \frac{1}{i} \right) \left(\sum_{\frac{34}{7} < i < \frac{51}{7}} \frac{1}{i} \right) \not\equiv 0 \pmod{17}.$$

3. $m = 15$, $p \equiv -15 \pmod{17}$.

In this case by Theorem 1 (i) we have

$$\frac{p+15}{34} \frac{2^{16}-1}{17} \equiv Q_{15,2}(A_{\frac{-1}{15}}, \dots, A_{\frac{-7}{15}}) \pmod{17}.$$

By computation we get

$$\frac{p+15}{34} \frac{2^{16}-1}{17} \equiv Q_{15,2}(10, 1, 5, 10, 2, 16, 12) \pmod{17},$$

hence

$$13 \frac{p+15}{34} \equiv 2 \pmod{17}.$$

By Theorem 2 I, (i) we have

$$10 \frac{p+15}{34} \equiv 7 \pmod{17},$$

—a contradiction.

4. $m = 25$, $p \equiv -25 \pmod{17}$.

By Corollary 10, it is enough to prove that

$$Q_{25,2}(10, 12, 5, 8, 5, 12, 10, 0, 1, 16, 2, 10) \not\equiv 0 \pmod{17}.$$

By computation we get

$$Q_{25,2}(10, 12, 5, 8, 5, 12, 10, 0, 1, 16, 2, 10) \equiv 6 \pmod{17}.$$

5. $m = 29$, $p \equiv -29 \pmod{17}$.

By Theorem 1 (i) we have

$$\frac{p+29}{34} \frac{2^{16}-1}{17} \equiv Q_{29,2}(A_{\frac{-1}{29}}, \dots, A_{\frac{-14}{29}}) \pmod{17},$$

$$\frac{p+29}{34} \frac{4^{16}-1}{17} \equiv Q_{29,2}(A_{\frac{-1}{29}}, \dots, A_{\frac{-14}{29}}) \pmod{17}.$$

By computation we get

$$13 \frac{p+29}{34} \equiv 11 \pmod{17},$$

$$9 \frac{p+29}{34} \equiv 8 \pmod{17},$$

—a contradiction.

6. $m = 31, p \equiv -31 \pmod{17}$.

By Theorem 1 (i) we have

$$\frac{p + 31}{34} \frac{2^{16} - 1}{17} \equiv Q_{31,2}(A_{\frac{-1}{31}}, \dots, A_{\frac{-15}{31}}) \pmod{17},$$

$$\frac{p + 31}{34} \frac{3^{16} - 1}{17} \equiv Q_{31,3}(A_{\frac{-1}{31}}, \dots, A_{\frac{-15}{31}}) \pmod{17}.$$

By computation we get two congruences

$$\frac{p + 31}{34} \frac{2^{16} - 1}{17} \equiv 13 \pmod{17},$$

$$\frac{p + 31}{34} \frac{3^{16} - 1}{17} \equiv 13 \pmod{17},$$

—a contradiction.

V. Case $q = 19$

By computation we get that $m = 1, 7, 9, 11, 13, 17, 21, 31, 33$.

1. $m = 7, p \equiv -7 \pmod{19}$.

By Corollary 3 it is enough to prove

$$\left(\sum_{\frac{19}{7} < i < \frac{38}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{38}{7} < i < \frac{57}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{19}{7} < i < \frac{38}{7}} \frac{1}{i} \right) \left(\sum_{\frac{38}{7} < i < \frac{57}{7}} \frac{1}{i} \right) \not\equiv 0 \pmod{19}.$$

By computation we have that the left side is equal to $13 \pmod{19}$.

2. $m = 9, p \equiv -9 \pmod{19}$.

By Corollary 6 it is enough to prove that

$$\left(\sum_{\frac{19}{9} < i < \frac{38}{9}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{38}{9} < i < \frac{76}{9}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{19}{9} < i < \frac{38}{9}} \frac{1}{i} \right) \left(\sum_{\frac{38}{9} < i < \frac{76}{9}} \frac{1}{i} \right) \not\equiv 0 \pmod{19}.$$

By computation we have that the left side is equal to $2 \pmod{19}$.

3. $m = 11, p \equiv -11 \pmod{19}$.

By Theorem 1 (i) we have

$$\frac{p + 11}{38} \frac{2^{18} - 1}{19} \equiv Q_{11,2}(A_{\frac{-1}{11}}, \dots, A_{\frac{-5}{11}}) \pmod{19}.$$

By computation we get that

$$Q_{11,2}(A_{\frac{-1}{11}}, \dots, A_{\frac{-5}{11}}) \equiv Q_{11,2}(11, 14, 1, 15, 5) \equiv 15 \pmod{19}.$$

By Theorem 2 II, (ii) we have

$$\frac{p + 11}{38} \frac{3^{18} - 1}{19} + \frac{2}{9} B_{17} \left(\frac{1}{3} \right) \equiv C_{11} \pmod{19},$$

where

$$C_{11} = \sum_{i=1}^5 A_{\frac{-i}{11}} - \sum_{i=1}^5 A_{\frac{-1}{11}} A_{\frac{-3}{11}+1} + \sum_{i=1}^{16} \frac{1}{\frac{-3i}{11} + 1} A_{\frac{-i}{11}} \equiv 17 \pmod{19},$$

$$B_{17} \left(\frac{1}{3} \right) \equiv 13 \pmod{19}.$$

Therefore

$$3 \frac{p+11}{38} \equiv 15 \pmod{19},$$

$$18 \frac{p+11}{38} \equiv 12 \pmod{19},$$

—a contradiction.

4. $m = 13$, $p \equiv -13 \pmod{19}$.

By Corollary 7 it is enough to prove

$$Q_{13,2}(A_{\frac{-1}{13}}, \dots, A_{\frac{-6}{13}}) \not\equiv 0 \pmod{19}.$$

But

$$Q_{13,2}(11, 14, 15, 3, 10, 1) \equiv 3 \pmod{19}.$$

5. $m = 17$, $p \equiv -17 \pmod{19}$.

By Corollary 8 it is enough to prove

$$Q_{17,2}(A_{\frac{-1}{17}}, \dots, A_{\frac{-8}{17}}) \not\equiv 0 \pmod{19},$$

but

$$Q_{17,2}(15, 1, 3, 11, 11, 5, 14, 10) \equiv 18 \pmod{19}.$$

6. $m = 21$, $p \equiv -21 \pmod{19}$.

By Theorem 1 (i) we have

$$\frac{p+21}{38} \frac{2^{18}-1}{19} \equiv Q_{21,2}(A_{\frac{-1}{21}}, \dots, A_{\frac{-10}{21}}) \pmod{19},$$

$$\frac{p+21}{38} \frac{4^{18}-1}{19} \equiv Q_{21,4}(A_{\frac{-1}{21}}, \dots, A_{\frac{-10}{21}}) \pmod{19}.$$

By computation we get

$$Q_{21,2}(13, 0, 15, 1, 3, 11, 11, 5, 10) \equiv 4 \pmod{19},$$

$$Q_{21,4}(13, 0, 15, 1, 3, 11, 11, 5, 14, 10) \equiv 3 \pmod{19},$$

which gives a contradiction.

7. $m = 31$, $p \equiv -31 \pmod{19}$.

By Theorem 1 (i) we have

$$\frac{p+31}{38} \frac{2^{18}-1}{19} \equiv Q_{31,2}(A_{\frac{-1}{31}}, \dots, A_{\frac{-15}{31}}) \pmod{19},$$

$$\frac{p+31}{38} \frac{3^{18}-1}{19} \equiv Q_{31,3}(A_{\frac{-1}{31}}, \dots, A_{\frac{-15}{31}}) \pmod{19},$$

$$Q_{31,2}(3, 5, 10, 11, 1, 13, 1, 11, 10, 5, 3, 0, 15, 11, 14) \equiv 4 \pmod{19},$$

$$Q_{31,3}(3, 5, 10, 11, 1, 13, 1, 11, 10, 5, 3, 0, 15, 11, 14) \equiv 3 \pmod{19}.$$

By computation we get

$$3 \frac{p+31}{38} \equiv 3 \pmod{19},$$

$$18 \frac{p+31}{38} \equiv 7 \pmod{19},$$

—a contradiction.

8. $m = 33, p \equiv -33 \pmod{19}$.

By Theorem 1 (i) we have

$$\frac{p+33}{38} \frac{2^{18}-1}{19} \equiv Q_{33,2}(A_{\frac{-1}{33}}, \dots, A_{\frac{-16}{33}}) \pmod{19},$$

$$\frac{p+33}{38} \frac{4^{18}-1}{19} \equiv Q_{33,2}(A_{\frac{-1}{33}}, \dots, A_{\frac{-16}{33}}) \pmod{19}.$$

By computation we get

$$Q_{33,2}(10, 15, 11, 11, 1, 14, 13, 14, 1, 11, 11, 15, 10, 0, 5, 3) \equiv 18 \pmod{19},$$

$$Q_{33,4}(10, 15, 11, 11, 1, 14, 13, 14, 1, 11, 11, 15, 10, 0, 5, 3) \equiv 1 \pmod{19},$$

—a contradiction.

VI. Case $q = 23$

The possible values for m are $m = 1, 3, 5, 7, 11, 15, 17, 25, 31, 35$.

1. $m = 3, p \equiv -3 \pmod{23}$.

By Theorem 1 (i) we have

$$\frac{p+3}{46} \frac{2^{22}-1}{23} \equiv 0 \pmod{23}.$$

By Theorem 2 I.(ii) we get

$$\frac{p+3}{46} \frac{3^{22}-1}{23} + \frac{2}{9} B_{21} \left(\frac{1}{3} \right) \equiv C_3 \pmod{23}.$$

By computation we obtain that $C_3 \equiv 19 \pmod{23}$, $B_{21} \left(\frac{1}{3} \right) \equiv 13 \pmod{23}$, —a contradiction.

2. $m = 7, p \equiv -7 \pmod{23}$.

By Corollary 3 it is enough to prove

$$\left(\sum_{\frac{23}{7} < i < \frac{46}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{46}{7} < i < \frac{69}{7}} \frac{1}{i} \right)^2 + \left(\sum_{\frac{23}{7} < i < \frac{46}{7}} \frac{1}{i} \right) \left(\sum_{\frac{46}{7} < i < \frac{69}{7}} \frac{1}{i} \right) \not\equiv 0 \pmod{23}.$$

By computation we get that the sum is different from zero $\pmod{23}$.

3. $m = 11, p \equiv -11 \pmod{23}$.

By Theorem 2 II.(i) we have

$$\frac{p+11}{46} \frac{3^{22}-1}{23} + \frac{1}{9} B_{21} \left(\frac{1}{3} \right) \equiv C_{11} \pmod{23},$$

where

$$C_{11} = \sum_{i=1}^5 A_{\frac{-i}{11}} - \sum_{i=1}^5 A_{\frac{-1}{11}} A_{\frac{-3}{11}+1} + \sum_{i=1}^{11} \frac{1}{\frac{-3i}{11}+1} A_{\frac{-i}{11}} \equiv 3 \pmod{23}.$$

Hence

$$\frac{p+11}{46} \frac{3^{22}-1}{23} \equiv 22 \pmod{23}.$$

By Theorem 1 (i) we have

$$\frac{p+11}{46} \frac{2^{22}-1}{23} \equiv Q_{11,2}(A_{\frac{-1}{11}}, \dots, A_{\frac{-5}{11}}) \pmod{23}.$$

Therefore

$$\frac{p+11}{38} \frac{2^{18}-1}{19} \equiv 17 \pmod{23},$$

$$\frac{p+11}{46} \frac{3^{22}-1}{23} \equiv 22 \pmod{23},$$

—a contradiction.

4. $m = 15$, $p \equiv -15 \pmod{23}$.

By Theorem 1 (i) we have

$$\frac{p+15}{46} \frac{2^{22}-1}{23} \equiv Q_{15,2}(A_{\frac{-1}{15}}, \dots, A_{\frac{-7}{15}}) \equiv 4 \pmod{23}.$$

By Theorem 2 I.(ii)

$$\frac{p+15}{46} \frac{3^{22}-1}{23} + \frac{2}{9} B_{21} \left(\frac{1}{3} \right) \equiv C_{15} \pmod{23}.$$

By computation we get a contradiction.

5. $m = 17$, $p \equiv -17 \pmod{23}$.

By Corollary 8 it is enough to prove

$$Q_{17,4}(A_{\frac{-1}{17}}, \dots, A_{\frac{-8}{17}}) \not\equiv 0 \pmod{23},$$

$$Q_{17,4}(A_{\frac{-1}{17}}, \dots, A_{\frac{-8}{17}}) \equiv 8 \pmod{23}.$$

6. $m = 25$, $p \equiv -25 \pmod{23}$.

By Corollary 10 it is enough to prove

$$Q_{25,4}(A_{\frac{-1}{25}}, \dots, A_{\frac{-12}{25}}) \not\equiv 0 \pmod{23},$$

$$Q_{25,4}(A_{\frac{-1}{25}}, \dots, A_{\frac{-12}{25}}) \equiv 11 \pmod{23}.$$

7. $m = 31$, $p \equiv -31 \pmod{23}$.

By Theorem 1 (i) we have

$$\frac{p+31}{46} \frac{2^{22}-1}{23} \equiv Q_{31,2}(A_{\frac{-1}{31}}, \dots, A_{\frac{-15}{31}}) \equiv 13 \pmod{23},$$

$$\frac{p+31}{46} \frac{3^{22}-1}{23} \equiv Q_{31,2}(A_{\frac{-1}{31}}, \dots, A_{\frac{-15}{31}}) \equiv 15 \pmod{23},$$

—a contradiction.

8. $m = 35$, $p \equiv -35 \pmod{23}$.

By Theorem 2 II. (i) we have

$$\frac{p+35}{46} \frac{3^{22}-1}{23} + \frac{1}{9} B_{21} \left(\frac{1}{3} \right) \equiv C_{35} \pmod{23},$$

by computation we get the congruence

$$\frac{p+15}{46} \frac{3^{22}-1}{23} \equiv 10 \pmod{23}.$$

By Theorem 1 (i) we have

$$\frac{p + 35 \cdot 2^{22} - 1}{46 \cdot 23} \equiv Q_{35,2}(A_{\frac{-1}{35}}, \dots, A_{\frac{-17}{35}}) \equiv 0 \pmod{23},$$

—a contradiction. Theorem 7 is proved. \square

Now we give the values of j such that $S_j \equiv 0 \pmod{q}$ for $q \leq 173$ (see Theorem 5)

- | | |
|---|---|
| 1. $q = 29, j = 4, 28, 30, 54$ | 16. $q = 101, j = 38, 100, 102, 164$ |
| 2. $q = 31, j = 30, 32$ | 17. $q = 103, j = 102, 104$ |
| 3. $q = 37, j = 36, 38$ | 18. $q = 107, j = 68, 92, 106, 108, 122, 146$ |
| 4. $q = 41, j = 40, 42$ | 19. $q = 109, j = 108, 110$ |
| 5. $q = 43, j = 34, 42, 44, 52$ | 20. $q = 113, j = 112, 114$ |
| 6. $q = 47, j = 46, 48$ | 21. $q = 127, j = 12, 26, 116, 126, 128, 138, 228, 242$ |
| 7. $q = 53, j = 14, 48, 52, 54, 58, 92$ | 22. $q = 131, j = 130, 132$ |
| 8. $q = 61, j = 36, 60, 62, 86$ | 23. $q = 137, j = 76, 80, 136, 138, 194, 198$ |
| 9. $q = 67, j = 66, 68$ | 24. $q = 139, j = 56, 138, 140, 222$ |
| 10. $q = 71, j = 70, 72$ | 25. $q = 149, j = 2, 126, 148, 150, 172, 196$ |
| 11. $q = 73, j = 72, 74$ | 26. $q = 151, j = 84, 150, 152, 218$ |
| 12. $q = 79, j = 78, 80$ | 27. $q = 157, j = 12, 156, 158, 302$ |
| 13. $q = 83, j = 82, 84$ | 28. $q = 163, j = 162, 164$ |
| 14. $q = 89, j = 88, 90$ | 29. $q = 167, j = 166, 168$ |
| 15. $q = 97, j = 96, 98$ | 30. $q = 173, j = 80, 172, 174, 266$ |

By Theorem 5, putting $n = 3$, we obtain that q does not divide h^+ for $q \leq 173$.

By computation it was verified that the assumption of Theorem 5 (putting $n = 3$) is satisfied for all $q \leq 857$.

ACKNOWLEDGEMENT

The author thanks the referee of this article for the important suggestions and for the short proof of Lemma 1.

REFERENCES

- [1] D. Davis, Computing the number of totally positive circular units which are squares, *J. Number Theory* **10** (1978), p. 1-9. MR **57**:16254
- [2] D.R. Estes, On the parity of the class number of the field of q -th roots of unity, *The Rocky Mountain J. of Math.* **19** (1989), p.675-681. MR **92b**:11078
- [3] A. Granville and Zhi-Wei Sun, Values of Bernoulli Polynomials, *Pacific Journal of Mathematics*, Vol. 172 No. 1, (1996) 117-137. CMP 96:09
- [4] S. Jakubec, On divisibility of class number or real Abelian fields of prime conductor, *Abh. Math. Sem. Univ. Hamburg* **63** (1993), p. 67-86. MR **95a**:11098
- [5] S.Jakubec, On divisibility of h^+ by the prime 3. *The Rocky Mountain J. of Math*, Volume 24, Number 4 (1994) 1467-1473. MR **95m**:11119
- [6] S. Jakubec, On divisibility of h^+ by the prime 5. *Math. Slovaca* 44,5,(1994) 650-700. MR **96f**:11140
- [7] S. Jakubec, Connection between Wieferich congruence and divisibility of h^+ . *Acta Arithmetica* 71.1 (1995) 55-64. MR **96e**:11138
- [8] S. Jakubec, Connection between congruences $n^{q-1} \equiv 1 \pmod{q^2}$ and divisibility of h^+ , *Abh. Math. Sem. Univ. Hamburg* 66 (1996) 151-158. CMP 97:03
- [9] P.L. Montgomery, New solutions of $a^{p-1} \equiv 1 \pmod{p^2}$, *Math. Comp.* 61 (1993), 361-363. MR **94d**:11003

- [10] L.C. Washington, Introduction to Cyclotomic Fields, Springer-Verlag, New York, 1982. MR **85g**:11001
- [11] Zhi-Mong Sun and Zhi-Wei Sun, Fibonacci numbers and Fermat's Last theorem, Acta Arith. 60 (1992), 371-388. MR **93e**:11025

MATHEMATICAL INSTITUTE OF THE SLOVAK ACADEMY OF SCIENCES, ŠTEFÁNIKOVA 49, 814 73
BRATISLAVA, SLOVAKIA

E-mail address: jakubec@mau.savba.sk