MATHEMATICS OF COMPUTATION
Volume 67, Number 222, April 1998, Pages 593–602
S 0025-5718(98)00917-X

CONVERGENCE OF A RANDOM WALK METHOD FOR A PARTIAL DIFFERENTIAL EQUATION

WEIDONG LU

ABSTRACT. A Cauchy problem for a one–dimensional diffusion–reaction equation is solved on a grid by a random walk method, in which the diffusion part is solved by random walk of particles, and the (nonlinear) reaction part is solved via Euler's polygonal arc method. Unlike in the literature, we do not assume monotonicity for the initial condition. It is proved that the algorithm converges and the rate of convergence is of order $O(h)$, where h is the spatial mesh length.

1. INTRODUCTION

In this paper we study the approximating solutions to the following Cauchy problem,

$$u_t = \nu u_{xx} + g(t,u), \qquad t \in [0,T], \tag{1}$$

$$u(0,x) = u_0(x), \qquad x \in [0,l], \tag{2}$$

$$u(t,0) = u(t,l) = 0, \qquad t \in [0,T], \tag{3}$$

by a random walk method, precisely, we use the random fractional method which was introduced by Chorin [1]. There are three main features that make our results different from those in the literature. First, we consider the initial–boundary problem, in fact, if only considering the initial problem, we can't complete the algorithm smoothly. Second, we do not assume the monotonicity of $u_0(x)$ in x, which is the main difficulty we have overcome in this paper. By Hald's method, one can't obtain the fact $|E(u_j^n - \phi_j^n)| \le Mh$ without this monotonicity condition. Third, we allow the t–dependence of reaction term $g(t,u)$, but this is not very important.

Hald [6] proved the convergence of a random walk algorithm for the Cauchy problem (1)–(2) with $g(t,u) = f(u)$ which is independent of t. Our method is very similar to his, which allows the creation of the particles and the solution is constrained on the grid. Puckett [10] proposed an algorithm for the same equation and proved the convergence of the algorithm for the Kolmogorov equation $u_t = \nu u_{xx} + u(1-u)$. We note that [6] and [10] had the same monotonicity assumption on the initial data, and the mass of the particles was not allowed to be negative. But we will see below that all of those are not necessary. We should mention that Robert

Received by the editor July 20, 1995 and, in revised form, December 11, 1996.
1991 *Mathematics Subject Classification.* Primary 65C05, 65M99.
Key words and phrases. Random walk method, partial differential equation, Cauchy problem.
This work is partially supported by the Chinese State Education Commission Natural Science Foundation.

[12] proved the convergence of a random walk algorithm for the Burgers equation $u_t = \nu u_{xx} + uu_x$. In [10] and [12] the position of the particles is determined by a grid-free random walk technique. Especially, when they approximated the solution of $u_t = \nu u_{xx}$ by a random walk method, the Gaussian distributed random variables were used and $u(t, x)$ was approximated by the total mass of the particles to the right of the point x. Now, let us explain the fractional method which we are going to apply. The first step is to approximate the solution of the ordinary differential equation (regard x as a parameter).

$$u_t = g(t, u), \tag{4}$$

$$u(0, x) = u_0(x). \tag{5}$$

The basic idea is to approximate the gradient of the solution by a collection of particles on a grid. The masses of the particles are determined by solving the above ordinary differential equation. Note that we will not approximate the solution by the total mass of a collection of particles on a grid directly because we want the variance of the computed solution to be small.

The second step is to solve the diffusion equation,

$$u_t = \nu u_{xx}, \tag{6}$$

$$u(0, x) = u_0(x). \tag{7}$$

We simulate the diffusion process by randomly perturbing the position of particles that generate the numerical solution, according to the correspondence between the distribution of the position of particles undergoing random walks and the solution of the diffusion equation (see Chorin [1], Chorin and Marsden [3] and Feller [4] for more details).

Combining the above two steps, we obtain our approximate solution to (1)–(3). We show that the expected value of the computed solution tends to the solution of the partial differential equation and the variance tends to zero. As a matter of fact, we obtain the rate of convergence for our algorithm to being of order $O(h)$, where h is the spatial mesh length.

Some similar methods have been developed to solve other problems which contain diffusion. Chorin [2] developed a random vortex sheet method to solve the Prandtl boundary layer equation. Convergence proofs for the random walk method in the absence of boundaries may be found in [5] and [8]. Robert [11] gave some estimates of the convergence rate for the method. For the problem with boundary conditions Hald [7] proved the convergence of a random walk method in which particles are created at the boundary.

2. The method

In this section we will present our random walk algorithm. We consider (1)–(3). Suppose that ν is a constant $(0 < \nu < 1)$, g is a continuously differentiable function and $g(t, 0) = 0$ for all $t \in [0, T]$.

First of all, let us introduce some notation. We denote $x_j = jh$, $t_n = nK$, with $0 < h, K < 1$ and approximate u by $u_j^n = u(t_n, x_j)$. We call K the time step length and h the space mesh length. Now, we present the following random walk algorithm.

Step 0. Set

$$u(t_n, 0) = u(t_n, l) = 0, \quad n = 0, 1, \cdots, T/K.$$

Step 1. Set $n = 0$ and

$$u_j^0 = u(0, x_j) = u_0(x_j), \qquad j = 0, 1, \cdots, l/h.$$

Step 2. Define

$$v_j^n = u_j^n + Kg(t_n, u_j^n), \qquad j = 0, 1, \cdots, l/h.$$

Step 3. Compute the following numerical differentiation:

$$\xi_0^n = \xi_{l/h}^n = 0,$$

$$\xi_j^n = \xi(t_n, x_j) = \frac{v_j^n - v_{j+1}^n}{h}, \quad j = 1, \cdots, l/h - 1.$$

Step 4. Let $\alpha = K^2$, (thus, $0 < \alpha < 1$). We choose a positive integer N_j such that

$$N_j - 1 < \frac{|h\xi_j^n|}{\alpha} \leq N_j.$$

At x_j we place N_j particles with mass $k_j^n = \frac{h\xi_j^n}{N_j}$ for each.

Step 5. We let the particles randomly walk following such a rule:

For each particle at the place x_j, we associate it with a random variable X taking values -1,0,1 with probabilities $\frac{\nu}{2}$, $1 - \nu$ and $\frac{\nu}{2}$, respectively.

If $X = 1$, the particle jumps to x_{j+1} from x_j; if $X = -1$, then the particle jumps to x_{j-1} from x_j; otherwise the particle still stays at x_j.

Step 6. Let

$$\begin{aligned} hL_j^{n+1} = &\sum_{\ell=1}^{N_{j-1}} I_1(X_\ell^{(j-1)})k_{j-1}^n + \sum_{\ell=1}^{N_j} I_0(X_\ell^{(j)})k_j^n \\ &+ \sum_{\ell=1}^{N_{j+1}} I_{-1}(X_\ell^{(j+1)})k_{j+1}^n, \end{aligned} \tag{8}$$

$$u_j^{n+1} = \sum_{\ell \geq j} hL_\ell^{n+1}, \tag{9}$$

where $j = 1, \cdots, l/h - 1$ and $X_\ell^{(j)}$ denotes the random variable corresponding to the ℓ-th particle at x_j, $I_i(x)$ is the indicator function, $I_i(x) = 1$ if $x = i$ and 0 otherwise. Clearly, hL_j^{n+1} is the total number of particles at x_j after the random walk, which consists of three parts: The particles moving from x_{j-1}, the particles moving from x_{j+1}, and the particles staying at x_j. Note that u_j^{n+1} is the total mass of the particles to the right of the point x_j.

Step 7. If $nK \leq T$, then $n := n + 1$ and go to Step 2, continues the same process, where T is the given time.

In fact, in the above procedure, Steps 2–3 represent a reaction process via which the total mass of particles at point x_j is changed and it provides the initial state for the later diffusion process. In Step 4, we split the particles into smaller ones. This will be helpful in estimating the variance of u_j^n, which is essential in proving the convergence of the algorithm. Steps 4–5 represent a diffusion process which redistributes the particles by means of random walk. This essentially approximates (6).

3. Main result and its proof

In this section we give the main result and its proof. Let us state our main result.

Theorem. *Assume that g and u_0 are continuously differentiable functions, and that $g(t,0)=0$ for all $t\in[0,T]$. Let $0<h,K<1$, with $\frac{K}{h^2}=\frac{1}{2}$. Then there exists a constant M independent of h and K, such that*

$$|E(u_j^n)-u(t_n,x_j)|\le Mh, \quad j=0,1,2,\cdots,l/h-1.$$

$$Var(u_j^n)\le Mh, \quad 1\le n\le T/K,$$

where u_j^n is the computed solution given by the previous section.

Our main idea is to prove that the computed solution converges to a difference scheme of the partial differential equation (1)-(3). At first we give the following lemma.

Lemma 1. *Let u_j^n, v_j^n be defined as in the algorithm. Then they satisfy the following relations:*

$$E(u_j^{n+1})=\frac{\nu}{2}E(v_{j+1}^n)+(1-\nu)E(v_j^n)+\frac{\nu}{2}E(v_{j-1}^n), \tag{10}$$

$$\begin{aligned} Var(u_j^{n+1})\le{}&(1+3K)[\frac{\nu}{2}Var(v_{j+1}^n)+(1-\nu)Var(v_j^n)+\frac{\nu}{2}Var(v_{j-1}^n)]\\ &+10K\nu(E|v_{j-1}^n-v_j^n|+E|v_j^n-v_{j+1}^n|), \end{aligned}\tag{11}$$

$$E(v_j^n)=E(u_j^n)+Kg(t_n,E(u_j^n))+KM_j^nVar(u_j^n), \tag{12}$$

$$Var(v_j^n)\le(1+KM)Var(u_j^n), \tag{13}$$

where $j=1,\cdots,l/h-1$, M is an absolute constant, and $|M_j^n|\le M$ for all j and n.

Proof. In what follows, M represents a generic constant (independent of h,K etc.), which could be different from line to line.

We first prove (10). From the definition, we have

(14)

$$
\begin{aligned}
u_j^{n+1} &= \sum_{\ell \geq j} h L_\ell^{n+1} \\
&= \sum_{\ell \geq j} \{ \sum_{i=1}^{N_{\ell-1}} I_1(X_i^{(\ell-1)}) k_{\ell-1}^n + \sum_{i=1}^{N_\ell} I_0(X_i^{(\ell)}) k_\ell^n + \sum_{i=1}^{N_{\ell+1}} I_{-1}(X_i^{(\ell+1)}) k_{\ell+1}^n \} \\
&= \sum_{\ell_1+1 \geq j} \sum_{i=1}^{N_{\ell_1}} I_1(X_i^{(\ell_1)}) k_{\ell_1}^n + \sum_{\ell_1 \geq j} \sum_{i=1}^{N_{\ell_1}} I_0(X_i^{(\ell_1)}) k_{\ell_1}^n + \sum_{\ell_1-1 \geq j} \sum_{i=1}^{N_{\ell_1}} I_{-1}(X_i^{(\ell_1)}) k_{\ell_1}^n \\
&= \sum_{i=1}^{N_{j-1}} I_1(X_i^{(j-1)}) k_{j-1}^n - \sum_{i=1}^{N_j} I_{-1}(X_i^{(j)}) k_j^n \\
&\quad + \sum_{\ell_1 \geq j} \sum_{i=1}^{N_{\ell_1}} [(I_1(X_i^{(\ell_1)}) + I_0(X_i^{(\ell_1)}) + I_{-1}(X_i^{(\ell_1)})] k_{\ell_1}^n \\
&= \sum_{i=1}^{N_{j-1}} I_1(X_i^{(j-1)}) k_{j-1}^n - \sum_{i=1}^{N_j} I_{-1}(X_i^{(j)}) k_j^n + \sum_{\ell \geq j} h \xi_\ell^n \\
&= v_j^n + \sum_{\ell=1}^{N_{j-1}} I_1(X_\ell^{(j-1)}) k_{j-1}^n - \sum_{\ell=1}^{N_j} I_{-1}(X_\ell^{(j)}) k_j^n,
\end{aligned}
$$

where the second term on the right is the total mass of particles moving from x_{j-1} to x_j, and the third term is the total mass of particles moving from x_j to x_{j-1}. Thus (14) can be written in the following form:

$$
u_j^{n+1} = \frac{\nu}{2} v_{j+1}^n + (1-\nu) v_j^n + \frac{\nu}{2} v_{j-1}^n + D_{j-1}^+ + D_j^- + D_j^0, \tag{15}
$$

where

$$
\begin{aligned}
D_j^- &= \sum_{\ell=1}^{N_j} \frac{1}{N_j} [I_{-1}(X_\ell^{(j)}) - E(I_{-1}(X_\ell^{(j)}))](v_j^n - v_{j+1}^n), \\
D_j^0 &= \sum_{\ell=1}^{N_j} \frac{1}{N_j} [I_0(X_\ell^{(j)}) - E(I_0(X_\ell^{(j)}))](v_j^n - v_{j+1}^n), \\
D_j^+ &= \sum_{\ell=1}^{N_j} \frac{1}{N_j} [I_1(X_\ell^{(j)}) - E(I_1(X_\ell^{(j)}))](v_j^n - v_{j+1}^n).
\end{aligned}
$$

Now we compute $E(D_j^-)$, $E(D_j^0)$, $E(D_j^+)$ and estimate $Var(D_j^-)$, $Var(D_j^0)$, $Var(D_j^+)$, separately. Here $E(D)$, $Var(D)$ are the expected value and the variance of the random variable D, respectively.

Let $Y_j = v_j^n - v_{j+1}^n$ and $Z_\ell = I_1(X_\ell^{(j)}) - E(I_1(X_\ell^{(j)}))$. We assume that Y_j takes values $y_1, y_2, \ldots, y_m$ with probabilities $p_1, p_2, \ldots, p_m$, and the corresponding values of N_j are $n_1, n_2, \ldots, n_m$, respectively. Then, we have

$$
D_j^+ = \sum_{i=1}^{m} \frac{1}{n_i} (Z_1 + Z_2 + \cdots + Z_{n_i}) y_i I_{y_i}(Y_j),
$$

where $I_y(Y_j)$ is the indicator function that $I_y(Y_j) = 1$ if $Y_j = y$ and 0 otherwise. Note that Y_j and $Z_i, i = 1, 2, \ldots,$ are independent. Consequently,

$$E(D_j^+) = \sum_{i=1}^{m} \frac{y_i}{n_i} \sum_{\ell=1}^{n_i} E(Z_\ell) p_i = 0.$$

Similarly $E(D_j^-) = E(D_j^0) = 0$. Thus, we can conclude from (15) that

$$E(u_j^{n+1}) = \frac{\nu}{2} E(v_{j-1}^n) + (1-\nu) E(v_j^n) + \frac{\nu}{2} E(v_{j+1}^n)$$

This completes the proof of (10). □

Next, we prove (11). Note that

$$\begin{aligned} Var(D_j^+) &= \sum_{i=1}^{m} \left(\frac{y_i}{n_i}\right)^2 \sum_{\ell=1}^{n_i} Var(Z_\ell) p_i = \sum_{i=1}^{m} \left(\frac{y_i}{n_i}\right)^2 n_i \frac{\nu}{2}(1 - \frac{\nu}{2}) p_i \\ &\le \frac{\alpha\nu}{2} \sum_{i=1}^{m} |y_i| p_i = \frac{\alpha\nu}{2} E|v_j^n - v_{j+1}^n|. \end{aligned} \tag{16}$$

Here, we note that by Step 4, $\frac{y_i}{n_i} \le \alpha$. Similarly, we have

$$Var(D_j^-) \le \frac{\alpha\nu}{2} E|v_j^n - v_{j+1}^n|, \tag{17}$$

$$Var(D_j^0) \le \alpha\nu E|v_j^n - v_{j+1}^n|. \tag{18}$$

From (15), it follows that

$$\begin{aligned} Var(u_j^{n+1}) = {} & Var(V) + Var(D_{j-1}^+) + Var(D_j^-) + Var(D_j^0) \\ & + 2Cov(V, D_{j-1}^+) + 2Cov(V, D_j^-) + 2Cov(V, D_j^0) \\ & + 2Cov(D_{j-1}^+, D_j^-) + 2Cov(D_{j-1}^+, D_j^0) + 2Cov(D_j^-, D_j^0), \end{aligned} \tag{19}$$

where $V = \frac{\nu}{2} v_{j+1}^n + (1-\nu) v_j^n + \frac{\nu}{2} v_{j-1}^n$.

Since $Cov(V, D) \le \sqrt{Var(V)}\sqrt{Var(D)}$, we see that (note (16))

$$\begin{aligned} 2Cov(V, D_{j-1}^+) &\le 2\sqrt{Var(V)} \sqrt{\frac{\alpha\nu}{2} E|v_{j-1}^n - v_j^n|} \\ &\le \sqrt{\frac{\alpha}{2}} \big(Var(V) + \nu E|v_{j-1}^n - v_j^n|\big). \end{aligned}$$

Similarly,

$$2Cov(V, D_j^-) \le \sqrt{\frac{\alpha}{2}} \big(Var(V) + \nu E|v_j^n - v_{j+1}^n|\big),$$

$$2Cov(V, D_j^0) \le \sqrt{\alpha} \big(Var(V) + \nu E|v_j^n - v_{j+1}^n|\big),$$

also,

$$2Cov(D_{j-1}^+, D_j^-) \le \frac{\alpha\nu}{2} (E|v_{j-1}^n - v_j^n| + E|v_j^n - v_{j+1}^n|),$$

$$2Cov(D_{j-1}^+, D_j^0) \le \frac{\alpha\nu}{\sqrt{2}} (E|v_{j-1}^n - v_j^n| + E|v_j^n - v_{j+1}^n|),$$

$$2Cov(D_j^-, D_j^0) \le \sqrt{2}\alpha\nu (E|v_j^n - v_{j+1}^n|).$$

Hence, (19) becomes

$$Var(u_j^{n+1}) \le (1+3\sqrt{\alpha})Var(V) + (2\alpha\nu + \sqrt{\alpha}\nu)E|v_{j-1}^n - v_j^n| + (5\alpha\nu + 2\sqrt{\alpha}\nu)E|v_j^n - v_{j+1}^n|.$$

Since $\alpha = K^2$ and $K < 1$, we have

$$\begin{aligned} Var(u_j^{n+1}) &\le (1+3K)Var(V) + 10K\nu(E|v_{j-1}^n - v_j^n| + E|v_j^n - v_{j+1}^n|) \\ &\le (1+3K)[\frac{\nu}{2}Var(v_{j+1}^n) + (1-\nu)Var(v_j^n) + \frac{\nu}{2}Var(v_{j-1}^n)] \\ &\quad + 10K\nu(E|v_{j-1}^n - v_j^n| + E|v_j^n - v_{j+1}^n|). \end{aligned}$$

This completes the proof of (11).

Finally, we prove (12) and (13). From Step 2, it follows that

$$E(v_j^n) = E(u_j^n) + KE[g(t_n, u_j^n)], \tag{20}$$

$$Var(v_j^n) = Var(u_j^n) + 2KCov(u_j^n, g(t_n, u_j^n)) + K^2 Var(g(t_n, u_j^n)). \tag{21}$$

Let u_j^n take values $u_1,u_2,\cdots,u_m$ with probabilities p_1, p_2, $\cdots,p_m$ and denote $\mu = E(u_j^n) = \sum_r u_r p_r$. Then we have

$$\begin{aligned} E(g(t_n, u_j^n)) &= \sum_r [g(t_n,\mu) + g'(t_n,\mu)(u_r - \mu) \\ &\qquad + \frac{1}{2}g''(t_n, \mu + \theta_r(u_r - \mu))(u_r - \mu)^2]p_r \\ &= g(t_n, E(u_j^n)) + M_j^n Var(u_j^n), \end{aligned}$$

where $M_j^n = \frac{1}{2}g''(t_n, \mu + \theta_r(u_r - \mu))$ is bounded. Further,

$$\begin{aligned} Cov(u_j^n, g(t_n, u_j^n)) &= E\{(u_j^n - \mu)[g(t_n, u_j^n) - Eg(t_n, u_j^n)]\} \\ &= \sum_r (u_r - \mu)[g'(t_n, \mu + \theta_r(u_r - \mu))(u_r - \mu) - \theta\frac{M}{2}Var(u_j^n)]p_r \\ &\le MVar(u_j^n). \end{aligned}$$

$$\begin{aligned} Var(g(t_n, u_j^n)) &= E(g^2(t_n, u_j^n)) - [E(g(t_n, u_j^n))]^2 \\ &= g^2(t_n, E(u_j^n)) + \tilde{M}_j^n Var(u_j^n) \\ &\qquad - [g(t_n, E(u_j^n)) + M_j^n Var(u_j^n)]^2 \\ &\le MVar(u_j^n). \end{aligned}$$

Combining the above two results with (20)–(21), we arrive at

$$E(v_j^n) = E(u_j^n) + Kg(t_n, E(u_j^n)) + M_j^n Var(u_j^n),$$

$$Var(v_j^n) \le (1 + KM)Var(u_j^n),$$

for some constant M. This completes the proof of Lemma 1.

Proof of the theorem. Substituting (13) into (11), we have

$$\begin{aligned} Var(u_j^{n+1}) &\le (1+KM)[\frac{\nu}{2}Var(u_{j+1}^n) + (1-\nu)Var(u_j^n) + \frac{\nu}{2}Var(u_{j-1}^n)] \\ &\quad + 10K\nu(E|v_{j-1}^n - v_j^n| + E|v_j^n - v_{j+1}^n|). \end{aligned} \tag{22}$$

From Step 2, it follows that

$$v_{j-1}^n - v_j^n = u_{j-1}^n - u_j^n + K[g(t_n, u_{j-1}^n) - g(t_n, u_j^n)].$$

According to the mean value theorem, there exists a constant M such that

$$|v_{j-1}^n - v_j^n| \le (1+KM)|u_{j-1}^n - u_j^n|. \tag{23}$$

Substituting (23) into (22) gives

$$\begin{aligned} Var(u_j^{n+1}) \le (1+KM)[\frac{\nu}{2}Var(u_{j+1}^n) + (1-\nu)Var(u_j^n) + \frac{\nu}{2}Var(u_{j-1}^n)] \\ + 10K\nu(1+KM)[E|u_{j-1}^n - u_j^n| + E|u_j^n - u_{j+1}^n]. \end{aligned} \tag{24}$$

Next, from (8) and (9) and Step 4, it follows that

$$\begin{aligned} E|u_{j-1}^{n+1} - u_j^{n+1}| = E|hL_{j-1}^{n+1}| \le E|\sum_{\ell=1}^{N_{j-2}} \frac{1}{N_{j-2}} I_1(X_\ell^{(j-2)}) h\xi_{j-2}^n| \\ + E|\sum_{\ell=1}^{N_{j-1}} \frac{1}{N_{j-1}} I_0(X_\ell^{(j-1)}) h\xi_{j-1}^n| + E|\sum_{\ell=1}^{N_j} \frac{1}{N_j} I_{-1}(X_\ell^{(j)}) h\xi_j^n|. \end{aligned} \tag{25}$$

We estimate the right hand side term by term. Let us consider the first term. Let $h\xi_{j-2}^n$ take values $y_1, y_2, \cdots, y_m$ with probabilities $p_1, p_2, \cdots, p_m$ and the corresponding values of N_{j-2} be $n_1, n_2, \cdots, n_m$. Thus, we conclude that

$$\begin{aligned} E|\sum_{\ell=1}^{N_{j-2}} \frac{1}{N_{j-2}} I_1(X_\ell^{(j-2)}) h\xi_{j-2}^n| &= E|\sum_{i=1}^{m}\sum_{\ell=1}^{n_i} \frac{y_i}{n_i} I_1(X_\ell^{(j-2)}) I_{y_i}(h\xi_{j-2}^n)| \\ &\le \sum_{i=1}^{m}\sum_{\ell=1}^{n_i} \frac{|y_i|}{n_i} E|I_1(X_\ell^{(j-2)}) I_{y_i}(h\xi_{j-2}^n)| \\ &= \sum_{i=1}^{m}\sum_{\ell=1}^{n_i} \frac{|y_i|}{n_i} \frac{\nu}{2} p_i = \frac{\nu}{2} E|h\xi_{j-2}^n|, \end{aligned}$$

where we have used the fact that $X_\ell^{(j-2)}$ is independent of $h\xi_{j-2}^n$ and $E(I_1(X_\ell^{(j-2)})) = \frac{\nu}{2}$. Similarly, we can estimate the last two terms on the right hand side of (25),

$$\begin{aligned} E|\sum_{\ell=1}^{N_{j-1}} \frac{1}{N_{j-1}} I_0(X_\ell^{(j-1)}) h\xi_{j-1}^n| &\le (1-\nu) E|h\xi_{j-1}^n|, \\ E|\sum_{\ell=1}^{N_j} \frac{1}{N_j} I_{-1}(X_\ell^{(j)}) h\xi_j^n| &\le \frac{\nu}{2} E|h\xi_j^n|. \end{aligned}$$

Combining (23) with the above, we obtain,

$$\begin{aligned} E|u_{j-1}^{n+1} - u_j^{n+1}| &\le \frac{\nu}{2}E|h\xi_{j-2}^n| + (1-\nu)E|h\xi_{j-1}^n| + \frac{\nu}{2}E|h\xi_j^n| \\ &= \frac{\nu}{2}E|v_{j-2}^n - v_{j-1}^n| + (1-\nu)E|v_{j-1}^n - v_j^n| + \frac{\nu}{2}E|v_j^n - v_{j+1}^n| \\ &\le (1+KM)\{\frac{\nu}{2}E|u_{j-2}^n - u_{j-1}^n| + (1-\nu)E|u_{j-1}^n - u_j^n| \\ &\qquad + \frac{\nu}{2}E|u_j^n - u_{j+1}^n|\}. \end{aligned} \tag{26}$$

Now, let

$$\delta^n = \max_j Var(u_j^n),$$

and

$$\eta^n = \max_j E|u_{j-1}^n - u_j^n|.$$

Combining (24) and (26), we have

$$\delta^{n+1} \le (1+MK)\delta^n + 20K\nu(1+KM)\eta^n, \tag{27}$$

$$\eta^{n+1} \le (1+KM)\eta^n. \tag{28}$$

From these two inequalities and recalling the fact that $nK \le T$, $\delta_0 = 0$, $\eta^0 \le 2hC$ ($C = \max|u_0'|$), we conclude that

$$\begin{aligned}
\delta^{n+1} &\le (1+KM)\delta^n + 20K\nu(1+KM)^n\eta^0 \\
&\le (1+KM)\delta^n + C_1K\eta^0 \\
&= (1+KM)^2\delta^{n-1} + (1+KM)C_1K\eta^0 + C_1K\eta^0 \\
&\le \cdots \le (e^{MT}-1)\frac{C_1}{M}2hC = Mh,
\end{aligned} \tag{29}$$

where the constant is M independent of K, h. Combining (10), (12) and (29), we arrive at

(30)

$$\begin{aligned}
E(u_j^{n+1}) = \frac{\nu}{2}E(u_{j+1}^n) &+ (1-\nu)E(u_j^n) + \frac{\nu}{2}E(u_{j-1}^n) \\
&+ K[\frac{\nu}{2}g(t_n, E(u_{j+1}^n)) + (1-\nu)g(t_n, E(u_j^n)) + \frac{\nu}{2}g(t_n, E(u_{j-1}^n))] \\
&+ KMM_j^n h.
\end{aligned}$$

According to [9], the solution of the following difference scheme

$$\begin{aligned}
\Phi_j^{n+1} = \frac{\nu}{2}\Phi_{j+1}^n &+ (1-\nu)\Phi_j^n + \frac{\nu}{2}\Phi_{j-1}^n \\
&+ K[\frac{\nu}{2}g(t_n, \Phi_{j+1}^n) + (1-\nu)g(t_n, \Phi_j^n) + \frac{\nu}{2}g(t_n, \Phi_{j-1}^n)],
\end{aligned} \tag{31}$$

satisfies that

$$|\Phi_j^n - u(t_n, x_j)| \le C_3K + C_4\nu h^2.$$

Let $\sigma^n = \max_j |E(u_j^n) - \Phi_j^n|$, From (30) and (31), it can be easily found that

$$\begin{aligned}
\sigma^{n+1} &\le \sigma^n + KM\sigma^n + KMh \\
&\le (1+KM)\sigma^{n-1} + (1+KM)KMh + KMh \le \cdots \le Mh,
\end{aligned}$$

where we have used the fact that M_j^n is uniform bounded for all n, j. Hence,

$$|E(u_j^n) - \Phi_j^n| \le Mh.$$

The above yields

$$|E(u_j^n - u(t_n, x_j)| \le Mh + C_3h + C_4\nu h^2 \le Mh,$$

for some constant M. This completes the proof. □

Acknowledgments

The author thanks Professors J. Yong and X. Li for their helpful discussions.

References

[1] A. J. Chorin, *Numerical study of slightly viscous flows, J. Fluid Mech.*, 57(1973), 785–796. MR **52:**16280

[2] A. J. Chorin, *Vortex sheet approximation of boundary layer, J. Comp. Phys*, 27(1978), 428–442.

[3] A. J. Chorin & J. Marsden, *A Mathematical Introduction to Fluid Mechanics, Springer - Verlag, New York*, 1979. MR **81m:**76001

[4] W. Feller, *An Introduction to Probability Theory and Its Applications, 2nd, Wiley, New York*, 1971. MR **42:**5292

[5] J. Goodman, *Convergence of the random vortex method, Comm. Pure Appl. Math.*, 40(1987), 189–220. MR **88d:**35159

[6] O. H. Hald, *Convergence of random methods for a reaction-diffusion equation, SIAM J. Sci. Stat. Comput.*, 2(1981), 85–94. MR **83c:**65210

[7] O. H. Hald, *Convergence of a random method with creation of vorticity, SIAM. J. Sci. Statist. Comput.*, 7(1986), 1373–1386. MR **88a:**65013

[8] D. G. Long, *Convergence of the random vortex method in one and two dimensions, Ph.D. Thesis, Univ. of California, Berkeley*, 1986.

[9] R. D. Morton, *Difference Methods for Initial-Value Problem, 2nd. ed.*, Interscience Pub. New York, 1967. MR **36:**3515

[10] E. G. Puckett, *Convergence of a random particle method to solutions of the Kolmogorov equation, Math. Comp., 52(1989)*, 615–645. MR **90h:**65008

[11] S. G. Roberts, *Accuracy of random vortex method for a problem with non-smooth initial conditions, J. Comput. Phys.* 58(1985), 29–43. MR **86f:**76022

[12] S. G. Roberts, *Convergence of a random walk method for the Burgers equation, Math. Comp.*, 52(1989), 647–673. MR **89i:**65090

Department of Mathematics, Fudan University, Shanghai, 200433, China