

ON THE NON-EXISTENCE OF ε -UNIFORM FINITE DIFFERENCE METHODS ON UNIFORM MESHES FOR SEMILINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper fitted finite difference methods on a uniform mesh with internodal spacing h , are considered for a singularly perturbed semilinear two-point boundary value problem. It is proved that a scheme of this type with a frozen fitting factor cannot converge ε -uniformly in the maximum norm to the solution of the differential equation as the mesh spacing h goes to zero. Numerical experiments are presented which show that the same result is true for a number of schemes with variable fitting factors.

1. INTRODUCTION

In this paper fitted finite difference methods on a uniform mesh are considered for a singularly perturbed semilinear two-point boundary value problem. Singularly perturbed differential equations are all pervasive in applications of mathematics to problems in the sciences and engineering. Among these are the Navier-Stokes equations of fluid flow at high Reynolds number, the drift-diffusion equations of semiconductor device physics [19], [10], the Michaelis-Menten theory for enzyme reactions [16], and mathematical models of liquid crystal materials and of chemical reactions [27].

The use of classical numerical methods for solving such problems may give rise to difficulties when the singular perturbation parameter ε is small. In particular, methods based on centered differences or upwinded differences on uniform meshes yield error bounds, in the maximum norm, which depend on an inverse power of ε . Similarly Brandt and Yavneh [1] demonstrated that anisotropic artificial viscosity in the first-order upwind finite difference scheme may result in inaccurate solutions, when $\varepsilon/h = O(1)$, where h is the mesh width. Two alternative approaches may be taken to the resolution of this problem. Either additional information about the solution may be used to produce accurate efficient methods, which may involve *a priori* modification of the mesh or operator, or an attempt may be made to produce *a posteriori* adaptive methods or black box methods.

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The latter approach leads to codes that are designed to handle a wider variety of problems than non-adaptive codes, usually at the expense of greater execution time. Moreover, such methods are less suitable than non-adaptive codes to implementation in a parallel environment. This is because the adaptation process inherent in *a posteriori* methods, introduces sequentiality to the solution process, which is absent in the *a priori* case. The *a priori* approach uses physical or mathematical knowledge about the problem to enhance the solution strategy. Such methods are widespread in the literature. These include fitted finite difference methods [2], finite element methods using special elements such as exponential elements [18], and methods which use *a priori* refined or special meshes [12]. Examples of these include methods for convection-diffusion problems devised by the British Central Electricity Generating Board [14], fluid flow in aerodynamics [3], semiconductor device physics [20], [5], [15], [13], chemical reactions [26], and hydrologic models for the Nash cascade model of flood routing [25].

It is of theoretical and practical interest to consider numerical methods for such problems, which exhibit ε -uniform convergence, that is, numerical methods for which there exists an N_0 , independent of ε , such that for all $N \geq N_0$, where N is the number of mesh elements, the error constant and rate of convergence in the maximum norm are independent of ε . Thus a numerical method is said to be ε -uniform of order p on the mesh $\Omega_N = \{x_i, i = 0, 1, \dots, N\}$ if there exists an N_0 independent of ε such that for all $N \geq N_0$

$$\sup_{0 < \varepsilon \leq 1} \max_{\Omega_N} |u(x) - u_N(x)| \leq CN^{-p},$$

where u is the solution of the differential equation, u_N is the numerical approximation to u , C and $p > 0$ are independent of ε and N .

Singularly perturbed boundary value problems for linear elliptic equations, which reduce for $\varepsilon = 0$ to zero-order equations, were examined in [9], [21], [22], [23], [24]. For such problems ε -uniform methods consisting of exponentially fitted finite difference operators on uniform meshes were thoroughly investigated and applied successfully to ordinary differential equations in [2], [8] and to linear partial differential equations in [9], [21], [22], [23]. A sufficient condition for ε -uniform convergence, for linear ordinary differential equations, is that the scheme be fitted with the appropriate constant fitting factor in the region of the boundary layer. This was shown for the non-selfadjoint case in [4]. Schemes with constant exponential fitting factors (a special case of the frozen fitting factor schemes considered in this paper) for the linear self-adjoint problem were considered in [2, Chap. 10], and shown there to be ε -uniform.

The semilinear problem considered in this paper exhibits an exponential boundary layer, which is asymptotically similar in behavior to the layers arising in self-adjoint linear ordinary differential equations. It has been an area of speculation in the community, which considers ε -uniformly convergent methods, whether results of the type available widely in the literature for linear problems could also be obtained for nonlinear equations using fitted finite difference methods on uniform meshes. Previous attempts in this direction include schemes which are ε -uniformly convergent in weaker norms, such as the ℓ_1 and ℓ_2 norm (cf. Nijima [17]). The key issue in this paper is to show that, even in the case of this very simple non-linearity, ε -uniform convergence cannot be achieved in the ℓ_∞ norm using fitted finite difference methods on uniform meshes.

In this paper, it is shown that a general class of fitted finite difference methods on a uniform mesh, which includes well known exponentially fitted finite difference methods [2], are not ε -uniform pointwise in the maximum norm for a singularly perturbed semilinear two-point boundary value problem. To be precise, in section 2, we shall prove this result for schemes with a *frozen* fitting factor. The fitting factor is said to be frozen if, at points x_i in a neighborhood of the boundary layer at $x = 0$, it is determined by the quantities given at $x = 0$ alone. In section 3 numerical results are given, which indicate that this result holds not only for schemes with a frozen fitting factor but also for some standard fitted schemes from the literature, the fitting factors of which are not frozen.

It should be noted however that the result does not indicate that fitted methods on non-uniform meshes cannot be ε -uniform. In fact, in [7], numerical methods, ε -uniform in the maximum norm, are constructed for a class of semilinear problems, using classical finite difference operators on special piecewise-uniform meshes. Thus ε -uniform methods can be constructed on special piecewise-uniform meshes even though it is not possible on uniform meshes.

2. THEORETICAL RESULT FOR FROZEN FITTING FACTORS

In this section the class \mathcal{C} of semilinear two-point boundary value problems on $\Omega = (0, 1)$ of the form

$$(P) \quad \begin{cases} \varepsilon^2 u''(x) - c(u(x))u(x) = 0, & x \in \Omega, \\ u(0) = 1, & u(1) = 0, \end{cases}$$

are considered. Here c is a smooth function satisfying

$$(2.1) \quad c(u(x)) \geq \alpha > 0, \quad x \in \Omega,$$

and the singular perturbation parameter ε satisfies $\varepsilon > 0$. When $\varepsilon \ll 1$ the solutions of such problems exhibit boundary layers in small neighborhoods of the boundary point $x = 0$. These boundary layers are the cause of significant numerical difficulties, some consequences of which are given in the theorem below.

A finite difference method is considered on a uniform mesh $\Omega_N = \{x_i\}_1^{N-1}$, where $x_i = ih$, $0 \leq i \leq N$ and $Nh = 1$. On this mesh the standard second order central difference operator δ_x^2 is used to approximate the second-order derivative, where δ_x^2 is defined by:

$$\delta_x^2 w(x_i) = \frac{w(x_{i+1}) - 2w(x_i) + w(x_{i-1}))}{h^2},$$

for any mesh function w . The discrete problem corresponding to continuous problem (P) is then

$$(P_h) \quad \begin{cases} \varepsilon^2 \gamma_i \delta_x^2 z(x_i) - c(z(x_i))z(x_i) = 0, & x_i \in \Omega_N, \\ z(0) = 1, & z(1) = 0, \end{cases}$$

where γ_i is the fitting factor. In general, the fitting factor γ_i is determined at each point $x_i \in \Omega_N$ by the quantities $\varepsilon, h, c(z(x_{i-1})), c(z(x_i))$ and $c(z(x_{i+1}))$. The fitting factor is said to be frozen if, at points x_i in a neighborhood of the boundary layer at $x = 0$, it is determined by the quantities $\varepsilon, h, c(z(0))$ alone.

The main theoretical result of this paper states that there is no fitted central

finite difference method (P_h) with a frozen fitting factor on a uniform mesh, whose solutions converge ε -uniformly to the solution of problem (P) .

Theorem 2.1. *Let u be the solution of any problem (P) in the class \mathcal{C} and z the solution of the corresponding discrete problem (P_h) on the uniform mesh Ω_N . Assume that the fitting factor γ depends continuously on its arguments and that it is frozen so that for all $x \in [0, 1/4)$, $\gamma(x) = \gamma(\varepsilon, h, c(z(0)))$. Then, there is no choice of the fitting factor γ for which the solutions of (P_h) converge ε -uniformly to the solution u of (P) , as $N \rightarrow \infty$ for all problems (P) in \mathcal{C} .*

Proof. The theorem is proved by assuming that it is false and then deriving a contradiction. Thus, it is assumed that for all problems (P) in \mathcal{C} , there is an ε -uniform fitted finite difference method (P_h) on the uniform mesh Ω_N , with a frozen fitting factor such that $\gamma(x) = \gamma(\varepsilon, h, c(z(0)))$ for all $x \in [0, 1/4)$, with γ depending continuously on its arguments. That is, there exists $\mu = \mu(h)$, independent of ε , such that $|u(x_i) - z(x_i)| \leq \mu(h)$ where $\mu(h) \rightarrow 0$ as $h \rightarrow 0$.

Under these assumptions it will be shown that for any choice of the fitting factor the error at the point x_1 , namely $u(x_1) - z(x_1)$, does not converge to zero as $N \rightarrow \infty$ for a sequence of problems in \mathcal{C} for which εN is held constant. This provides the required contradiction.

It suffices to consider problems in \mathcal{C} corresponding to the following two choices of the coefficient c ,

$$(2.2) \quad c = c_s(u(x)) = 2 - s + su(x), \quad s = 0, 1.$$

The corresponding solutions of (P) and (P_h) are denoted by u_s and z_s respectively. It will be shown that either $u_0(x_1) - z_0(x_1)$ or $u_1(x_1) - z_1(x_1)$ does not converge to zero as $N \rightarrow \infty$ for the sequence of problems with $\varepsilon N = 1$.

It is clear that the coefficient in (2.2) fulfills condition (2.1) for the linear problem corresponding to $s = 0$. That the same is true when $0 < s \leq 1$ may be verified by a standard argument using the maximum principle.

It is more convenient to work with the following auxiliary problems in the semi-infinite domain $[0, \infty)$:

$$\begin{aligned} \varepsilon^2 v_s''(x) - c_s(v_s(x))v_s(x) &= 0, \quad x \in [0, \infty), \\ v_s(0) &= 1, \quad v_s(\infty) = 0. \end{aligned}$$

The exact solution of the linear problem corresponding to $s = 0$ is $v_0(x) = e^{-\sqrt{2}\varepsilon^{-1}x}$. Again, using a standard maximum principle argument, it is not hard to show that for $0 < s \leq 1$ the solutions $v_s(x)$ satisfy

$$(2.3) \quad e^{-\sqrt{2}\varepsilon^{-1}x} \leq v_s(x) \leq e^{-\sqrt{2-s}\varepsilon^{-1}x}, \quad x \in [0, \infty).$$

Moreover, on the interval $\bar{\Omega} = [0, 1]$, the difference between the solution u_s and the solution v_s of the corresponding auxiliary problem decreases as $\varepsilon \rightarrow 0$ in the sense that

$$(2.4) \quad |u_s(x) - v_s(x)| \leq \nu(\varepsilon), \quad x \in \bar{\Omega}, \quad s = 0, 1,$$

where $\nu(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Changing from x to the new variable $\eta(x) = x/\varepsilon$ the auxiliary problems become

$$\begin{aligned} w_s''(\eta) - c_s(w_s(\eta))w_s(\eta) &= 0, \quad \eta \in [0, \infty), \\ w_s(0) &= 1, \quad w_s(\infty) = 0, \end{aligned}$$

where $w_s(\eta) = v_s(x)$. Letting $w_s(\eta)$ denote the solution of this problem, the function $\beta(s)$ is chosen to satisfy the difference equation

$$\beta(s)\delta_\eta^2 w_s(\eta) - c_s(w_s(0))w_s(\eta) = 0,$$

at the point $\eta = \varepsilon^{-1}h$. Since $w_s(0) = 1$ it follows that $c_s(w_s(0)) = c_s(1) = 2$ for all $0 \leq s \leq 1$ and $\beta(s)$ satisfies

$$(2.5) \quad \beta(s)[\delta_\eta^2 w_s(\eta)]_{\eta=\ell} - 2w_s(\ell) = 0,$$

where $\ell = \varepsilon^{-1}h$.

To obtain an approximate expression for $\beta(s)$ as a power series in ℓ , the function $w_s(\eta)$ is expanded as a power series in η

$$(2.6) \quad w_s(\eta) = 1 + \sum_{r=1}^{\infty} k_r(s)\eta^r.$$

Then, by (2.5) and (2.6),

$$(2.7) \quad \begin{aligned} \beta(s) &= \frac{2(1 + \sum_{r=1}^{\infty} k_r(s)\ell^r)}{\sum_{r=2}^{\infty} k_r(s)[\delta_\eta^2 \eta^r]_{\eta=\ell}} \\ &= \frac{1 + k_1(s)\ell + k_2(s)\ell^2 + O(\ell^3)}{k_2(s) + 3k_3(s)\ell + 7k_4(s)\ell^2 + O(\ell^3)}. \end{aligned}$$

Expressions for the functions $k_r(s)$ are now obtained by substituting the expansion of $w_s(\eta)$ into the differential equation satisfied by $w_s(\eta)$, namely

$$w_s''(\eta) = (2 - s + sw_s(\eta))w_s(\eta),$$

and so

$$\left(\sum_{r=0}^{\infty} k_r(s)\eta^r\right)'' = \left(2 - s + s\sum_{r=0}^{\infty} k_r(s)\eta^r\right)\left(\sum_{r=0}^{\infty} k_r(s)\eta^r\right).$$

Equating coefficients of $1, \eta$ and η^2 on both sides and simplifying, the relations

$$(2.8) \quad k_2(s) = 1, \quad 6k_3(s) = (2 + s)k_1(s), \quad 12k_4(s) = 2 + s + sk_1^2(s)$$

are obtained.

Writing $w_s(2\ell) = e^{-\kappa(s)\ell}$, for some $\kappa(s)$, and expanding as a power series in ℓ , we obtain

$$w_s(2\ell) = 1 + \sum_{r=1}^{\infty} (-1)^r \frac{\kappa^r(s)\ell^r}{r!}.$$

Comparing this with the expansion (2.6) when $\eta = 2\ell$, and using $k_2(s) = 1$ from (2.8), it follows that

$$(2.9) \quad k_1(s) = -\frac{\kappa(s)}{2} + \left(\frac{\kappa^2(s)}{4} - 2\right)\ell + O(\ell^2).$$

Using (2.8) and (2.9), the expression (2.7) for $\beta(s)$ becomes

$$\begin{aligned}
\beta(s) &= \frac{1 + [-\frac{\kappa(s)}{2} + (\frac{\kappa^2(s)}{4} - 2)\ell]\ell + \ell^2 + O(\ell^3)}{1 + \frac{1}{2}(2+s)[-\frac{\kappa(s)}{2} + (\frac{\kappa^2(s)}{4} - 2)\ell]\ell + \frac{7}{12}[2+s + s\frac{\kappa^2(s)}{4}]\ell^2 + O(\ell^3)} \\
&= [1 - \frac{\kappa(s)\ell}{2} + (-1 + \frac{\kappa^2(s)}{4})\ell^2 + O(\ell^3)][1 - \frac{(2+s)\kappa(s)\ell}{4} + \\
&\quad + (-\frac{5(2+s)}{12} + \frac{(12+13s)\kappa^2(s)}{48})\ell^2 + O(\ell^3)]^{-1} \\
&= [1 - \frac{\kappa(s)\ell}{2} + (-1 + \frac{\kappa^2(s)}{4})\ell^2 + O(\ell^3)][1 + \frac{(2+s)\kappa(s)\ell}{4} - \\
&\quad - (-\frac{5(2+s)}{12} + \frac{(12+13s)\kappa^2(s)}{48})\ell^2 + \frac{(2+s)^2\kappa^2(s)\ell^2}{16} + O(\ell^3)] \\
&= 1 - \frac{\ell^2}{6} + \frac{\ell}{4} \left\{ s\kappa(s) + \ell \left[\frac{5s}{3} + \kappa^2(s) \left(-1 - \frac{13s}{12} + 1 + s + \frac{s^2}{4} - 1 - \frac{s}{2} + 1 \right) \right] \right\} \\
&\quad + O(\ell^3) \\
&= 1 - \frac{\ell^2}{6} + \frac{\ell s}{4} \left\{ \kappa(s) + \ell \left[\frac{5}{3} - \frac{7\kappa^2(s)}{12} + \frac{s\kappa^2(s)}{4} \right] \right\} + O(\ell^3) \\
&= \bar{\beta}(s) + O(\ell^3),
\end{aligned}$$

where

$$(2.10) \quad \bar{\beta}(s) = 1 - \frac{\ell^2}{6} + \frac{\ell s}{4} \left\{ \kappa(s) + \ell \left[\frac{5}{3} - \frac{7}{12}\kappa^2(s) + \frac{s\kappa^2(s)}{4} \right] \right\}.$$

The following bounds for $\kappa(s)$ are obtained from (2.3)

$$(2.11) \quad 2\sqrt{2-s} \leq \kappa(s) \leq 2\sqrt{2}.$$

From (2.10)

$$\begin{aligned}
\bar{\beta}(0) &= 1 - \frac{\ell^2}{6}, \\
\bar{\beta}(1) &= 1 - \frac{\ell^2}{6} + \frac{\ell}{4} \left\{ \kappa(1) + \ell \left[\frac{5}{3} - \frac{7}{12}\kappa^2(1) + \frac{\kappa^2(1)}{4} \right] \right\}
\end{aligned}$$

and so

$$\bar{\beta}(1) - \bar{\beta}(0) = \frac{\kappa(1)\ell}{4} + O(\ell^2).$$

It follows from (2.11) that $\kappa(1) \geq 2$ and thus, for all sufficiently small ℓ ,

$$(2.12) \quad \bar{\beta}(1) - \bar{\beta}(0) \geq \frac{\ell}{4}.$$

Putting

$$(2.13) \quad \beta^* = \bar{\beta}(0) + \frac{\ell}{8},$$

the difference equation (2.5) can be rewritten as

$$\frac{\beta^*}{2(\beta^* + \ell^2)}(w_s(0) + w_s(2\ell)) = w_s(\ell) + \frac{(\beta^* - \beta(s))\ell^2}{\beta(s)(\beta^* + \ell^2)}w_s(\ell).$$

Then, for all sufficiently small ℓ and some constant $m_0 > 0$, from (2.10), (2.13) and (2.3), it follows that

$$(2.14) \quad \frac{\beta^*}{2(\beta^* + \ell^2)}(w_0(0) + w_0(2\ell)) \geq w_0(\ell) + m_0\ell^3$$

and, from (2.10), (2.12), (2.13) and (2.3),

$$(2.15) \quad \frac{\beta^*}{2(\beta^* + \ell^2)}(w_1(0) + w_1(2\ell)) \leq w_1(\ell) - m_0\ell^3.$$

Consider the difference scheme (P_h) , with frozen fitting factor $\tilde{\gamma}$, applied to problems (P) with the coefficient c_s that is

$$(\tilde{P}_h) \quad \begin{cases} \varepsilon^2 \tilde{\gamma} \delta_x^2 z_s(x_i) - c_s(z_s(x_i))z_s(x_i) = 0, & x_i \in \Omega_N, \\ z_s(0) = 1, \quad z_s(1) = 0, \end{cases}$$

At the point x_1 , this can be written in the form

$$z_s(x_1) = \frac{\tilde{\gamma}}{2\tilde{\gamma} + \ell^2(2 - s + sz_s(x_1))}(z_s(x_2) + z_s(0)),$$

where $\tilde{\gamma} = \tilde{\gamma}(\varepsilon, h, c_s(z_s(0))) = \tilde{\gamma}(\varepsilon, h, c_s(1)) = \tilde{\gamma}(\varepsilon, h, 2)$. Note that $\tilde{\gamma}$ is independent of s . We first show that ε -uniform convergence implies $\tilde{\gamma} \geq 0$ for all sufficiently small h and ε , satisfying $l = h/\varepsilon = \text{const}$. From \tilde{P}_h , it is clear that $\tilde{\gamma} \geq 0$, if $z_0(x_1) \geq 0$ and $\delta_x^2 z_0(x_1) > 0$.

By the maximum principle, $u_0(x) \geq 0, x \in (0, 1/4]$. Now, by uniform convergence,

$$|u_s(x) - z_s(x)| \leq \mu(h),$$

where $\mu(h) \rightarrow 0$ as $h \rightarrow 0$. Hence

$$z_s(x_i) \geq 0, \quad x_i \in (0, 1/4], \quad h \text{ sufficiently small.}$$

It remains to show $\delta_x^2 z_0(x_1) \geq 0$. To do this rewrite the equations for u_s and z_s in scaled coordinates $\eta(x) = x/\varepsilon$ thus:

$$\frac{d^2}{d\eta^2} \tilde{u}_s(\eta) - c_s(\tilde{u}_s(\eta_i))\tilde{u}_s(\eta) = 0,$$

$$\tilde{u}_s(0) = 1 \quad , \quad \tilde{u}_s(1/\varepsilon) = 0 \quad ,$$

and

$$\tilde{\gamma} \delta_\eta^2 \tilde{z}_s(\eta_i) - c_s(\tilde{z}_s(\eta_i))\tilde{z}_s(\eta_i) = 0,$$

$$\tilde{z}_s(0) = 1 \quad , \quad \tilde{z}_s(1/\varepsilon) = 0 \quad .$$

Recalling $l = h/\varepsilon$, we have

$$(2.16) \quad \begin{aligned} \delta_\eta^2 \tilde{z}_s(\eta_i) &= \{\tilde{z}_s(\eta_i + l) - 2\tilde{z}_s(\eta_i) + \tilde{z}_s(\eta_i - l)\}/l^2 \\ &= \{\tilde{z}_s(\eta_i + l) - \tilde{u}_s(\eta_i + l)\}/l^2 - 2\{\tilde{z}_s(\eta_i) - \tilde{u}_s(\eta_i)\}/l^2 \\ &\quad + \{\tilde{z}_s(\eta_i - l) - \tilde{u}_s(\eta_i - l)\}/l^2 + \delta_\eta^2 \tilde{u}_s(\eta_i) \\ &\geq -4\mu(h)/l^2 + \delta_\eta^2 \tilde{u}_s(\eta_i). \end{aligned}$$

Now

(2.17)

$$\delta_\eta^2 \tilde{u}_s(\eta_i) = \frac{d^2}{d\eta^2} \tilde{u}_s(\eta_i + \theta_i) = c_s(\tilde{u}_s) \tilde{u}_s(\eta_i + \theta_i) \geq \alpha \tilde{u}_s(\eta_i + \theta_i), \quad 0 \leq \theta_i \leq l.$$

Using (2.4), we have

$$(2.18) \quad |\tilde{u}_s(\eta) - w_s(\eta)| = |u_s(x) - v_s(x)| \leq \nu(\varepsilon),$$

where $\nu(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We now restrict our attention to the case required, that is $u_0(x)$. We have

$$w_0(\eta_1 + \theta_1) = v_0(x_1/\varepsilon + \theta_1) \leq v_0(2h/\varepsilon) = v_0(2l) = e^{-2\sqrt{2}l}.$$

Using (2.18), we now have

$$\tilde{u}_0(\eta_1 + \theta_1) \geq w_0(\eta_1 + \theta_1) - \nu(\varepsilon) = e^{-2\sqrt{2}l} - \nu(\varepsilon).$$

Thus, for ε sufficiently small,

$$(2.19) \quad \tilde{u}_0(\eta_1 + \theta_1) \geq e^{-4l} > 0.$$

Hence, from (2.16), (2.17) and (2.19), since l is a constant, we have

$$\delta_\eta^2 \tilde{z}_0(\eta_1) \geq -4\mu(h)/l^2 + e^{-4l} > 0$$

for h sufficiently small, and the required result, $\tilde{\gamma} \geq 0$ follows.

The argument is now divided into the two possible cases $\tilde{\gamma} \geq \beta^*$ and $\tilde{\gamma} \leq \beta^*$.

Suppose first that $\tilde{\gamma} \geq \beta^*$ and consider problems corresponding to $s = 0$. Then, for all sufficiently small ℓ , using the assumption of ε -uniform convergence, (2.4) and (2.14) we obtain

$$\begin{aligned} z_0(x_1) &= \frac{\tilde{\gamma}}{2(\tilde{\gamma} + \ell^2)} (z_0(x_2) + z_0(0)) \\ &\geq \frac{\tilde{\gamma}}{2(\tilde{\gamma} + \ell^2)} (u_0(x_2) + u_0(0) - \mu(h)) \\ &\geq \frac{\tilde{\gamma}}{2(\tilde{\gamma} + \ell^2)} (v_0(x_2) + v_0(0) - \nu(\varepsilon) - \mu(h)) \\ &= \frac{\tilde{\gamma}}{2(\tilde{\gamma} + \ell^2)} (w_0(x_2) + w_0(0) - \nu(\varepsilon) - \mu(h)) \\ &\geq \frac{\tilde{\gamma}}{2(\tilde{\gamma} + \ell^2)} \left[\frac{2(\beta^* + \ell^2)}{\beta^*} (w_0(\ell) + m_0 \ell^3) - \nu(\varepsilon) - \mu(h) \right] \\ &\geq w_0(\ell) + m_0 \ell^3 - \nu(\varepsilon) - \mu(h) \\ &\geq u_0(x_1) + m_0 \ell^3 - \nu(\varepsilon) - \mu(h), \end{aligned}$$

since $\tilde{\gamma} \geq \beta^*$ implies that $\frac{\tilde{\gamma}(\beta^* + \ell^2)}{(\tilde{\gamma} + \ell^2)\beta^*} \geq 1$. Fixing ℓ sufficiently small, and considering the sequence of problems corresponding to $\varepsilon = \frac{h}{N\ell} = \frac{1}{N\ell}$, it follows that

$$z_0(x_1) - u_0(x_1) \geq \frac{1}{2} m_0 \ell^3,$$

for all sufficiently small h , which contradicts the assumption of ε -uniform convergence of the method for these problems.

On the other hand if $\tilde{\gamma} \leq \beta^*$, using the assumption of ε -uniform convergence, (2.4) and (2.15), a similar argument for problems corresponding to $s = 1$ gives for all sufficiently small ℓ that

$$\begin{aligned} z_1(x_1) &= \frac{\tilde{\gamma}}{2\tilde{\gamma} + \ell^2(1 + z_1(x_1))}(z_1(x_2) + z_1(0)) \\ &\leq \frac{\tilde{\gamma}}{2\tilde{\gamma} + \ell^2(1 + z_1(x_1))} \left[\frac{2(\beta^* + \ell^2)}{\beta^*}(w_1(\ell) - m_0\ell^3) + \nu(\varepsilon) + \mu(h) \right] \\ &\leq u_1(\ell) - m_0\ell^3 + \nu(\varepsilon) + \mu(h), \end{aligned}$$

since $\tilde{\gamma} \leq \beta^*$ implies that $\frac{\tilde{\gamma}(\beta^* + \ell^2)}{(\tilde{\gamma} + \ell^2)\beta^*} \leq 1$.

Again, fixing ℓ sufficiently small, and considering the sequence of problems corresponding to $\varepsilon = \frac{h}{\ell} = \frac{1}{N\ell}$, it follows that

$$u_1(x_1) - z_1(x_1) \geq m_0\ell^3 - \nu(\varepsilon) - \mu(h) \geq \frac{1}{2}m_0\ell^3,$$

for all sufficiently small h , which contradicts the assumption of ε -uniform convergence.

Thus it has been shown that the assumption of ε -uniform convergence leads to a contradiction in all cases, which completes the proof of the theorem. \square

3. NUMERICAL RESULTS

We shall now examine numerically a number of fitted schemes on uniform meshes for the continuous problem (P) and related problems with two boundary layers. We shall first consider schemes of the form

$$(P_h) \quad \begin{cases} \varepsilon\gamma_i\delta_x^2 z_N(x_i) - c(z_N(x_i))z_N(x_i) = 0, & x_i \in \Omega_N, \\ z_N(0) = 1, \quad z_N(1) = 0, \end{cases}$$

where $\gamma_i \equiv \gamma(\varepsilon, h, c(z_N(0)))$ is the frozen fitting factor. Note that, in order to preserve consistency with schemes given in the literature, this and subsequent problems have an ε as coefficient of the second derivative, rather than the ε^2 considered previously.

The nonlinear finite difference method (P_h) is linearized using a continuation method of the form :

$$L_t^h u_N \equiv \varepsilon\gamma_i\delta_x^2 u_N(x, t_j) - c(u_N(x, t_{j-1}))u_N(x, t_j) - D_t^- u_N(x, t_j) = 0, \quad j = 1, \dots, K,$$

$$u_N(0, t_j) = u(0), \quad u_N(1, t_j) = u(1) \text{ for all } j,$$

$$u_N(x, 0) = u_{init}(x),$$

where $\gamma_i \equiv \gamma(\varepsilon, h, c(u_N(0, t_{j-1}))) \equiv \gamma(\varepsilon, h, c(u(0)))$ is the frozen fitting factor. Various starting values $u_{init}(x)$ are chosen. The number of iterations K and the choice of uniform time step $h_t = t_j - t_{j-1}$ are discussed below. With the definition

$$e(j) \equiv \max_{1 \leq i \leq N} |u_N(x_i, t_j) - u_N(x_i, t_{j-1})|/h_t, \text{ for } j = 1, 2, \dots, K,$$

the time step h_t is chosen sufficiently small so that

$$(3.1) \quad e(j) \leq e(j-1), \text{ for } 1 < j \leq K,$$

and the number of iterations K is chosen such that

$$(3.2) \quad e(K) \leq \text{TOL},$$

where TOL is some prescribed small tolerance.

The numerical solution is obtained as follows:

Start with $h_t = 0.0625$. If, at some value of j , (3.1) is not satisfied, then halve the time step until (3.1) is satisfied. Continue the iterations until either (3.2) is satisfied or until $K = 90$. If (3.2) is not satisfied, then repeat the entire process starting with $h_t = 0.03125$.

The resulting values of $u_N(x, K)$ are taken as approximations to the solution of the continuous problem.

The problem is solved on a sequence of meshes, with $N = 8, 16, 32, 64, 128, 256, 512, 1024$ and for $\varepsilon = 2^{-n}$, $n = 1, 2, \dots, j_{red}$, where j_{red} is chosen so that ε is a value at which the rate of convergence stabilizes, which normally occurs when, to machine accuracy, we are solving the reduced problem.

The errors $|u_N(x_i, K) - u(x_i)|$ are approximated on each mesh for successive values of ε by $e_{\varepsilon, N}(i) = |u_N(x_i, K) - u^I(x_i, K)|$, where $u^I(x, K)$ is defined by linear interpolation on each subinterval $[y_{j-1}, y_j]$ by

$$u^I(x, K) = u^*(y_{j-1}, K) + (u^*(y_j, K) - u^*(y_{j-1}, K)) \frac{x - y_{j-1}}{y_j - y_{j-1}}, \quad 1 \leq j \leq 1024,$$

where the nodal values $\{u^*(y_j, K)\}_{j=0}^{1024}$ are obtained from the solution of the finite difference method L_t^h with $N = 1024$. For each ε and each N the maximum nodal error is approximated by

$$E_{\varepsilon, N} = \max_i e_{\varepsilon, N}(i).$$

For each N , the ε -uniform maximum nodal error is approximated by

$$E_N = \max_{\varepsilon} E_{\varepsilon, N}.$$

A numerical method for solving (P) is ε -uniform of order p on the mesh $\Omega_N = \{x_i, i = 0, 1, \dots, N\}$ if

$$\sup_{0 < \varepsilon \leq 1} \max_{\Omega_N} |u(x) - u_N(x, K)| \leq CN^{-p},$$

where u is the solution of (P) , u_N is the numerical approximation to u , C and $p > 0$ are independent of ε and N . An approximation to p , the ε -uniform rate of convergence, was determined using a variation of the double mesh method described in [6]. This involves calculating the double mesh error

$$D_{\varepsilon, N} = \max_{\Omega_N} |u_N(x_i, K) - u_{2N}^I(x_i, K)|,$$

which is the difference between the values of the solution on a mesh of N points and the interpolated value for the solution, at the same point, on a mesh of $2N$

points. For each value of N the quantities

$$D_N = \max_{\varepsilon} D_{\varepsilon, N}, \quad p_N = \log_2 \left(\frac{D_N}{D_{2N}} \right),$$

are computed. The values of p_N are the approximations to p .

We now present numerical results, first for a problem of the form (P), and secondly for a generalization of that problem. All calculations were carried out in double-precision FORTRAN 77 on an Hewlett-Packard/Apollo 730. The first scheme we consider is the unfitted central difference scheme, where $\gamma_i \equiv 1$, and the second is a modification to the single layer case of the constant fitting factor version of the scheme of Miller [11] proposed in [2, Ch. 10, p. 156]. This gives a frozen fitting factor method for the problem

$$\varepsilon u''(x) - c(u(x))u(x) = 0, x \in \Omega$$

with boundary layer at $x = 0$. In this case the frozen fitting factor is given by:

$$(3.3) \quad \gamma_i = \frac{c(u(0))h^2}{4\varepsilon} \sinh^{-2} \frac{\sqrt{c(u(0))h}}{2\sqrt{\varepsilon}}.$$

Table 1 gives uniform errors and rates of uniform convergence for the centered difference method for the problem

$$(3.4) \quad \varepsilon \frac{d^2}{dx^2} u(x) - u - u^2 = 0, x \in (0, 1),$$

$$u(0) = 1, \quad u(1) = 0.$$

TABLE 1. Maximum errors E_N and rate of convergence p_N for the Centered Difference scheme

Boundary Conditions: $u(0) = 1.0, u(1) = 0.0$						
Initial Guess : $u_{init} = u(x) = u(0) + (u(1) - u(0))x$						
N	8	16	32	64	128	256
E_N	.048165	.048145	.048089	.047872	.047021	.043776
p_N	.00	.00	.00	.00	.00	.00

Tables 2 and 3 give errors and rates of uniform convergence for the scheme with frozen fitting factor given by (3.3).

We now show numerically that the result of Theorem 2.1 also holds for fitted methods with frozen and non-frozen fitting factors for the problem

$$(P2) \quad \begin{cases} \varepsilon u''(x) - c(u(x))u(x) = 0, x \in \Omega, \\ u(0) = A, u(1) = B, \end{cases}$$

where c satisfies (2.1). This has, in general, boundary layers at both $x = 0$ and $x = 1$. The nature of these boundary layers and the behavior of the derivatives of u in the neighborhood of $x = 0$ and $x = 1$ is similar to that of the layer in (P).

TABLE 2. Errors $E_{\epsilon, N}$ and E_N for Frozen Fitting Factor scheme (3.3)

Boundary Conditions: $u(0) = 1.0$, $u(1) = 0.0$							
Initial Guess : $u_{init} = u(x) = u(0) + (u(1) - u(0))x$							
ϵ	Number of Mesh Points N						
	8	16	32	64	128	256	512
1/ 2	.000468	.000120	.000030	.000008	.000002	.000000	.000000
1/ 4	.000819	.000212	.000054	.000013	.000003	.000001	.000000
1/ 8	.001460	.000391	.000101	.000025	.000006	.000001	.000000
1/ 16	.002606	.000743	.000196	.000049	.000012	.000003	.000001
1/ 32	.003724	.001429	.000383	.000098	.000024	.000006	.000001
1/ 64	.003293	.002592	.000739	.000194	.000049	.000012	.000002
1/ 128	.003525	.003719	.001427	.000381	.000097	.000023	.000005
1/ 256	.007963	.003289	.002590	.000737	.000192	.000046	.000009
1/ 512	.011277	.003529	.003715	.001423	.000377	.000092	.000019
1/ 1024	.006868	.007967	.003283	.002580	.000728	.000183	.000037
1/ 2048	.001845	.011277	.003528	.003699	.001404	.000358	.000074
1/ 4096	.000195	.006864	.007969	.003261	.002544	.000692	.000146
1/ 8192	.000006	.001842	.011269	.003521	.003637	.001331	.000285
1/ 16384	.000001	.000194	.006851	.007970	.003177	.002400	.000548
1/ 32768	.000000	.000006	.001834	.011239	.003491	.003395	.001046
1/ 65536	.000000	.000000	.000192	.006803	.007966	.002861	.001852
1/ 131072	.000000	.000000	.000006	.001801	.011111	.003362	.002516
1/ 262144	.000000	.000000	.000000	.000183	.006609	.007868	.001833
1/ 524288	.000000	.000000	.000000	.000005	.001673	.010508	.002760
1/ 1048576	.000000	.000000	.000000	.000000	.000151	.005824	.006751
1/ 2097152	.000000	.000000	.000000	.000000	.000003	.001255	.007748
1/ 4194304	.000000	.000000	.000000	.000000	.000000	.000082	.003395
1/ 8388608	.000000	.000000	.000000	.000000	.000000	.000001	.000503
1/ 16777216	.000000	.000000	.000000	.000000	.000000	.000000	.000021
1/ 33554432	.000000	.000000	.000000	.000000	.000000	.000000	.000000
E_N	.011277	.011277	.011269	.011239	.011111	.010508	.007748

TABLE 3. Maximum errors E_N and rate of convergence p_N for Frozen Fitting Factor scheme (3.3)

Boundary Conditions: $u(0) = 1.0$, $u(1) = 0.0$						
Initial Guess : $u_{init} = u(x) = u(0) + (u(1) - u(0))x$						
N	8	16	32	64	128	256
E_N	.011277	.011277	.011269	.011239	.011111	.010508
p_N	.00	.00	.00	.00	.00	.00

We present numerical results for the analog of (3.4) above, that is

$$(3.5) \quad \varepsilon \frac{d^2}{dx^2} u(x) - u - u^2 = 0, \quad x \in (0, 1),$$

$$u(0) = A, \quad u(1) = B.$$

We first consider the method proposed in [2, Ch. 10, p. 159]. This gives a piecewise constant frozen fitting factor method for the problem (P2). The fitting factor is

given by:

$$\begin{aligned}
 \gamma_i &= \frac{c(u(0))h^2}{4\varepsilon} \sinh^{-2} \frac{\sqrt{c(u(0))h}}{2\sqrt{\varepsilon}}, & 0 \leq x_i \leq 1/2, \\
 \gamma_i &= \frac{c(u(1))h^2}{4\varepsilon} \sinh^{-2} \frac{\sqrt{c(u(1))h}}{2\sqrt{\varepsilon}}, & 1/2 < x_i \leq 1.
 \end{aligned}
 \tag{3.6}$$

Table 4 gives uniform convergence rates for this scheme.

TABLE 4. Maximum errors E_N and rate of convergence p_N for Frozen Fitting Factor scheme (3.6)

Boundary Conditions: $u(0) = 0.5, u(1) = 0.7$						
Initial Guess : $u_{init} = u(x) = u(0) + (u(1) - u(0))x$						
N	8	16	32	64	128	256
E_N	.006028	.006027	.006024	.006010	.005952	.005669
p_N	.00	.00	.00	.00	.00	.00

Table 5 gives rates for the scheme of Miller [11] discussed in [2, Ch. 6] which has the following (non-frozen) fitting factor:

$$\gamma_i = \frac{c(u(x_i))h^2}{4\varepsilon} \sinh^{-2} \frac{\sqrt{c(u(x_i))h}}{2\sqrt{\varepsilon}}.
 \tag{3.7}$$

TABLE 5. Maximum errors E_N and rate of convergence p_N for the scheme of Miller (3.7)

Boundary Conditions: $u(0) = 0.5, u(1) = 0.7$						
Initial Guess : $u_{init} = u(x) = u(0) + (u(1) - u(0))x$						
N	8	16	32	64	128	256
E_N	.007895	.007891	.007882	.007846	.007705	.007163
p_N	.00	.00	.00	.00	.00	.00

As can be seen from these tables, none of the standard fitted schemes from the literature, on uniform meshes, are uniformly ε -convergent for the test problems. As remarked in the introduction, numerical methods, ε -uniform in the maximum norm, were constructed in [7] for a class of semilinear problems, which includes the class of problems considered here. These use classical finite difference operators on special piecewise-uniform meshes condensed or refined in the boundary layers. Thus ε -uniform methods can be constructed on special piecewise uniform meshes although it is not possible on uniform meshes.

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