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MULTILEVEL ADDITIVE SCHWARZ METHOD FOR THE h-p VERSION OF THE GALERKIN BOUNDARY ELEMENT METHOD

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ABSTRACT. We study a multilevel additive Schwarz method for the h-p version of the Galerkin boundary element method with geometrically graded meshes. Both hypersingular and weakly singular integral equations of the first kind are considered. As it is well known the h-p version with geometric meshes converges exponentially fast in the energy norm. However, the condition number of the Galerkin matrix in this case blows up exponentially in the number of unknowns M. We prove that the condition number $\kappa(P)$ of the multilevel additive Schwarz operator behaves like $O(\sqrt{M}\log^2 M)$. As a direct consequence of this we also give the results for the 2-level preconditioner and also for the h-p version with quasi-uniform meshes. Numerical results supporting our theory are presented.

1. INTRODUCTION

In this paper, we deal with a multilevel additive Schwarz method for the h-p version of the Galerkin boundary element method applied to both hypersingular and weakly singular integral equations. The boundary integral operators under consideration are the single layer potential operator and the normal derivative of the double layer potential operator on a polygon Γ in $\mathbb{R}^2$. That means we have to consider weak formulations of the form

$$a(u,v) := \langle Au, v\rangle = \langle g, v\rangle \quad \text{for all } v \in \tilde{H}^{s+\alpha/2}(\Gamma),$$

where $A : \tilde{H}^{s+\alpha/2}(\Gamma) \to H^{s-\alpha/2}(\Gamma)$ is symmetric and positive definite. (The definitions of the Sobolev spaces $\tilde{H}^{s+\alpha/2}(\Gamma)$ and $H^{s-\alpha/2}(\Gamma)$ are given in §2.) Here, $\langle\cdot,\cdot\rangle$ denotes the $L^2(\Gamma)$ inner product. In order to obtain an approximant u_M to u we solve the equation on a finite dimensional subspace $V_M \subset \tilde{H}^{s+\alpha/2}(\Gamma)$, that is we will find $u_M \in V_M$ satisfying

$$\langle Au_M, v\rangle = \langle g, v\rangle \quad \text{for all } v \in V_M. \tag{1.1}$$

The condition number of the above linear system grows at least like $h^{-1}p^2$ if the h-p version is used for quasi-uniform meshes, and exponentially for geometric meshes.

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Here h is the mesh size and p is the degree of the splines used in the Galerkin scheme. In order to use the conjugate gradient algorithm to solve the system efficiently we need a preconditioner. We shall study in this paper additive Schwarz methods as preconditioners for the above system.

Additive Schwarz methods were originally designed for finite element discretizations of differential equations (see e.g. [1], [2], [6], [5], [10], [18], [21], [29], [30]). The applicability of the method with overlapping domains was recently investigated in [13] for the h version of the boundary element method. Non-overlapping 2-level and multilevel methods (also for the h version) were considered in [26]. For the p version corresponding first results (of the 2-level method) for weakly singular integral operators were presented in [15]. A thorough investigation of the method was discussed in [24] for both hypersingular and weakly singular operators. The results in [26], [24] can be summarized as follows. For the h version the 2-level and multilevel additive Schwarz methods yield preconditioned systems which have bounded condition numbers. For the p version considered in [24] the condition number is proved to behave like $(1 + \log p)^2$.

In this paper we prove the efficiency of the 2-level and multilevel methods for the h-p version, with both quasi-uniform and geometric meshes. Our main results concern the multilevel additive Schwarz method for the h-p version with geometric meshes, cf. Theorems 3.1 and 4.1. More precisely, we will prove that when the 2-level and multilevel methods are used with quasi-uniform meshes, the condition number of the resulting system behaves like $(1 + \log p)^2$ and $p\,(1 + \log p)^2$, respectively. When the above two methods are used with geometric meshes, the improvement is even more. Instead of an exponential increase in the condition number, we will have the behavior $\log^2 M$ and $\sqrt{M}\log^2 M$ for the 2-level and multilevel methods, respectively. Here M is the number of unknowns of the system. We note here that even though the condition number of the multilevel method increases faster than that of the 2-level method, which may lead to a bigger number of iterations to solve the linear system, the implementation of the multilevel method is recommended since for each iteration it is actually the diagonal preconditioner, and is therefore cheaper for each iteration.

The paper is organized as follows. In §2 we describe the three versions of the boundary element method for the hypersingular and weakly singular integral equations and give the general setting of the additive Schwarz method. Section 3 gives the analysis of the multilevel additive Schwarz method for the hypersingular integral operator. We also collect the results for the 2-level method and for quasi-uniform meshes which are implicitly covered by the theory of the multilevel method for the h-p version with geometric meshes. In §4 we treat the weakly singular integral operator in the same manner. Numerical results are given in §5 to underline our analysis.

In this paper, c denotes a generic constant and may take different values at different occurrences.

2. Preliminaries

In this section we briefly introduce the different versions of the boundary element method and describe the additive Schwarz method for preconditioning the arising linear systems.

The hypersingular and weakly singular integral equations we consider are, respectively,

$$Du(x) := -\frac{1}{\pi} f.p. \int_\Gamma \frac{u(y)}{|x-y|^2}\, ds_y = g(x), \quad x \in \Gamma, \tag{2.1}$$

and

$$Vu(x) := -\frac{1}{\pi} \int_\Gamma u(y) \log|x-y|\, ds_y = g(x), \quad x \in \Gamma, \tag{2.2}$$

where f.p. denotes a finite part integral in the sense of Hadamard. The right hand side g is given in $H^{-1/2}(\Gamma)$ in the case of the hypersingular operator, and in $H^{1/2}(\Gamma)$ in the case of the weakly singular operator.

In the following we introduce the needed Sobolev spaces for a general, possibly open, Lipschitz curve $\Gamma \subset \mathbb{R}^2$. Let $\tilde{\Gamma}$ be an arbitrary closed curve containing Γ. We define, as in [12], the Sobolev spaces

$$H^s(\tilde{\Gamma}) = \begin{cases} \{\phi|_{\tilde{\Gamma}}\,;\ \phi \in H^{s+1/2}_{loc}(\mathbb{R}^2)\} & (s>0), \\ L^2(\tilde{\Gamma}) & (s=0), \\ (H^{-s}(\tilde{\Gamma}))' \quad \text{(dual space)} & (s<0). \end{cases}$$

Further, we define for positive s

$$H^s(\Gamma) = \{\phi|_\Gamma\,;\ \phi \in H^s(\tilde{\Gamma})\},$$
$$\tilde{H}^s(\Gamma) = \{\phi \in H^s(\tilde{\Gamma})\,;\ \mathrm{supp}(\phi) \subset \bar{\Gamma}\},$$

and for negative s

$$H^s(\Gamma) = (\tilde{H}^{-s}(\Gamma))', \quad \tilde{H}^s(\Gamma) = (H^{-s}(\Gamma))'.$$

In this paper we consider polygons Γ for the integral equations (2.1) and (2.2). Since then Γ is closed the spaces $\tilde{H}^s(\Gamma)$ and $H^s(\Gamma)$ coincide.

The operator $V:\ \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is continuous and positive definite which can always be achieved by an appropriate scaling of Γ, cf. [8], [25]. The operator $D:\ \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is continuous (see [8]) and due to the relation

$$\langle Dv, v\rangle = \langle Vv', v'\rangle, \quad v \in \tilde{H}^{1/2}(\Gamma)$$

(see [19]), D is positive definite on

$$\tilde{H}^{1/2}_0(\Gamma) = \{v \in \tilde{H}^{1/2}(\Gamma);\ \langle v, 1\rangle = 0\},$$

see also [7]. Hence the operators V and D define norms which are equivalent to the $\tilde{H}^{-1/2}(\Gamma)$- and $\tilde{H}^{1/2}(\Gamma)$-norms, respectively. These equivalences are frequently used in the proofs. Since D is positive definite only on $\tilde{H}^{1/2}_0(\Gamma)$ we always assume in the following that, in the case of the hypersingular operator, the space V_M in (1.1) is a subspace of $\tilde{H}^{1/2}_0(\Gamma)$. We note that instead of this conformity condition one can also consider a modified system for D by introducing another scalar unknown, cf. [9].

Different choices of the trial space V_M in (1.1) give rise to different types of the Galerkin scheme, namely to the h, p, and h-p versions defined as follows. Consider a uniform mesh of size h on Γ defined by the nodes

$$\{x_j,\ j = 0, \dots, N\} \quad \text{with} \quad |x_j - x_{j-1}| = h \quad \text{and} \quad x_0 = x_N. \tag{2.3}$$

We assume that the corners of the polygon Γ coincide with some nodes.

The h version. We define on the mesh (2.3) the space V^h of continuous piecewise linear functions (piecewise constant functions, respectively) for the hypersingular operator (weakly singular operator, respectively). For the h version of the Galerkin scheme the solution of (1.1) is approximated by functions in V^h and the accuracy is increased by reducing h.

The p version. On the mesh (2.3) the space V^p is now defined as the space of functions on Γ whose restrictions on $\Gamma_j := (x_{j-1}, x_j)$, $j = 1, \ldots, N$, are polynomials of degree at most p. For the hypersingular operator it is required that these functions are continuous. For the p version of the Galerkin scheme the solution of (1.1) is approximated by functions in V^p and the accuracy of the approximation is increased not by reducing h (which is fixed) but by increasing p.

The h-p version. If we combine the above two versions we end up with the h-p version on quasi-uniform meshes. This version works even more efficiently if one uses geometrical mesh grading towards the corners of Γ (where in general singularities of the exact solution occur) and appropriately chooses the polynomial degree on each subinterval. In doing so, one obtains exponentially fast convergence in the energy norms, i.e. for the Galerkin solution u_M of (1.1) there holds

$$\|u - u_M\|_{\tilde{H}^{1/2}(\Gamma)} \leq Ce^{-b\sqrt{M}}$$

in the case of the hypersingular operator (u is the exact solution of (2.1)) and

$$\|u - u_M\|_{\tilde{H}^{-1/2}(\Gamma)} \leq Ce^{-b\sqrt{M}}$$

in the case of the weakly singular operator (u is the exact solution of (2.2)). The positive constants C and b are independent of the dimension M of the ansatz space. Due to this exponentially fast convergence the h-p version with geometric meshes is superior to the standard h, p and h-p versions with quasi-uniform meshes, which converge only algebraically. Concerning the convergence theory we refer to [28], [25], [22], [23], [4], [3], [17], [16].

We will define the ansatz space for the h-p version with geometric meshes on $(-1, 0)$. Here we refine the meshes in a geometric manner towards -1. On $(-1, 1)$ the geometric meshes are defined by symmetry. The space on the whole polygon Γ can then be obtained by affine mappings of $(-1, 1)$ onto the edges of Γ. On the interval $(-1, 0)$ we use the partition

$$-1 = x_0 < x_1 < \ldots < x_K = 0, \tag{2.4}$$

where $x_i = -1 + \sigma^{K-i}$, $i = 1, \ldots, K$, for a constant $0 < \sigma < 1$. The mesh grading parameter $\sigma \in (0, 1)$ steers the geometrical grading towards the corner (in this case towards -1). The number K is usually called the number of levels of the geometric mesh. The ansatz space V_σ^K (on $(-1, 0)$) is now spanned by piecewise polynomial functions on $(-1, 0)$ whose restrictions on (x_{j-1}, x_j) are polynomials of degree at most p_j, $j = 1, \ldots, K$. The degree p_j is defined as $p_j = j$ for the hypersingular operator, and as $p_j = j - 1$ for the weakly singular operator. On Γ the space V_σ^K is obtained by affine mappings as mentioned above. For the hypersingular operator it is also required that the trial functions are continuous.

In order to be clear we recall the following notations. In general M is the dimension of the ansatz space which is denoted, by relating to the method under consideration, by V_M, V^h, V^p, and V_σ^K, simultaneously. The number of elements Γ_j of the mesh is denoted by N, i.e. $\bar{\Gamma} = \bigcup_{j=1}^N \bar{\Gamma}_j$. In the case of geometric meshes we have the variable K which is the number of levels in (2.4) and which is, for simplicity, assumed to be the same for all the corners of the polygon Γ. Since the number of edges of Γ is fixed for a specific problem we have the equivalence $K \simeq N$.

We are now giving the general setting for the additive Schwarz method for preconditioning the linear system (1.1). Let

$$V_M = V_0 + V_1 + \cdots + V_N \tag{2.5}$$

denote a decomposition of V_M into subspaces V_j. The additive Schwarz method (ASM) consists in solving, by an iterative method, the equation

$$Pu_M := (P_0 + P_1 + \cdots + P_N)u_M = f_M, \tag{2.6}$$

where the projections $P_j : V_M \to V_j$, $j = 0, \ldots, N$, are defined for any $v_M \in V_M$ by

$$a(P_j v_M, \phi_j) = a(v_M, \phi_j) \quad \text{for any } \phi_j \in V_j. \tag{2.7}$$

The right hand side of (2.6), $f_M = \sum_{j=0}^{N} P_j u_M$, can be computed without knowing the solution u_M of (1.1) by

$$a(P_j u_M, \phi_j) = \langle g, \phi_j \rangle \quad \text{for any } \phi_j \in V_j,\ j = 0, \ldots, N. \tag{2.8}$$

The following lemma is standard in proving bounds for the maximum and minimum eigenvalues of the additive Schwarz operator P defined by (2.5), (2.6), (2.7), and (2.8), see e.g. [18], [20], [21], [29], [30].

Lemma 2.1. *Let $v_M = \sum_{j=0}^{N} v_{M,j}$, where $v_{M,j} \in V_j$, be a representation of an element of $V_M = V_0 + \cdots + V_N$.*

(i) *If a representation can be chosen such that, for some $C_1 > 0$,*

$$\sum_{j=0}^{N} a(v_{M,j}, v_{M,j}) \le C_1^{-1} a(v_M, v_M),$$

then $\lambda_{\min}(P) \ge C_1$.

(ii) *If there exists $C_2 > 0$ such that for any representation of v_M*

$$a(v_M, v_M) \le C_2 \sum_{j=0}^{N} a(v_{M,j}, v_{M,j}),$$

then $\lambda_{\max}(P) \le C_2$.

3. Preconditioners for the hypersingular integral equation

In this section we deal with the h-p version of the boundary element method for solving the first kind integral equation (2.1) with the hypersingular integral operator. The h-p version with geometrically graded meshes yields an exponentially fast convergence in the energy norm, cf. [4], [3], [17], [16]. However, such a fast convergence is at the expense of an exponentially fast increasing condition number of the Galerkin system. Therefore, it is necessary to design preconditioners in order to solve the systems efficiently.

We first investigate a multilevel additive Schwarz preconditioner for the h-p version on geometric meshes (Theorem 3.1). A direct consequence are the results for the 2-level method (Corollary 3.3) and for the 2-level and multilevel methods for the h-p version with quasi-uniform meshes (Corollaries 3.4 and 3.5). Since the additive Schwarz method is generally defined by (2.5), (2.6), (2.7), and (2.8) it suffices to give for each preconditioner the decomposition of the ansatz space corresponding to (2.5).

Let us start with the decomposition

$$V_\sigma^K = V^1 \oplus V^p$$

of the ansatz space V_σ^K of the h-p version with a geometric mesh. Here V^1 denotes the piecewise linear part of V_σ^K and $V^p \subset V_\sigma^K$ denotes the space spanned by the piecewise polynomials of higher degrees q, $q \geq 2$. The theory of standard h version preconditioners cannot be applied to the block corresponding to the subspace V^1 since the mesh is non-quasi-uniform. However, we note that $\dim V^1 = N$ (the number of elements) and

$$\dim V^p = \sum_{j=1}^{N}(p_j - 1) \simeq \sum_{j=1}^{K}(p_j - 1) = \sum_{j=1}^{K}(j-1) \simeq K^2 \simeq N^2. \tag{3.1}$$

Therefore, the size of the h-block (which is proportional to $N \times N$) is small compared to the size of the p-block (which is proportional to $N^2 \times N^2$). For that reason it is practical to directly invert the block corresponding to V^1.

The block corresponding to V^p is first decomposed with respect to the elements Γ_j, i.e.

$$V_\sigma^K = V^1 \oplus V_1^{p_1} \oplus \cdots \oplus V_N^{p_N}. \tag{3.2}$$

This can be considered as a 2-level method. The space $V_j^{p_j}$ is spanned by the affine images onto Γ_j of the functions $\mathcal{L}_2, \ldots, \mathcal{L}_{p_j}$, where $\mathcal{L}_k$ is defined as $\mathcal{L}_k(x) = \int_{-1}^{x} L_{k-1}(y)\,dy$ with L_{k-1} being the Legendre polynomial of degree $k-1$. Note that $\mathcal{L}_k$ vanishes at ± 1 and therefore functions in $V_j^{p_j}$ vanish at the endpoints of Γ_j. These functions are then extended to be 0 outside Γ_j. We also note that, for $i, j = 1, \ldots, N$, $i \neq j$, $V^1 \cap V_j^{p_j} = \{0\}$ (because of the difference in the polynomial degrees) and $V_i^{p_i} \cap V_j^{p_j} = \{0\}$ (because of the disjoint supports).

In the next step we define the multilevel decomposition by separating the basis functions on the different elements,

$$V_\sigma^K = V^1 \oplus \Big(\bigoplus_{j=1}^{N} \bigoplus_{k=2}^{p_j} \tilde{V}_j^k\Big). \tag{3.3}$$

Here, the space $\tilde{V}_j^k$ is spanned by the affine image onto Γ_j of $\mathcal{L}_k$.

For the multilevel additive Schwarz operator there holds the following theorem.

Theorem 3.1. *The additive Schwarz operator P associated with the multilevel decomposition* (3.3) *for the h-p version with geometric meshes has condition number bounded as*

$$\kappa(P) \leq C\,K\,\log^2 K \simeq \sqrt{M}\,\log^2 M.$$

Here, K, which is proportional to the square root of the number of unknowns M, is the number of levels of the geometric mesh which is also the highest degree of the polynomials used. The positive constant C is independent of K.

Before proving Theorem 3.1 we present a lemma which will be used to split boundary element functions with respect to the polynomial degrees to prove bounds for the maximum and minimum eigenvalues.

Lemma 3.2. *Let $u_k = c_k \mathcal{L}_k^*$ where $\mathcal{L}_k^* = \mathcal{L}_k / \|\mathcal{L}_k\|_{L^2(I)}$ with $I = (-1,1)$, and $u = \sum_{k=2}^p u_k$. Then there exist positive constants C_1 and C_2 independent of the polynomial degree such that*

$$C_1 \|u\|^2_{\tilde{H}^{1/2}(I)} \le \sum_{k=2}^p \|u_k\|^2_{\tilde{H}^{1/2}(I)} \le C_2 p \, \|u\|^2_{\tilde{H}^{1/2}(I)}.$$

Proof. Since

$$\|u_k\|^2_{\tilde{H}^1(I)} = c_k^2 \|\mathcal{L}_k^*\|^2_{\tilde{H}^1(I)} = c_k^2 \frac{\|L_{k-1}\|^2_{L^2(I)}}{\|\mathcal{L}_k\|^2_{L^2(I)}}$$

and

$$\begin{aligned}\|u\|^2_{\tilde{H}^1(I)} &= \Big\|\sum_{k=2}^p u_k\Big\|^2_{\tilde{H}^1(I)} = \Big\|\sum_{k=2}^p \frac{c_k}{\|\mathcal{L}_k\|_{L^2(I)}} L_{k-1}\Big\|^2_{L^2(I)} \\ &= \sum_{k=2}^p \frac{c_k^2}{\|\mathcal{L}_k\|^2_{L^2(I)}} \|L_{k-1}\|^2_{L^2(I)},\end{aligned}$$

we have

$$\|u\|^2_{\tilde{H}^1(I)} = \sum_{k=2}^p \|u_k\|^2_{\tilde{H}^1(I)}. \tag{3.4}$$

We will prove

$$c\,\|u\|^2_{L^2(I)} \le \sum_{k=2}^p \|u_k\|^2_{L^2(I)} \le C\,p^2 \|u\|^2_{L^2(I)}. \tag{3.5}$$

It is easy to check that (cf. [21], [24])

$$\langle \mathcal{L}_k^*, \mathcal{L}_l^* \rangle_{L^2(I)} = \begin{cases} 1 & \text{for } k = l \ge 2, \\ -\frac{1}{2}\sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}} & \text{for } l = k+2,\ k \ge 2, \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\begin{aligned}\|u\|^2_{L^2(I)} &= \sum_{k=2}^p \sum_{l=2}^p c_k c_l \langle \mathcal{L}_k^*, \mathcal{L}_l^* \rangle_{L^2(I)} = \sum_{k=2}^p c_k^2 + 2\sum_{k=2}^{p-2} c_k c_{k+2} \langle \mathcal{L}_k^*, \mathcal{L}_{k+2}^* \rangle_{L^2(I)} \\ &= \sum_{k=2}^p c_k^2 - \sum_{k=2}^{p-2} c_k c_{k+2} \sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}}.\end{aligned} \tag{3.6}$$

Therefore we have

$$\|u\|^2_{L^2(I)} \le \sum_{k=2}^p c_k^2 + \frac{1}{2}\sum_{k=2}^{p-2} c_k^2 + \frac{1}{2}\sum_{k=2}^{p-2} c_{k+2}^2 \le 2\sum_{k=2}^p c_k^2 = 2\sum_{k=2}^p \|u_k\|^2_{L^2(I)},$$

which proves the left inequality in (3.5). To prove the right inequality in (3.5) we observe that from (3.6) we have

$$\begin{aligned}
\|u\|^2_{L^2(I)} &\geq \sum_{k=2}^{p} c_k^2 - \Big(\sum_{k=2}^{p-2} c_k^2 \sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}}\Big)^{1/2} \Big(\sum_{k=2}^{p-2} c_{k+2}^2 \sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}}\Big)^{1/2} \\
&= \sum_{k=2}^{p} c_k^2 - \Big(\sum_{k=2}^{p-2} c_k^2 \sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}}\Big)^{1/2} \Big(\sum_{k=4}^{p} c_k^2 \sqrt{\frac{(2k-7)(2k+1)}{(2k-5)(2k-1)}}\Big)^{1/2} \\
&\geq \sum_{k=2}^{p} c_k^2 - \frac{1}{2}\sum_{k=2}^{p-2} c_k^2 \sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}} - \frac{1}{2}\sum_{k=4}^{p} c_k^2 \sqrt{\frac{(2k-7)(2k+1)}{(2k-5)(2k-1)}},
\end{aligned}$$

and therefore we need to show that there exists a constant $c > 0$ independent of p such that

$$\begin{aligned}
\sum_{k=2}^{p} c_k^2 - \frac{1}{2}\sum_{k=2}^{p-2} c_k^2 \sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}} - \frac{1}{2}\sum_{k=4}^{p} c_k^2 \sqrt{\frac{(2k-7)(2k+1)}{(2k-5)(2k-1)}} \\
\geq c\sum_{k=2}^{p} k^{-2} c_k^2 \geq c\, p^{-2}\sum_{k=2}^{p} c_k^2.
\end{aligned}$$

This can be seen by noting that

$$\lim_{k\to\infty} k^2\Bigg(1 - \frac{1}{2}\sqrt{\frac{(2k-3)(2k+5)}{(2k-1)(2k+3)}} - \frac{1}{2}\sqrt{\frac{(2k-7)(2k+1)}{(2k-5)(2k-1)}}\Bigg) = \frac{3}{2}.$$

Thus we showed the right inequality of (3.5). Applying interpolation to (3.4) and (3.5) we obtain the desired results. Therefore the lemma is proved. □

Proof of Theorem 3.1. Let

$$u_\sigma^K = u_1 + \sum_{j=1}^{N}\sum_{k=2}^{p_j} u_{k,j} \ \in V_\sigma^K$$

where $u_{k,j} \in \tilde{V}_j^k$, accordingly to (3.3). Since (3.3) is a direct sum decomposition this representation is unique. Let $w_p := \sum_{j=1}^N \sum_{k=2}^{p_j} u_{k,j}$ and $u^j := \sum_{k=2}^{p_j} u_{k,j}$, $j = 1,\dots,N$. Then $u^j \in \tilde{H}^{1/2}(\Gamma_j)$ is the restriction of w_p onto Γ_j.

As the first step of the proof we note that there holds with $p_{\max} := \max\{p_j;\ j = 1,\dots,N\}$

$$\begin{aligned}
&C_1\,(1+\log p_{\max})^{-2}\big(a(u_1,u_1) + \sum_{j=1}^{N} a(u^j,u^j)\big) \\
&\qquad \leq a(u_\sigma^K, u_\sigma^K) \leq C_2\big(a(u_1,u_1) + \sum_{j=1}^{N} a(u^j,u^j)\big).
\end{aligned} \tag{3.7}$$

The right inequality of (3.7) is due to the norm property

$$\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\sum_{j=1}^{N} \|u|_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \tag{3.8}$$

for general decompositions of Γ into subintervals Γ_j and for functions $u \in \tilde{H}^{1/2}(\Gamma)$ such that $u|_{\Gamma_j} \in \tilde{H}^{1/2}(\Gamma_j)$ (cf. [14, Lemma 4] and [27, Lemma 3.3]). The constant

c is independent of the function u and of the number of subintervals N. The left inequality of (3.7) has been proved in [24, Lemma 3.6] to bound the minimum eigenvalue of an additive Schwarz operator for the p version of the boundary element method. The quasi-uniformity of the mesh was not required in that proof. Therefore this estimate holds also for the geometric mesh by inserting as the worst case the maximum polynomial degree $p_{\max}$.

The second step of our proof consists in separating the estimates in (3.7) with respect to the polynomial degrees. By using Lemma 3.2, and considering only the reference interval I, we obtain

$$c_1 p_j^{-1} \sum_{k=2}^{p_j} \|u_{k,j}\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \leq \Big\|\sum_{k=2}^{p_j} u_{k,j}\Big\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \leq c_2 \sum_{k=2}^{p_j} \|u_{k,j}\|^2_{\tilde{H}^{1/2}(\Gamma_j)}.$$

This is equivalent to the relations

$$c_1 p_j^{-1} \sum_{k=2}^{p_j} a(u_{k,j}, u_{k,j}) \leq a(u^j, u^j) \leq c_2 \sum_{k=2}^{p_j} a(u_{k,j}, u_{k,j}). \tag{3.9}$$

Combining (3.7) and (3.9) we obtain

$$C_1(1+\log p_{\max})^{-2} p_{\max}^{-1}\Big(a(u_1,u_1) + \sum_{j=1}^{N}\sum_{k=2}^{p_j} a(u_{k,j}, u_{k,j})\Big) \leq a(u_\sigma^K, u_\sigma^K)$$

and

$$a(u_\sigma^K, u_\sigma^K) \leq C_2\Big(a(u_1,u_1) + \sum_{j=1}^{N}\sum_{k=2}^{p_j} a(u_{k,j}, u_{k,j})\Big).$$

Therefore, using Lemma 2.1, we obtain

$$\lambda_{\min}(P) \geq C_1\, p_{\max}^{-1}\,(1+\log p_{\max})^{-2} \quad \text{and} \quad \lambda_{\max}(P) \leq C_2,$$

i.e.

$$\kappa(P) \leq (C_2/C_1)\, p_{\max}\,(1+\log p_{\max})^2 \simeq K\ \log^2 K.$$

Concerning the number of unknowns we note that

$$M = \dim V_\sigma^K = \dim V^1 + \dim V^p \simeq K^2,$$

cf. (3.1). Therefore the theorem is proved. □

When omitting the decomposition with respect to the polynomial degrees we just have the 2-level method given by (3.2). Then we directly obtain from (3.7) the following estimate for the condition number.

Corollary 3.3. *The additive Schwarz operator P associated with the* 2*-level decomposition* (3.2) *for the h-p version with geometric meshes has condition number bounded as*

$$\kappa(P) \leq C\ \log^2 K \simeq \log^2 M.$$

The positive constant C is independent of K.

We note that even though the behavior of the condition number of the multilevel operator is not as good as that of the 2-level operator, it is worth applying the multilevel method since it is actually the diagonal preconditioner for the p-block of the Galerkin matrix.

Obviously the above 2-level and multilevel additive Schwarz methods can be applied to the *h-p* version with quasi-uniform meshes as well. We just have to assume that the polynomial degrees are everywhere the same, $p_j = p$ $(j = 1, \dots, N)$, and that the mesh is quasi-uniform. For the 2-level method we obtain the following corollary.

Corollary 3.4. *The additive Schwarz operator P associated with the* 2*-level decomposition* (3.2) *for the h-p version with quasi-uniform meshes has condition number bounded as*

$$\kappa(P) \le C\,(1 + \log p)^2.$$

The positive constant C is independent of p.

When considering quasi-uniform meshes we can also use a multilevel decomposition of the space V^1 of the piecewise linear functions. This decomposition was considered in [26]; we briefly recall it here. We consider a sequence of mesh sizes h_l, $l = 1, \dots, L$, where $h_{l-1} = 2h_l$, $l = 2, \dots, L$ and $h_L = h$. Let N_{h_l} denote the number of elements of the mesh with size h_l. The decomposition of V^1 is then given by

$$V^1 = V_{h_1} + \sum_{l=2}^{L} \bigl(V_{h_l,1} + \cdots + V_{h_l,N_{h_l}}\bigr). \tag{3.10}$$

Here V_{h_1} is the coarse grid space of continuous piecewise linear functions on the mesh with size h_1 and $V_{h_l,j} = \operatorname{span}\{\phi_{h_l,j}\}$ where $\phi_{h_l,j}$ is the hat function which takes the value 1 at the jth node and the value 0 at the other nodes of the mesh with size h_l, $l = 2, \dots, L$. The result is the following.

Corollary 3.5. *The additive Schwarz operator P associated with the multilevel decomposition given by the combination of* (3.3) *and* (3.10) *for the h-p version with quasi-uniform meshes has condition number bounded as*

$$\kappa(P) \le C\,p\,(1 + \log p)^2.$$

The positive constant C is independent of p and L.

Proof. Again, we apply Theorem 3.1 to a uniform polynomial degree distribution and a quasi-uniform mesh and note that the multilevel additive Schwarz preconditioner given by the decomposition (3.10) is optimal independently of the number of levels L and the mesh size h, cf. [26, Theorem 2.9 and Remark 2.10]. □

4. Preconditioners for the weakly singular integral equation

In this section we consider additive Schwarz preconditioners for the Galerkin equations discretizing the weakly singular integral equation (2.2). As in the previous section the main result concerns a multilevel additive Schwarz preconditioner for the *h-p* version with geometric meshes (Theorem 4.1). Again we also collect the results for the 2-level method (Corollary 4.2) and for the *h-p* version with quasi-uniform meshes (Corollaries 4.3 and 4.4). Recall that we are now dealing with functions in $\tilde{H}^{-1/2}(\Gamma)$ which need not be continuous.

First let us consider a 2-level and a multilevel decomposition of the ansatz space V_σ^K which consists of piecewise polynomials of different degrees on a geometric mesh with K levels as described in §1. The 2-level method is again based on a decomposition into the elements Γ_j,

$$V_\sigma^K = V^0 \oplus V_1^{p_1} \oplus \cdots \oplus V_N^{p_N}. \tag{4.1}$$

The space V^0 contains the piecewise constant functions. The space $V_j^{p_j}$ is spanned by the affine images onto Γ_j of the Legendre polynomials $L_1, \ldots, L_{p_j}$, $j = 1, \ldots, N$. All functions are assumed to be extended by 0 onto the whole boundary Γ. The decomposition (4.1) is obviously a direct one. For the multilevel method we further split the subspaces $V_j^{p_j}$, $j = 1, \ldots, N$, with respect to the degrees,

$$V_\sigma^K = V^0 \oplus \Big(\bigoplus_{j=1}^{N} \bigoplus_{k=1}^{p_j} \tilde{V}_j^k\Big). \tag{4.2}$$

The space $\tilde{V}_j^k$ is spanned by the affine image onto Γ_j of the Legendre polynomial of degree k, $j = 1, \ldots, N$, $k = 1, \ldots, p_j$.

For the multilevel additive Schwarz preconditioner there holds the following result.

Theorem 4.1. *The additive Schwarz operator P associated with the multilevel decomposition* (4.2) *for the h-p version with geometric meshes has condition number bounded as*

$$\kappa(P) \le C\, K \,\log^2 K \simeq \sqrt{M}\, \log^2 M.$$

Here, K is the number of levels of the geometric mesh which is proportional to the square root of the number of unknowns M. The positive constant C is independent of K.

Proof. Let

$$u_\sigma^K = u_0 + \sum_{j=1}^{N} \sum_{k=1}^{p_j} u_{k,j} \ \in V_\sigma^K$$

where $u_{k,j} \in \tilde{V}_j^k$, accordingly to (4.2). Since (4.2) is a direct sum decomposition this representation is unique. Let $w_p := \sum_{j=1}^{N} \sum_{k=1}^{p_j} u_{k,j}$ and $u^j := \sum_{k=1}^{p_j} u_{k,j}$, $j = 1, \ldots, N$. Then $u^j \in \tilde{H}^{-1/2}(\Gamma_j)$ is the restriction of w_p onto Γ_j.

We first note that there holds

$$\begin{aligned} &C_1\,(1 + \log p_{\max})^{-2}\big(a(u_0, u_0) + \sum_{j=1}^{N} a(u^j, u^j)\big) \\ &\qquad \le a(u_\sigma^K, u_\sigma^K) \le C_2\big(a(u_0, u_0) + \sum_{j=1}^{N} a(u^j, u^j)\big). \end{aligned} \tag{4.3}$$

The right inequality holds due to the same argument as in the proof of Theorem 3.1 by noting that an estimate analogous to (3.8) is also true of the $\tilde{H}^{-1/2}$-norm. The left inequality has been proved in [24, Lemma 4.3] for the p version. Of course, in order to deal with a non-uniform h-p version, we have to insert the maximum polynomial degree $p_{\max}$.

To split with respect to the polynomial degrees we proceed as follows. We note that there holds (cf. [13])

$$\|v\|_{\tilde{H}^{-1/2}(\Gamma)} \simeq \|Iv\|_{\tilde{H}^{1/2}(\Gamma)} \tag{4.4}$$

for all functions $v \in \tilde{H}_0^{-1/2}(\Gamma)$ where

$$\tilde{H}_0^{-1/2}(\Gamma) := \{v \in \tilde{H}^{-1/2}(\Gamma);\ \langle v, 1\rangle = 0\},$$

and I denotes the antiderivative operator which is bounded on $\tilde{H}_0^{-1/2}(\Gamma)$ and has bounded inverse. Therefore, instead of dealing with the $\tilde{H}^{-1/2}$-norm of piecewise Legendre polynomials, we can consider the $\tilde{H}^{1/2}$-norm of antiderivatives of Legendre polynomials which are in fact the basis functions used for the hypersingular operator. Let us consider the function $u^j = w_p|_{\Gamma_j}$ which can be expanded by affine images $L_{k,j}$ of the Legendre polynomials L_k, $k = 1, \dots, p_j$. We normalize these basis functions by the factors $1/\|\mathcal{L}_{k+1}\|_{L^2(-1,1)}$ ($\mathcal{L}_{k+1}$ is the antiderivative of L_k, cf. §3). The expansion looks like

$$u^j = \sum_{k=1}^{p_j} c_k \frac{L_{k,j}}{\|\mathcal{L}_{k+1}\|_{L^2(-1,1)}}.$$

Due to (4.4) we obtain, by denoting the affine image of $\mathcal{L}^*_{k+1} = \mathcal{L}_{k+1}/\|\mathcal{L}_{k+1}\|_{L^2(I)}$ onto Γ_j by $\mathcal{L}^*_{k+1,j}$,

$$\|u^j\|^2_{\tilde{H}^{-1/2}(\Gamma_j)} \simeq \Big\|\sum_{k=1}^{p_j} c_k \frac{IL_{k,j}}{\|\mathcal{L}_{k+1}\|_{L^2(-1,1)}}\Big\|^2_{\tilde{H}^{1/2}(\Gamma_j)} = \Big\|\sum_{k=1}^{p_j} c_k \mathcal{L}^*_{k+1,j}\Big\|^2_{\tilde{H}^{1/2}(\Gamma_j)}.$$

Together with Lemma 3.2 and the relation $u_{k,j} = c_k \mathcal{L}^*_{k+1,j}$ this yields

$$c_1 p_j^{-1} \sum_{k=1}^{p_j} \|u_{k,j}\|^2_{\tilde{H}^{-1/2}(\Gamma_j)} \le \Big\|\sum_{k=1}^{p_j} u_{k,j}\Big\|^2_{\tilde{H}^{-1/2}(\Gamma_j)} \le c_2 \sum_{k=1}^{p_j} \|u_{k,j}\|^2_{\tilde{H}^{-1/2}(\Gamma_j)}$$

which is equivalent to

$$c_1 p_j^{-1} \sum_{k=1}^{p_j} a(u_{k,j}, u_{k,j}) \le a(u^j, u^j) \le c_2 \sum_{k=1}^{p_j} a(u_{k,j}, u_{k,j}). \tag{4.5}$$

Combining (4.3) and (4.5) and proceeding as in the proof of Theorem 3.1 we obtain the desired estimate for the condition number. □

As for the hypersingular operator we also consider the 2-level decomposition and directly obtain the following bound for the corresponding condition number.

Corollary 4.2. *The additive Schwarz operator P associated with the* 2*-level decomposition* (4.1) *for the h-p version with geometric meshes has condition number bounded as*

$$\kappa(P) \le C \log^2 K \simeq \log^2 M.$$

The positive constant C is independent of K.

When applying the 2-level method to the h-p version with quasi-uniform meshes the result is given by the next corollary.

Corollary 4.3. *The additive Schwarz operator P associated with the* 2*-level decomposition* (4.1) *for the h-p version with quasi-uniform meshes has condition number bounded as*

$$\kappa(P) \le C\,(1 + \log(p+1))^2.$$

The positive constant C is independent of p.

For the multilevel method in the case of quasi-uniform meshes we also decompose the subspace V^0 of piecewise constant functions by

$$V^0 = V_{h_1} + \sum_{l=2}^{L} \big(V_{h_l,1} + \cdots + V_{h_l,N_{h_l}}\big). \tag{4.6}$$

Here, h_l, $l = 1, \ldots, L$, is a sequence of mesh sizes as in §3, i.e. $h_{l-1} = 2h_l$ and $h_L = h$, and N_{h_l} is the number of elements of the mesh with size h_l. In this case V_{h_1} is the space of piecewise constant functions on the coarse mesh with size h_1 and $V_{h_l,j} = \operatorname{span}\{\psi_{h_l,j}\}$ where $\psi_{h_l,j}$ is the derivative of the hat function which takes the value 1 at the jth node and the value 0 at the other nodes of the mesh with size h_l. These functions, which are called Haar basis functions, are orthogonal to 1 in the $L^2(\Gamma)$ inner product. Therefore the antiderivative operator I can be used to carry over the results from the multilevel h version for the hypersingular operator to the multilevel h version for the weakly singular operator. For more details we refer to [26]. The result is the following.

Corollary 4.4. *The additive Schwarz operator P associated with the multilevel decomposition given by the combination of* (4.2) *and* (4.6) *for the h-p version with quasi-uniform meshes has condition number bounded as*

$$\kappa(P) \le C\,(p+1)\,(1 + \log(p+1))^2.$$

The positive constant C is independent of p and L.

Proof. We apply Theorem 4.1 to a uniform polynomial degree distribution and a quasi-uniform mesh and note that the multilevel additive Schwarz preconditioner given by the decomposition (4.6) has bounded condition number independently of the number of levels L and the mesh size h, cf. [26, Theorem 3.10, Remark 3.11]. □

5. Numerical results

In this section we present numerical results for the h-p version of the boundary element method for solving the weakly singular integral equation (2.2). The underlying problem is the Dirichlet problem for the L-shaped domain Ω shown in Figure 1,

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

where f is chosen such that

$$u(x,y) = \Im(z^{2/3}) \quad \text{for } z = x + iy.$$

This problem is re-formulated to the integral equation (2.2). The finite dimensional subspace V_M of $\tilde{H}^{-1/2}(\Gamma)$ is constructed by the affine images of the Legendre polynomials on each subinterval of Γ. We start with a uniform mesh consisting of 8 elements of equal length on Γ. The sequence of subspaces V_M for the h-p version with quasi-uniform meshes is constructed by halving the elements and increasing the polynomial degrees by 1 from step to step.

For the h-p version with geometric meshes we use a geometric mesh-grading just towards the reentrant corner since there occurs the only singularity in our example. On the remaining edges of the polygon Γ the mesh is identical to that of the p version. The polynomial degrees on the geometric mesh are increasing from 0 at $(0,0)$ up to $K-1$ at $(0,1/2)$ and $(1/2,0)$. Again, K is the number of levels of

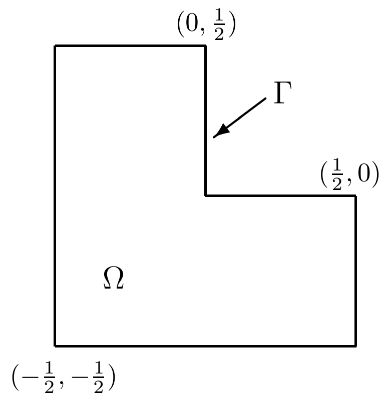

FIGURE 1. The L-shaped domain for the Dirichlet problem.

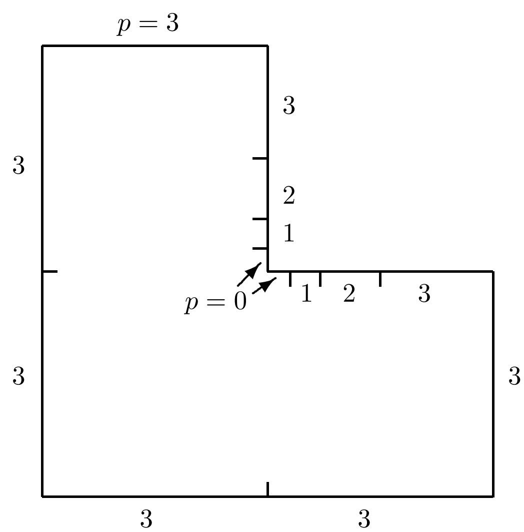

FIGURE 2. A sample of the geometric mesh and the used polynomial degrees ($K = 4$, $\sigma = 0.5$).

the geometric mesh, i.e. the number of elements on each of the edges attaching the reentrant corner. On the other edges the constant polynomial degree $K - 1$ is used. A sample of the geometric mesh is shown in Figure 2. For more details and numerical results for the different versions of the boundary element method we refer to [11].

Figures 3 and 4 show the condition numbers of the h-p version with quasi-uniform and geometric meshes, respectively. In both cases we present the numbers obtained by the 2-level and multilevel additive Schwarz preconditioners to compare with the numbers given by the unpreconditioned methods. The figures clearly demonstrate the improvement of the behavior of the condition numbers of the linear systems when a preconditioner is used, as expected from our theory.

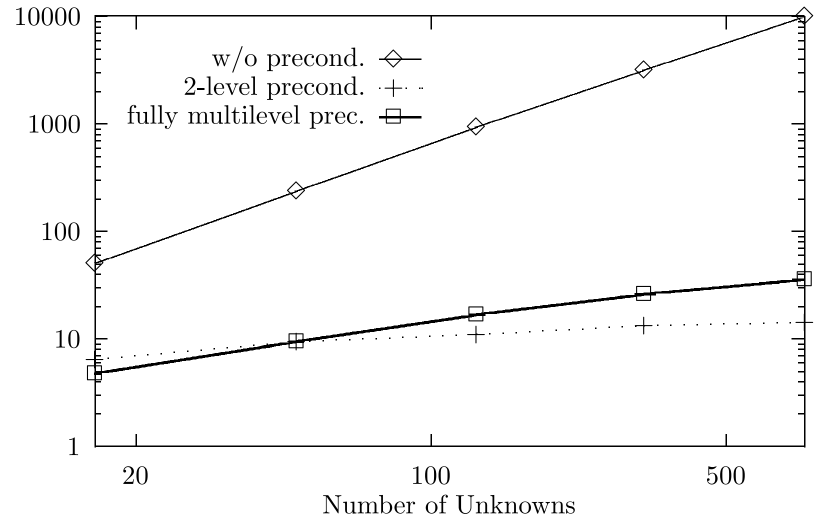

FIGURE 3. The condition numbers for the h-p version with quasi-uniform meshes.

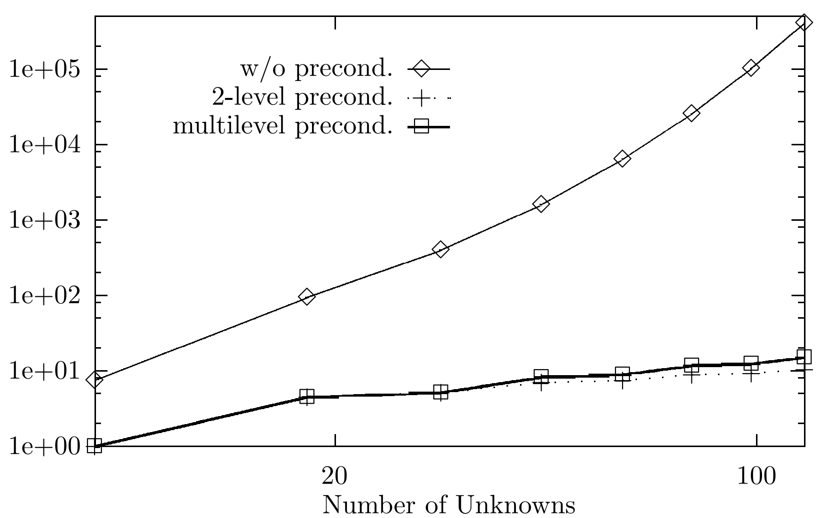

FIGURE 4. The condition numbers for the h-p version with geometric meshes ($\sigma = 0.5$).

Tables 2 and 3 present explicitly the condition numbers κ of different additive Schwarz operators for the h-p version. The given numbers α reflect the numerically found "convergence rates" of the numbers κ as predicted by the theory in the previous sections and collected in Table 1. More precisely, the α's are the computed numbers in the assumption $\kappa = \tilde{\kappa}^{\alpha}$ where the $\tilde{\kappa}$'s are given in Table 1.

Table 2 confirms the theoretical results for the h-p version with quasi-uniform meshes. In case of the 2-level method the growth of κ in p is slower than linear and in case of the multilevel method faster than p and slower than p^2. For the

TABLE 1. The theoretically expected behavior of the condition numbers κ.

BEM version	precond. version	κ (theoretically)	$\tilde{\kappa}$	Reference
h-p, quasi-uniformly	2-level	$\log^2 p$	p	Cor. 4.3
	multilevel	$p \log^2 p$	p	Cor. 4.4
h-p, geom. mesh	2-level	$\log^2 K$	K	Cor. 4.2
	multilevel	$K \log^2 K$	K	Thm. 4.1

TABLE 2. The condition numbers for the h-p version with quasi-uniform meshes.

			2-level		multilevel	
M	$4/h$	p	κ	α	κ	α
16	8	1	6.50		4.73	
				0.55		1.01
48	16	2	9.51		9.51	
				0.35		1.42
128	32	3	10.97		16.89	
				0.67		1.54
320	64	4	13.32		26.29	
				0.34		1.38
768	128	5	14.38		35.74	

TABLE 3. The condition numbers for the h-p version with geometric meshes ($\sigma = 0.5$).

		2-level		multilevel	
M	K	κ	α	κ	α
18	2	4.47		4.47	
			0.65		0.87
44	4	7.00		8.17	
			0.57		0.88
78	6	8.82		11.66	
			0.53		0.90
120	8	10.27		15.09	

multilevel preconditioner for the block of piecewise constant functions we use the mesh consisting of 8 elements as the coarse mesh, i.e. $\dim V_{h_1} = 8$ in (4.6).

The results for the h-p version with geometric meshes are shown in Table 3. For the 2-level preconditioner the α's are decreasing which is consistent with the logarithmic behavior in K of the condition number predicted by Corollary 4.2. Also for the multilevel method the computed condition numbers are in agreement with the theoretical bound $\kappa(P) \leq CK \log^2 K$ given by Theorem 4.1.

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