

## NONEXISTENCE CONDITIONS OF A SOLUTION FOR THE CONGRUENCE $x_1^k + \cdots + x_s^k \equiv N \pmod{p^n}$

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ABSTRACT. We obtain nonexistence conditions of a solution for of the congruence  $x_1^k + \cdots + x_s^k \equiv N \pmod{p^n}$ , where  $k \geq 2$ ,  $s \geq 2$  and  $N$  are integers, and  $p^n$  is a prime power. We give nonexistence conditions of the form  $(s, N \pmod{p^n})$  for  $k = 2, 3, 4, 5, 7$ , and of the form  $(s, p^n)$  for  $k = 11, 13, 17, 19$ . Furthermore, we complete some tables concerned with Waring's problem in  $p$ -adic fields that were computed by Hardy and Littlewood.

### 1. INTRODUCTION

It is well known that there is no solution to the Diophantine equation  $x^3 + y^3 + z^3 = n$  where  $n \equiv \pm 4 \pmod{9}$  [6]. Furthermore, if  $n \equiv 2 \pmod{7}$ , then there is no solution such that  $x \equiv 3, 5, 6 \pmod{7}$ . In this paper, we consider more general congruences and their conditions for nonexistence of a solution. The analysis and computer search is not only interesting in itself, but is also useful for some number theoretic sieves that efficiently solve some Diophantine equations [5].

Here, we discuss nonexistence conditions of a solution for the following congruence:

$$(1) \quad x_1^k + \cdots + x_s^k \equiv N \pmod{p^n},$$

where  $k \geq 2$ ,  $s \geq 2$  and  $N$  are integers, and  $p^n$  is a prime power.

As described in Section 4, for a sufficiently large  $p$  that depends on  $k$  (we can compute the bound), we can completely describe whether there exists a solution for the congruence (1) through theoretical analysis. Therefore, we consider the following problem.

**Problem.** For a given integer  $k \geq 2$ , find all integers  $s \geq 2$  and prime powers  $p^n$  (we are interested in the least  $n$  for each pair  $(s, p)$ ) such that the congruence (1) has no solution for an integer  $N$ . In addition, find all the values of  $N \pmod{p^n}$ .

After the preliminary Section 2, we consider some special cases in which the nonexistence conditions can be obtained through the theoretical analysis in Section 3. We will show some theoretical results that our search algorithm depends on to obtain all nonexistence conditions in Section 4. Using these results, we describe our search algorithm in Section 5. In Section 6, by theoretical analysis and computer search, we will show the nonexistence conditions of a solution for the

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congruence (1) of the form  $(s, N \bmod p^n)$  for  $k = 2, 3, 4, 5, 7$ , and of the form  $(s, p^n)$  for  $k = 11, 13, 17, 19$ . Finally in Section 7, using the computer search, we consider Waring's problem in  $p$ -adic fields; we complete some tables of Hardy and Littlewood, and correct some of their computation errors in [4].

## 2. PRELIMINARIES

If  $k$  is a positive integer and  $p$  is a prime we can write  $k = p^\tau dl$ , where  $d = (k, p - 1)$  and  $p \nmid l$ . We write

$$\nu = \begin{cases} \tau + 1, & p \text{ odd,} \\ \tau + 2, & p = 2. \end{cases}$$

If the congruence

$$(2) \quad x_1^k + \cdots + x_s^k \equiv M \pmod{p^\nu}$$

has a primitive solution, then for all positive integers  $m$ , the congruence

$$x_1^k + \cdots + x_s^k \equiv M \pmod{p^m}$$

has a primitive solution. This statement follows from the fact that for an integer  $a \not\equiv 0 \pmod{p}$ , if the congruence

$$x^k \equiv a \pmod{p^\nu}$$

has a solution, then for all positive integers  $m$ , the congruence

$$(3) \quad x^k \equiv a \pmod{p^m}$$

has a solution. Notably, the congruence (3) has a solution for any integer  $a \not\equiv 0 \pmod{p}$  and any positive integer  $m$  if and only if  $\tau = 0$  and  $d = 1$ .

The following lemmas are obvious.

**Lemma 1.** *When  $k \geq 2$ , then  $k \geq \nu$  if and only if  $(k, p) \neq (2, 2)$ . The equality holds if and only if  $(k, p) = (4, 2)$ .*

**Lemma 2.** *Suppose that  $(k, p) \neq (2, 2), (4, 2)$  and  $M \not\equiv 0 \pmod{p^\nu}$ . If the congruence (2) has a solution, then it is primitive.*

**Lemma 3.** *Let  $p$  be a prime,  $k \geq 2$ , and  $(k, p) \neq (2, 2), (4, 2)$ . If the congruence*

$$x_1^k + \cdots + x_s^k \equiv 0 \pmod{p^\nu}$$

*has no primitive solution, then for any integer  $t \not\equiv 0 \pmod{p}$ , the congruence*

$$x_1^k + \cdots + x_s^k \equiv p^\nu t \pmod{p^{\nu+1}}$$

*has no solution.*

## 3. NONEXISTENCE CONDITIONS THROUGH THEORETICAL ANALYSIS

In this section, we discuss the cases we can treat analytically. First, we can obtain all nonexistence conditions for the following congruence:

$$(4) \quad x_1^k + \cdots + x_s^k \equiv N \pmod{2^n}.$$

**Theorem 1.**

1. *When  $k$  is odd, the congruence  $x_1^k + x_2^k = M \pmod{2^m}$  has a primitive solution for any integer  $M$  and any positive integer  $m$ .*

2. When  $k$  is even, the nonexistence conditions of a solution for the congruence (4) are as follows:

(a) If  $k = 2$ , then  $(2, 3 \pmod{4})$ ,  $(3, 7 \pmod{8})$ .

(b) If  $k = 4$ , then

$$\begin{aligned} &(2, 3 \pmod{4}), \\ &(i, j \pmod{8}) \quad \text{for } 3 \leq i \leq 6, i + 1 \leq j \leq 7, \\ &(i, j \pmod{16}) \quad \text{for } 7 \leq i \leq 14, i + 1 \leq j \leq 15. \end{aligned}$$

(c) If  $k \neq 2, 4$ , then

$$\begin{aligned} &(2, 3 \pmod{4}), \\ &(i, j \pmod{2^m}) \quad \text{for } 3 \leq m \leq \nu, 2^{m-1} - 1 \leq i \leq 2^m - 2, \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad i + 1 \leq j \leq 2^m - 1, \\ &(2^\nu - 1, 2^\nu \pmod{2^{\nu+1}}). \end{aligned}$$

*Proof.* Suppose that  $k$  is even (when  $k$  is odd, Theorem 1 is clear).

When  $k = 2$ ,

$$(\mathbb{Z}/2^\nu\mathbb{Z})^k = (\mathbb{Z}/8\mathbb{Z})^2 = \{0 \pmod{8}, 1 \pmod{8}, 4 \pmod{8}\}.$$

Therefore, for  $s = 2, 3$  the statement holds. For  $s = 4$ , we will show that the congruence (4) has a solution for any integer  $N$  and any positive integer  $n$ . The statement holds for  $N \not\equiv 0 \pmod{8}$  clearly. For  $N \equiv 0 \pmod{8}$ , put  $N = 4^e N'$ , where  $4 \nmid N'$ ; then the congruence

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv N' \pmod{8}$$

has a primitive solution, and therefore, for any  $n$ , the congruence

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv N' \pmod{2^n}$$

has a solution  $x_i \equiv a_i \pmod{2^n}$ . Therefore,

$$(2^e a_1)^2 + (2^e a_2)^2 + (2^e a_3)^2 + (2^e a_4)^2 \equiv N \pmod{2^n},$$

that is, Theorem 1 holds for  $k = 2$ .

Next we consider the case  $k \neq 2$ . Note that

$$(\mathbb{Z}/2^\nu\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{\tau}\mathbb{Z}$$

and

$$(\mathbb{Z}/2^\nu\mathbb{Z})^k = \{0 \pmod{2^\nu}, 1 \pmod{2^\nu}\}.$$

The latter holds from the former and from  $2^k \equiv 0 \pmod{2^\nu}$ , which follows from Lemma 1. Therefore, for  $s = 2, 3, \dots, 2^\nu - 2$ , Theorem 1 holds.

When  $k = 4$ , then  $2^\nu = 16$ . For any integer  $N$  and  $s = 15$ , we will show that the congruence (4) has a solution for all positive integers  $n$ . The statement clearly holds for  $N \not\equiv 0 \pmod{16}$ . For  $N \equiv 0 \pmod{16}$ , put  $N = 16^e N'$ , where  $16 \nmid N'$ ; the proof is similar to the case  $k = 2$ ,  $s = 4$  and  $N \equiv 0 \pmod{8}$ .

Finally we consider the case  $k \neq 2, 4$  and  $s = 2^\nu - 1$ . When  $N \not\equiv 0 \pmod{2^\nu}$ , the congruence (4) has a primitive solution for any integer  $N$  from Lemma 2; therefore the congruence has a solution for any integer  $n$ . When  $N \equiv 0 \pmod{2^\nu}$ , the congruence has only a trivial solution, and the statement follows from Lemma 3.  $\square$

Next, we consider nonexistence conditions of a solution for the congruence

$$(5) \quad x_1^k + \dots + x_s^k \equiv N \pmod{p^n}$$

for odd primes of  $p$ .

**Theorem 2.** *Let  $p$  be an odd prime.*

1. *When  $(k, p - 1) = p - 1$ , the nonexistence conditions of a solution for the congruence (5) are as follows:*

$$\begin{aligned} &(i, j \pmod p) \text{ for } 2 \leq i \leq p - 2, \ i + 1 \leq j \leq p - 1, \\ &(i, j \pmod{p^m}) \text{ for } 2 \leq m \leq \nu, \ p^{m-1} - 1 \leq i \leq p^m - 2, \\ &\hspace{15em} i + 1 \leq j \leq p^m - 1, \\ &(p^\nu - 1, p^\nu t \pmod{p^{\nu+1}}) \text{ for } 1 \leq t \leq p - 1. \end{aligned}$$

2. *In the case where  $(k, p - 1) = (p - 1)/2$ :*
  - (a) *If  $(p, \tau) = (3, 0)$ , the congruence  $x_1^k + x_2^k \equiv M \pmod{3^m}$  has a primitive solution for any integer  $M$  and any positive integer  $m$ .*
  - (b) *If  $(p, \tau) \neq (3, 0)$ , the nonexistence conditions of a solution for the congruence (5) are as follows:*

$$\begin{aligned} &(i, j \pmod p) \text{ for } 2 \leq i \leq (p - 3)/2, \ i + 1 \leq j \leq p - i - 1, \\ &(i, j \pmod{p^m}) \text{ for } 2 \leq m \leq \nu, \ (p^{m-1} - 1)/2 \leq i \leq (p^m - 3)/2, \\ &\hspace{15em} i + 1 \leq j \leq p^m - i - 1. \end{aligned}$$

*Proof.* Note that  $(\mathbb{Z}/p^\nu\mathbb{Z})^\times \cong \mathbb{Z}/(p - 1)p^\tau\mathbb{Z}$ .

1. When  $(k, p - 1) = p - 1$ ,

$$(\mathbb{Z}/p^\nu\mathbb{Z})^k = \{0 \pmod{p^\nu}, 1 \pmod{p^\nu}\}$$

follows from the above fact and from  $p^k \equiv 0 \pmod{p^\nu}$ . Therefore, for  $s = 2, 3, \dots, p^\nu - 2$ , Theorem 2 holds. For  $s = p^\nu - 1$ , the congruence

$$x_1^k + \dots + x_s^k \equiv M \pmod{p^\nu}$$

has a solution for any integer  $M$ . Therefore, from Lemma 2, for  $M \not\equiv 0 \pmod{p^\nu}$ , the solution is primitive. For  $M \equiv 0 \pmod{p^\nu}$ , the above congruence has only a trivial solution, and when  $s = p^\nu$ , the congruence has a primitive solution. Therefore, the statement holds by Lemma 3.

2. When  $(k, p - 1) = (p - 1)/2$ , the proof is similar to the case  $(k, p - 1) = p - 1$  except that

$$(\mathbb{Z}/p^\nu\mathbb{Z})^k = \{0 \pmod{p^\nu}, 1 \pmod{p^\nu}, -1 \pmod{p^\nu}\},$$

and the congruence

$$x_1^k + x_2^k \equiv 0 \pmod{p^\nu}$$

has a primitive solution. □

*Remark 1.* When (a)  $p = 2$  or (b)  $p$  is odd and  $(k, p - 1) \geq (p - 1)/2$ , using Theorems 1 and 2, we can find the minimal  $s$  such that the congruence

$$x_1^k + \dots + x_s^k \equiv M \pmod{p^m}$$

has a primitive solution for any integer  $M$  and any positive integer  $m$  as follows:

- (a)  $k = 2,$  then 4,  
 $k = 4,$  then 15,  
 $k$  is even,  $\neq 2, 4,$  then  $2^\nu,$   
 $k$  is odd, then 2,
- (b)  $(k, p - 1) = p - 1,$  then  $p^\nu,$   
 $(p, \tau) = (3, 0),$  then 2,  
 $(k, p - 1) = (p - 1)/2, (p, \tau) \neq (3, 0),$  then  $(p^\nu - 1)/2.$

Hardy and Littlewood obtained these values in [4].

#### 4. FINITENESS OF SEARCH FOR A FIXED $k$

In this section, we show three kinds of theoretical results that our search algorithm depends on to obtain all nonexistence conditions for the congruence

$$x_1^k + \dots + x_s^k \equiv N \pmod{p^n}.$$

The first kind of theoretical result shows that for a fixed  $k$  we can obtain all nonexistence conditions in a finite number of steps (Theorem 5 and Corollary 2). The second kind is for efficiency (Proposition 1, Lemma 4 and Corollaries 1, 3 and 5). The third kind is for Waring’s problem in  $p$ -adic integers, which is used in Section 7 (Theorem 6).

First, we observe solutions with modulus  $p$ . For  $s = 2$ , the following famous theorem by Weil [7] clearly shows that the necessary search is finite.

**Theorem 3** ([7]). *Let  $C$  be a nonsingular projective curve over a finite field  $\mathbb{F}_p$ . Let  $L$  be the number of  $\mathbb{F}_p$ -rational points, and let  $g$  be the genus of  $C$ . Then,*

$$|L - p - 1| \leq 2g\sqrt{p}.$$

From Theorem 3, we obtain the following corollary, which is used for the efficient search.

**Corollary 1.** *Let  $p$  be a prime and  $d$  be  $(k, p - 1)$ . We write  $k = p^\tau dl$ , where  $(p, l) = 1$ , and write  $d = 2^f d'$ , where  $d'$  is odd. Put*

$$c = \begin{cases} d & p = 2 \text{ or } p \equiv 1 \pmod{2^{f+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

*If  $p$  satisfies the inequality*

$$p + 1 - c > (dl - 1)(dl - 2)\sqrt{p},$$

*then for any integer  $M$  the following congruence has a solution:*

$$(6) \quad x_1^k + x_2^k \equiv M \pmod{p}.$$

*Proof.* It is sufficient to prove the case  $M \not\equiv 0 \pmod{p}$ . Note that the congruence (6) has a solution if and only if the congruence

$$x_1^{dl} + x_2^{dl} \equiv M \pmod{p}$$

has a solution. Apply Theorem 3 to the nonsingular projective curve over  $\mathbb{F}_p$  defined by the equation

$$x^{dl} + y^{dl} - Mz^{dl} = 0.$$

The genus of the curve is  $(dl - 1)(dl - 2)/2$ . The number of  $\mathbb{F}_p$ -rational points whose  $z$  coordinates are 0 is  $d$ , if there exists a nontrivial solution for  $x^{dl} + y^{dl} = 0$  in  $\mathbb{F}_p$ , and otherwise 0. Therefore, Corollary 1 follows from the next lemma.  $\square$

**Lemma 4.** *The congruence  $x_1^k + x_2^k \equiv 0 \pmod{p}$  has a nontrivial solution if and only if  $p = 2$  or  $p \equiv 1 \pmod{2^{f+1}}$ , where  $k = 2^f k'$  and  $k'$  is odd.*

The following corollary is also derived from Theorem 3, and it gives a computable bound  $A_k$  such that for any prime  $p > A_k$  and any integer  $M$ , the congruence (6) has a solution.

**Corollary 2.** *Let  $k \geq 2$  be an integer and let  $A_k$  be*

$$\frac{1}{2} \left( (k-1)^2(k-2)^2 + 2(k-1) + (k-1)(k-2)\sqrt{(k-1)^2(k-2)^2 + 4(k-1)} \right).$$

*Then for any prime number  $p > A_k$  and for any integer  $M$ , the congruence (6) has a solution. The order of magnitude of  $A_k$  is  $k^4$ .*

*Proof.* In Corollary 1,  $c \leq k$  and  $dl \leq k$ .  $\square$

For  $s \geq 3$ , we can obtain similar results. The following theorem corresponds to Theorem 3 (see [8] for an example).

**Theorem 4** ([8]). *Let  $L$  be the number of solutions of the congruence*

$$a_1 x_1^{k_1} + \dots + a_s x_s^{k_s} \equiv 0 \pmod{p},$$

*where  $p$  does not divide  $a_1, \dots, a_s$ . Then*

$$|L - p^{s-1}| \leq D(p-1)p^{s/2-1},$$

*where  $D = \prod_{i=1}^s (d_i - 1)$ ,  $d_i = (k_i, p - 1)$ .*

The following two corollaries give conditions when the congruence

$$(7) \quad x_1^k + \dots + x_s^k \equiv M \pmod{p}$$

has a solution for any integer  $M$ ; Corollary 3 corresponds to Corollary 1 and Corollary 4 corresponds to Corollary 2.

**Corollary 3.** *Let  $k \geq 2$  and  $s \geq 2$  be integers,  $p$  be a prime, and  $d$  be  $(k, p - 1)$ . If  $p$  satisfies the inequality*

$$(8) \quad p^{s/2} > (d-1)^s \left( (d-1)p^{1/2} + 1 \right),$$

*then for any integer  $M$ , the congruence (7) has a solution.*

*Proof.* It is sufficient to prove this for  $M \not\equiv 0 \pmod{p}$ . The congruence (7) has a solution if and only if the congruence

$$(9) \quad x_1^k + \dots + x_s^k - M x_{s+1}^k \equiv 0 \pmod{p}$$

has a solution such that  $x_{s+1} \not\equiv 0 \pmod{p}$ . From Theorem 4 the number of solutions of the congruence (9) is at least

$$p^s - (d-1)^{s+1}(p-1)p^{(s-1)/2},$$

and that of the congruence (9) such that  $x_{s+1} \equiv 0 \pmod{p}$  is at most

$$p^{s-1} + (d-1)^s(p-1)p^{s/2-1},$$

since the latter is equal to the number of solution of the congruence

$$x_1^k + \dots + x_s^k \equiv 0 \pmod{p}.$$

Therefore, if the following inequality holds, then the congruence (7) has a solution:

$$(10) \quad p^s - (d - 1)^{s+1}(p - 1)p^{(s-1)/2} > p^{s-1} + (d - 1)^s(p - 1)p^{s/2-1}$$

Corollary 3 follows from the inequality (10). □

**Corollary 4.** *Let  $k \geq 2$  and  $s \geq 3$  be integers, and let*

$$A_k(s) = (k - 1)^{2s/(s-1)}k^{2/(s-1)}.$$

*Then for any prime number  $p \geq A_k(s)$  and for any integer  $M$ , the congruence (7) has a solution.*

*Proof.* Since  $d \leq k$  and  $1 < p^{1/2}$ , if  $p$  satisfies the inequality

$$(11) \quad p^{s/2} \geq (k - 1)^s k p^{1/2},$$

then  $p$  satisfies the inequality (8) in Corollary 3. Corollary 4 follows from the inequality (11). □

*Remark 2.* For small values of  $k$  and  $s$ , the bound  $A_k(s)$  is larger than  $A_k$ . The pairs  $(k, s)$  such that the inequality  $A_k(s) > A_k$  holds are as follows: (a)  $(k, 3)$  where  $k \geq 2$ , (b)  $(3, 4), (3, 5), (3, 6), (4, 4)$ .

The following two corollaries give conditions when the congruence

$$(12) \quad x_1^k + \dots + x_s^k \equiv 0 \pmod{p}$$

has a nontrivial zero for an even  $k$ ; Corollary 5 corresponds to Corollary 1 and Corollary 6 corresponds to Corollary 2.

**Corollary 5.** *Let  $k \geq 2$  be even,  $s \geq 3$  be an integer,  $p$  be a prime and  $d$  be  $(k, p - 1)$ . If  $p$  satisfies the inequality*

$$p^{s/2} - (d - 1)^s(p - 1) > 0,$$

*then for any integer  $M$  the congruence (12) has a primitive solution.*

*Proof.* If the inequality

$$(13) \quad p^{s-1} - (d - 1)^s p^{s/2-1} > 0$$

holds, then the congruence (12) has a nontrivial solution since the number of solutions for this congruence is not less than  $p^{s-1} - (d - 1)^s(p - 1)p^{s/2-1}$ . Corollary 5 immediately follows from the inequality (13). □

**Corollary 6.** *Let  $k \geq 2$  be even,  $s \geq 3$  be an integer, and let  $B_k(s)$  be  $(k - 1)^{2s/(s-2)}$ . Then for any prime number  $p \geq B_k(s)$ , the congruence (12) has a primitive solution.*

*Remark 3.* The congruence

$$(14) \quad x_1^k + x_2^k + x_3^k \equiv 0 \pmod{p}$$

has a nontrivial solution if and only if the congruence  $x_1^k + x_2^k \equiv -1 \pmod{p}$  has a solution. Therefore, if  $p > A_k$  then the congruence (14) has a nontrivial solution. The inequality  $A_k < B_k(s)$  holds if and only if  $k \geq 4$  and  $s = 3, 4$ .

Next, we are concerned with higher prime powers  $p^n$ . Thanks to the fact described in Section 2, we must only examine  $n \leq \nu$ .

If  $s = 2$  and  $p|k$ , then there exists an integer  $N$  such that the congruence has no solution with modulus  $p^2$ .

**Proposition 1.** *Let  $k \geq 2$  be an integer. If a prime  $p$  divides  $k$ , then there exists an integer  $N$  such that the congruence*

$$(15) \quad x_1^k + x_2^k \equiv N \pmod{p^2}$$

has no solution.

*Proof.* It is sufficient to prove that  $k = p$ . Since  $\#\{a^p \mid a \in \mathbb{Z}/p^2\mathbb{Z}\} = p$ ,

$$\begin{aligned} \#\{a^p + b^p \mid a, b \in \mathbb{Z}/p^2\mathbb{Z}\} &\leq \#\{\{a^p, b^p\} \mid a, b \in \mathbb{Z}/p^2\mathbb{Z}\} = \frac{p^2 + p}{2} \\ &< p^2 = \#\mathbb{Z}/p^2\mathbb{Z}. \end{aligned}$$

Therefore, there exists at least one  $N \in \mathbb{Z}/p^2\mathbb{Z}$  such that the congruence (15) has no solution.  $\square$

With modulus  $p^n$ , we need only examine  $s < k$ , thanks to the next theorem.

**Theorem 5** ([4]). *Suppose  $k$  is of the form  $k = p^\tau dl$ , where  $d = (k, p - 1)$ ,  $p \nmid l$  and  $p$  is an odd prime. If  $d < (p - 1)/2$ , then for  $s \geq k$  the congruence*

$$(16) \quad x_1^k + \cdots + x_s^k \equiv M \pmod{p^m}$$

has a primitive solution for any integer  $M$  and any positive integer  $m$ .

*Remark 4.* Improvements of the form “if  $s \geq k^c$  ( $c < 1$ ) then the congruence has a solution” exist: Birch [1], Dodson [3], Bovey [2], etc. However, their results contain a condition “for all sufficiently large  $k$ .”

From Corollaries 4, 6 and Remarks 2, 3, we obtain the following theorem, which is a supplement to Theorem 5.

**Theorem 6.** *Let  $k \geq 4$  and  $3 \leq s \leq k - 1$  be integers, and  $p$  be a prime. Then the congruence (16) has a primitive solution for any integer  $M$  and any positive integer  $m$  if the following condition is satisfied, where  $A_k$ ,  $A_k(s)$  and  $B_k(s)$  are described as in Corollaries 2, 4 and 6, respectively.*

1. *When  $k$  is odd, then*
  - $p > A_k$  for  $s = 3$ ,
  - $p \geq A_k(s)$  for  $s \geq 4$ .
2. *When  $k$  is even, then*
  - $p > A_4 = 21 + 12\sqrt{3} = 41.78 \dots$  for  $(k, s) = (4, 3)$ ,
  - $p \geq B_k(s)$  for other  $(k, s)$ .

*Proof.* First note that if  $p > A_k$  (resp.  $p \geq A_k(s)$  or  $p \geq B_k(s)$ ) then  $p \nmid k$ . Therefore, we need only examine with modulus  $p$  whether the congruence has a primitive solution for any integer  $M$ .

When  $k$  is odd, the congruence has a nontrivial zero even if  $s = 2$ . Therefore, Theorem 6 follows from Corollary 4 and Remark 2.



When  $k$  is even, first we will show that in the range  $3 \leq s \leq k - 1$  the inequality  $A_k(s) < B_k(s)$ , i.e., the inequality

$$(17) \quad \frac{A_k(s)}{B_k(s)} = \left( \frac{k^{s-2}}{(k-1)^s} \right)^{2/(s-1)(s-2)} < 1$$

holds. But this follows from the following inequalities:

$$\frac{k^{s-2}}{(k-1)^s} = \frac{1}{k^2} \left( 1 + \frac{1}{k-1} \right)^s \leq \frac{1}{k^2} \left( 1 + \frac{1}{k-1} \right)^{k-1} < \frac{e}{k^2} \leq \frac{e}{16} < 1.$$

Therefore, by Corollaries 4, 6 and Remark 3, Theorem 6 holds. □

### 5. ALGORITHM

Using the results in Section 4, we will show that for a fixed  $k$ , we can obtain all nonexistence conditions for the congruence

$$(18) \quad x_1^k + \dots + x_s^k \equiv N \pmod{p^n}$$

in a finite number of steps. Note that if  $p \nmid k$  and  $(k, p-1) = 1$ , then the congruence

$$x_1^k + x_2^k \equiv M \pmod{p^m}$$

has a primitive solution for any  $M$  and  $m$ . Therefore, we only examine primes  $p$  such that  $p|k$  or  $d = (k, p-1) > 1$ .

**Algorithm** (finding all nonexistence conditions).

**Input:** An integer  $k \geq 2$ .

**Output:** All nonexistence conditions for the congruence (18).

1. For  $p = 2$  and odd primes  $p$  such that  $d \geq (p-1)/2$ , we completely determine the nonexistence conditions of a solution for the congruence (18) from Theorems 1 and 2.
2. For other primes  $p$ , we need only examine  $s < k$  from Theorem 5.
  - (a) The number of odd primes  $p$  that satisfy  $p|k$  and  $1 \leq d < (p-1)/2$  is finite, and it is sufficient to examine whether the congruence (18) has a solution in the range  $n \leq \nu$  for such a prime  $p$ , using Lemmas 3, 4, Proposition 1, and Corollaries 1, 3, 5, or by a computer search.
  - (b) There are an infinite number of odd primes  $p$  such that  $p \nmid k$  and  $1 < d < (p-1)/2$ , by Dirichlet's Theorem. Since  $\nu = 1$  for these  $p$ , we need only examine whether the congruence

$$(19) \quad x_1^k + \dots + x_s^k \equiv N \pmod{p}$$

has a primitive solution (higher powers of  $p^n$  are not necessary).

For any prime  $p > A_k$  and any integer  $N$ , the congruence (19) has a solution. When  $p \nmid N$  the solution is primitive from Lemma 2.

Suppose that  $p > A_k$  and  $p|N$ . When  $s > 2$  the congruence (19) has a primitive solution (set  $x_s = 1$ ). When  $s = 2$  we completely determine whether (19) has a primitive solution from Lemma 4. For primes  $p$  where (19) has no primitive solution, we completely determine the values  $N \pmod{p^2}$  so that the congruence  $x_1^k + x_2^k \equiv N \pmod{p^2}$  has no solution from Lemma 3.

Therefore, we must only examine  $s < k$  and odd primes  $p$  that satisfy  $p \leq A_k$ ,  $p \nmid k$  and  $1 < d < (p-1)/2$  to determine whether the congruence (19)

has a primitive solution using Corollaries 3 and 5, or by a computer search.

We illustrate the algorithm when  $k = 5$ .

**Example** ( $k = 5$ ). We examine primes  $p$  such that  $p|5$  or  $d = (5, p - 1) > 1$ ; the latter condition is equivalent to  $p \equiv 1 \pmod{10}$ .

1. Among the above primes, 11 is the only one that satisfies  $d \geq (p - 1)/2$ . From Theorem 2, we completely determine the nonexistence conditions of a solution for the congruence (18).
2. For other primes, we must only examine  $s < 5$ .
  - (a) 5 is the only prime that divides  $k$ . Note that  $\nu = 2$  for  $p = 5$ . Since  $a^5 \equiv a \pmod{5}$  for any  $a$ , the congruence (19) has a solution for any  $s \geq 2$  and any integer  $N$ . When  $s = 2$ , there exists an integer  $N$  such that the congruence

$$x_1^5 + x_2^5 \equiv N \pmod{5^2}$$

has no solution, by Proposition 1. We search for them and obtain  $N \equiv 3, 4, 5, 9, 10, 12, 13, 15, 16, 20, 21, 22 \pmod{5^2}$ . When  $s = 3$ , for the above values  $N$  we find that the congruence

$$x_1^5 + x_2^5 + x_3^5 \equiv N \pmod{5^2}$$

has a primitive solution by a computer search.

- (b) Since the bound  $A_5$  is  $76 + 24\sqrt{10} = 151.89\dots$ , the primes we must examine are 31, 41, 61, 71, 101, 131, 151. Since 5 is odd, the congruence

$$x_1^5 + x_2^5 \equiv M \pmod{p}$$

has a primitive solution for any integer  $M$  and any prime  $p > 151$ .

Set  $p = 31$  and  $s = 2$ . The pair  $(k, p) = (5, 31)$  does not satisfy the condition in Corollary 1. Therefore, by a computer search, we find that for  $N \equiv 3, 8, 9, 13, 14, 15, 16, 17, 18, 22, 23, 28 \pmod{31}$ , the congruence

$$x_1^5 + x_2^5 \equiv N \pmod{31}$$

has no solution. Next set  $s = 3$ . Since  $(k, s, p) = (5, 3, 31)$  does not satisfy the condition in Corollary 3, we must examine whether the congruence

$$(20) \quad x_1^5 + x_2^5 + x_3^5 \equiv N \pmod{31}$$

has a solution for the above values of  $N$  by a computer search. We confirm that for these values of  $N$ , the congruence (20) has a primitive solution. Similar procedures are carried out for  $p = 41, 61, 71, 101, 131, 151$ , and we obtain the results described in Table 2.

## 6. TABLES DERIVED BY COMPUTER SEARCH

Using the algorithm in Section 5, for a fixed  $k$ , we obtained the nonexistence conditions of the form  $(s, N \pmod{p^n})$ .

Fortunately, for  $k = 2$  and 3, no computer search is necessary.

**Case  $k = 2$ :** The bound  $A_2$  is 1. The nonexistence conditions  $(s, N \pmod{p^n})$  are:

- (2, 3 mod 4), from Theorem 1,
- (2,  $pt \pmod{p^2}$ ) for  $p \equiv 3 \pmod{4}$  and  $1 \leq t \leq p - 1$ , from Lemmas 3 and 4,
- (3, 7 mod 8), from Theorem 1.

**Case  $k = 3$ :** The bound  $A_3$  is  $7.46\dots$ . The nonexistence conditions  $(s, N \pmod{p^n})$  are:

- $(2, i \pmod 9)$ , for  $3 \leq i \leq 6$ , from Theorem 2,
- $(2, i \pmod 7)$ , for  $i = 3, 4$ , from Theorem 2,
- $(3, i \pmod 9)$ , for  $i = 4, 5$ , from Theorem 2.

For other values of  $k$ , we completed tables by computer search. We obtained the nonexistence conditions for  $4 \leq k \leq 10$  and all primes  $k \leq 47$ . To save space, we do not show all of the results; Tables 1, 2 and 3 show the nonexistence conditions  $(s, N \pmod{p^n})$  for  $k = 4, 5$  and  $7$ , respectively. For  $k = 7$ , integers  $N \pmod{p^n}$  are represented by means of a generator of the cyclic group  $((\mathbb{Z}/p^n\mathbb{Z})^\times)^7$  and representatives of the cosets  $(\mathbb{Z}/p^n\mathbb{Z})^\times / ((\mathbb{Z}/p^n\mathbb{Z})^\times)^7$ . For example, the first row means the congruence

$$x_1^7 + x_2^7 \equiv N \pmod{7^2}$$

has no solution for  $N \equiv 31^i t \pmod{7^2}$ , where  $1 \leq i \leq 6$  and  $t = 3, 9, 27, 43$ .

Table 4 shows the nonexistence conditions  $(s, p^n)$  for  $k = 11, 13, 17, 19$ . To save space, this table shows only the maximal  $s$  for each pair  $(k, p^n)$ . For example, the row “ $k = 11, s = 3, p^n = 11^2, 89$ ” means there exist integers  $N_1$  and  $N_2$  such that the congruences

$$\begin{aligned} x_1^{11} + \dots + x_s^{11} &\equiv N_1 \pmod{11^2} \\ x_1^{11} + \dots + x_s^{11} &\equiv N_2 \pmod{89} \end{aligned}$$

have no solution for  $s \leq 3$ .

The computer search was carried out on a DEC Alpha Server 4100/5/400 (400 MHz, 256 MB memory). We show the CPU times for some values of  $k$ : less than 0.1 seconds for  $k = 7$ , approximately 40 seconds for  $k = 19$ , approximately 20 minutes for  $k = 29$ , approximately 2 hours for  $k = 37$ , and approximately 13 hours for  $k = 47$ .

TABLE 1. Nonexistence conditions  $(s, N \pmod{p^n})$  for  $k = 4$ .

| $s$ | $N \pmod{p^n}$                             |
|-----|--------------------------------------------|
| 2   | 7, 8, 11 mod 13                            |
|     | 6, 7, 10, 11 mod 17                        |
|     | 4, 5, 6, 9, 13, 22, 28 mod 29              |
|     | $pt \pmod{p^2}$ ( $1 \leq t \leq p - 1$ )* |
|     | $37t \pmod{37^2}$ ( $1 \leq t \leq 36$ )   |
| 3   | $29t \pmod{29^2}$ ( $1 \leq t \leq 28$ )   |

\*:  $p = 7, 11, 19, 23, 31$ .

- $A_4 = 41.78\dots$ ,
- mod  $2^n$  : see Theorem 1,
- mod  $3^n, \pmod{5^n}$  : see Theorem 2,
- mod  $p^2$  for  $p \geq 43, \equiv 3 \pmod{4}$  : see Lemmas 3 and 4,
- mod  $p^2$  for  $p \geq 53, \equiv 5 \pmod{8}$  : see Lemmas 3 and 4.

TABLE 2. Nonexistence conditions  $(s, N \bmod p^n)$  for  $k = 5$ .

| $s$ | $N \bmod p^n$                                                                                                                                                                                           |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 2   | 3, 4, 5, 9, 10, 12, 13, 15, 16, 20, 21, 22 mod $5^2$<br>3, 8, 9, 13, 14, 15, 16, 17, 18, 22, 23, 28 mod 31<br>7, 16, 19, 20, 21, 22, 25, 34 mod 41<br>4, 5, 6, 9, 17, 23, 38, 44, 52, 55, 56, 57 mod 61 |

mod  $11^n$ : see Theorem 2.

TABLE 3. Nonexistence conditions  $(s, N \bmod p^n)$  for  $k = 7$ .

| $s$ | $N \bmod p^n$                                        |                                                                                                |
|-----|------------------------------------------------------|------------------------------------------------------------------------------------------------|
|     | generator of $((\mathbb{Z}/p^n\mathbb{Z})^\times)^7$ | representatives of $(\mathbb{Z}/p^n\mathbb{Z})^\times / ((\mathbb{Z}/p^n\mathbb{Z})^\times)^7$ |
| 2   | 31 mod $7^2$                                         | 3, 9, 27, 43 mod $7^2$                                                                         |
|     | 12 mod 29                                            | 3, 4, 6, 8 mod 29                                                                              |
|     | 37 mod 43                                            | 3, 9, 27, 28 mod 43                                                                            |
|     | 14 mod 71                                            | 7 mod 71                                                                                       |
|     | 40 mod 113                                           | 81 mod 113                                                                                     |
|     | 28 mod 127                                           | 116 mod 127                                                                                    |
| 3   | 31 mod $7^2$                                         | 9, 27 mod $7^2$                                                                                |
|     | 12 mod 29                                            | 8 mod 29                                                                                       |
|     | 37 mod 43                                            | 27 mod 43                                                                                      |

TABLE 4. Nonexistence conditions  $(s, p^n)$  for  $k = 11, 13, 17, 19$ .

| $k$ | $s$ | $p^n$                                                                                  |
|-----|-----|----------------------------------------------------------------------------------------|
| 11  | 2   | 199, 331, 353, 419, 463, 617                                                           |
|     | 3   | $11^2, 89$                                                                             |
|     | 4   | 67                                                                                     |
| 13  | 2   | $13^2, 131, 157, 313, 443, 521, 547, 599, 677, 859, 911, 937, 1171$                    |
|     | 4   | 79                                                                                     |
|     | 5   | 53                                                                                     |
| 17  | 2   | 239, 307, 409, 443, 613, 647, 919, 953, 1021, 1123, 1259, 1327, 1361, 1531, 1667       |
|     | 3   | $17^2, 137$                                                                            |
|     | 5   | 103                                                                                    |
| 19  | 2   | $19^2, 457, 571, 647, 761, 1103, 1217, 1483, 1559, 1597, 1787, 2053, 2129, 2357, 2927$ |
|     | 3   | 191, 229, 419                                                                          |

mod  $23^n$  for  $k = 11$ : see Theorem 2.

7. WARING'S PROBLEM IN  $p$ -ADIC FIELDS

Through theoretical analysis and computer search, for several values of  $k$  we obtained the nonexistence conditions of a solution for the congruence

$$x_1^k + \dots + x_s^k \equiv N \pmod{p^n}.$$

These results are closely related to Waring's problem in  $p$ -adic fields, namely, the problem of representing any  $p$ -adic integer by a sum of  $s$   $k$ th powers of  $p$ -adic integers. The problem is equivalent to finding a primitive solution of the congruence

$$(21) \quad x_1^k + \dots + x_s^k \equiv M \pmod{p^\nu}$$

for any rational integer  $M$ , except in the case  $(k, p) = (4, 2)$  (the least  $s$  such that any 2-adic integer can be represented as  $s$  4th powers of 2-adic integers is 15, however, the least  $s$  such that the congruence (21) has a primitive solution for any rational integer  $M$  is 16). We define the number  $\Gamma_p(k)$  as the least positive integer  $s$  such that the congruence (21) has a primitive solution for all rational integers  $M$ . The number  $\Gamma(k)$  is defined as  $\max\{\Gamma_p(k)\}$ , where  $p$  runs through all prime numbers.

We utilize the algorithm described in Section 5 for computing  $\Gamma(k)$ . Obtaining the value of  $\Gamma(k)$  is easier than obtaining all nonexistence conditions for  $k$ , since the bound, up to which we must examine primes  $p$ , decreases whenever we find a prime  $p$  such that  $\Gamma_q(k) < \Gamma_p(k)$  for all primes  $q < p$ . Significantly, if we find a prime  $p$  such that  $\Gamma_p(k) \geq k$  in Step 1, then Step 2 is not necessary, by Theorem 5.

We illustrate how the bound decreases while computing  $\Gamma(34)$ . In Step 1, the largest value of  $\Gamma_p(34)$  is 8 for  $p = 2$ . Since  $8 < k = 34$ , by Theorem 6, we must examine whether there exists a prime  $p < B_{34}(8) = 11203.93\dots$  such that  $\Gamma_p(34) \geq 9$  using Corollaries 3, 5 or by a computer search. The bound decreases to  $B_{34}(10) = 6255.82\dots$  when we find that  $\Gamma_{103}(34) = 10$ . Note that we must examine primes  $p \leq A_{34} = 115201.99\dots$  to obtain all nonexistence conditions for  $k = 34$ .

In [4], Hardy and Littlewood considered the number  $\Gamma(k)$ ; however, their notations were slightly different from ours. Tables 5 and 6 correspond to Tables 1 and 3 in [4], respectively. In the row in Table 5 and in the column in Table 6, " $p$ " refers to the least  $p$  such that  $\Gamma_p(k) = \Gamma(k)$ . There were undecided results (e.g., the entry of  $\Gamma(37)$  was " $\geq 9$ ") and several errors in [4]. The symbols " $*$ " and " $\dagger$ " refer to the undecided results and errors in [4], respectively. It took approximately 6 seconds to obtain Table 5. Note that it took a much longer time to obtain all nonexistence conditions for a single value of  $k$  ( $k = 29$ , as described in Section 6,

TABLE 5.  $\Gamma(k)$  for  $3 \leq k \leq 36$  (corresponding to Table 1 in [4]).

|             |   |    |    |   |   |    |    |    |    |    |    |    |    |    |     |    |               |
|-------------|---|----|----|---|---|----|----|----|----|----|----|----|----|----|-----|----|---------------|
| $k$         | 3 | 4  | 5  | 6 | 7 | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17  | 18 | 19            |
| $\Gamma(k)$ | 4 | 16 | 5  | 9 | 4 | 32 | 13 | 12 | 11 | 16 | 6  | 14 | 15 | 64 | 6   | 27 | 4             |
| $p$         | 3 | 2  | 11 | 3 | 7 | 2  | 3  | 5  | 23 | 2  | 53 | 29 | 31 | 2  | 103 | 3  | $\dagger$ 191 |

  

|             |    |    |    |    |    |             |    |    |    |    |    |               |     |    |              |    |              |
|-------------|----|----|----|----|----|-------------|----|----|----|----|----|---------------|-----|----|--------------|----|--------------|
| $k$         | 20 | 21 | 22 | 23 | 24 | 25          | 26 | 27 | 28 | 29 | 30 | 31            | 32  | 33 | 34           | 35 | 36           |
| $\Gamma(k)$ | 25 | 24 | 23 | 23 | 32 | 10          | 26 | 40 | 29 | 29 | 31 | 5             | 128 | 33 | $\dagger$ 10 | 35 | $\dagger$ 37 |
| $p$         | 5  | 7  | 23 | 47 | 2  | $\dagger$ 5 | 53 | 3  | 29 | 59 | 31 | $\dagger$ 373 | 2   | 67 | 103          | 71 | $\dagger$ 37 |

TABLE 6.  $\Gamma(k)$  for  $37 \leq k \leq 200$  (corresponding to Table 3 in [4]).

| $k$ | $\Gamma(k)$ | $p$ | $k$ | $\Gamma(k)$ | $p$  | $k$ | $\Gamma(k)$ | $p$  | $k$ | $\Gamma(k)$ | $p$  |
|-----|-------------|-----|-----|-------------|------|-----|-------------|------|-----|-------------|------|
| 37  | *9          | 149 | 78  | 84          | 13   | 119 | 119         | 239  | 160 | *128        | 2    |
| 38  | *9          | 229 | 79  | *13         | 317  | 120 | 120         | 241  | 161 | *23         | 47   |
| 39  | 39          | 79  | 80  | †64         | 2    | 121 | *16         | 727  | 162 | †243        | 3    |
| 40  | 41          | 41  | 81  | 121         | 3    | 122 | *21         | 367  | 163 | *21         | 653  |
| 41  | 41          | 83  | 82  | 83          | 83   | 123 | *41         | 83   | 164 | *83         | 83   |
| 42  | 49          | 7   | 83  | 83          | 167  | 124 | *20         | 373  | 165 | 165         | 331  |
| 43  | †12         | 173 | 84  | *49         | 7    | 125 | 125         | 251  | 166 | 167         | 167  |
| 44  | 44          | 89  | 85  | *7          | 1021 | 126 | 127         | 127  | 167 | *5          | 2339 |
| 45  | *15         | 31  | 86  | 86          | 173  | 127 | *21         | 509  | 168 | 168         | 337  |
| 46  | 47          | 47  | 87  | *29         | 59   | 128 | 512         | 2    | 169 | *25         | 677  |
| 47  | *10         | 283 | 88  | 89          | 89   | 129 | †*12        | 173  | 170 | *20         | 1021 |
| 48  | 64          | 2   | 89  | 89          | 179  | 130 | 131         | 131  | 171 | *180        | 19   |
| 49  | *13         | 197 | 90  | 90          | 181  | 131 | 131         | 263  | 172 | 173         | 173  |
| 50  | †62         | 5   | 91  | *13         | 547  | 132 | *67         | 67   | 173 | 173         | 347  |
| 51  | 51          | 103 | 92  | *47         | 47   | 133 | *7          | 1597 | 174 | 174         | 349  |
| 52  | 53          | 53  | 93  | *17         | 373  | 134 | 134         | 269  | 175 | *35         | 71   |
| 53  | 53          | 107 | 94  | *18         | 283  | 135 | 135         | 271  | 176 | †176        | 353  |
| 54  | 81          | 3   | 95  | 95          | 191  | 136 | 144         | 17   | 177 | *21         | 709  |
| 55  | 60          | 11  | 96  | 128         | 2    | 137 | *17         | 823  | 178 | 179         | 179  |
| 56  | 56          | 113 | 97  | *16         | 389  | 138 | 139         | 139  | 179 | 179         | 359  |
| 57  | *14         | 229 | 98  | 98          | 197  | 139 | *18         | 557  | 180 | 181         | 181  |
| 58  | 59          | 59  | 99  | 99          | 199  | 140 | 140         | 281  | 181 | *19         | 1087 |
| 59  | †5          | 709 | 100 | 125         | 5    | 141 | 141         | 283  | 182 | *26         | 53   |
| 60  | 61          | 61  | 101 | *16         | 607  | 142 | *19         | 569  | 183 | 183         | 367  |
| 61  | *11         | 367 | 102 | 103         | 103  | 143 | *18         | 859  | 184 | *47         | 47   |
| 62  | *12         | 373 | 103 | *16         | 619  | 144 | *73         | 73   | 185 | *9          | 149  |
| 63  | 63          | 127 | 104 | †53         | 53   | 145 | *29         | 59   | 186 | 186         | 373  |
| 64  | 256         | 2   | 105 | 105         | 211  | 146 | 146         | 293  | 187 | *22         | 1123 |
| 65  | 65          | 131 | 106 | 107         | 107  | 147 | 171         | 7    | 188 | *22         | 1129 |
| 66  | *67         | 67  | 107 | *15         | 643  | 148 | 149         | 149  | 189 | 189         | 379  |
| 67  | *12         | 269 | 108 | *109        | 109  | 149 | *8          | 1193 | 190 | 191         | 191  |
| 68  | 68          | 137 | 109 | *6          | 1091 | 150 | 151         | 151  | 191 | 191         | 383  |
| 69  | 69          | 139 | 110 | 121         | 11   | 151 | *19         | 907  | 192 | 256         | 2    |
| 70  | 71          | 71  | 111 | 111         | 223  | 152 | *32         | 2    | 193 | *21         | 773  |
| 71  | *6          | 569 | 112 | 113         | 113  | 153 | 153         | 307  | 194 | 194         | 389  |
| 72  | 73          | 73  | 113 | 113         | 227  | 154 | *23         | 23   | 195 | *65         | 131  |
| 73  | *16         | 293 | 114 | 114         | 229  | 155 | 155         | 311  | 196 | *197        | 197  |
| 74  | 74          | 149 | 115 | *23         | 47   | 156 | 169         | 13   | 197 | *5          | 3547 |
| 75  | 75          | 151 | 116 | 116         | 233  | 157 | †*7         | 1571 | 198 | 199         | 199  |
| 76  | *16         | 2   | 117 | *39         | 79   | 158 | 158         | 317  | 199 | *25         | 797  |
| 77  | *14         | 463 | 118 | †*17        | 709  | 159 | †53         | 107  | 200 | 200         | 401  |

took approximately 20 minutes). The results in Table 6 were obtained in approximately 33 hours. Almost all CPU time was for  $k = 167$  and 197 (approximately 9 hours for  $k = 167$  and approximately 23 hours for  $k = 197$ ).

Hardy and Littlewood proved that  $\Gamma(k) \geq 3$ . After declaring they could not prove that  $\Gamma(k) \geq 4$ , although no case of  $\Gamma(k) = 3$  was known, they wrote (p. 539 in [4]):

We have explored the possibilities  $\Gamma(k) = 3$  and  $\Gamma(k) = 4$  in the range  $2 < k \leq 3000$ . Our results are that  $\Gamma(k) > 3$  in all cases; and  $\Gamma(k) > 4$  except for  $k = 2, 3, 7, 19$ , for which it is 4, and possibly (but very improbably) for  $k = 1163, 1637, 1861, 1997, 2053$ .

Therefore, we computed  $\Gamma(k)$  for  $k = 1163, 1637, 1861, 1997, 2053$ . In approximately one minute, we obtained

$$\begin{aligned} \Gamma_{37217}(1163) &= 6, & \Gamma_{62207}(1637) &= 7, & \Gamma_{74441}(1861) &= 5, \\ \Gamma_{87869}(1997) &= 5, & \Gamma_{94439}(2053) &= 5, \end{aligned}$$

where the primes  $p = 37217, 62207, 74441, 87869, 94439$  were the least primes satisfying the condition  $p \equiv 1 \pmod{k}$  for  $k = 1163, 1637, 1861, 1997, 2053$ , respectively. That is,  $\Gamma(k) \geq 6, 7, 5, 5, 5$  for  $k = 1163, 1637, 1861, 1997, 2053$ , and we confirmed that  $\Gamma(k) > 4$  in the range  $19 < k \leq 3000$ .

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