MATHEMATICS OF COMPUTATION
Volume 68, Number 228, Pages 1739–1747
S 0025-5718(99)01117-5
Article electronically published on May 24, 1999

PRIME CLUSTERS AND CUNNINGHAM CHAINS

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ABSTRACT. We discuss the methods and results of a search for certain types of prime clusters. In particular, we report specific examples of prime 16-tuplets and Cunningham chains of length 14.

INTRODUCTION

We are mainly interested in finding large, maximally dense clusters of primes. Let k be an integer greater than one. Generalizing the notion of prime twins, we define a prime k-tuplet as a sequence of k consecutive primes such that in some sense the difference between the first and the last is as small as possible. More precisely, we first define $s(k)$ to be the smallest number s for which there exist a set of k integers $\{b_1 = 0, b_2, \ldots, b_k\}$ such that $b_k = s$ and, for every prime q, not all the residues classes modulo q are represented by $\{0, b_2, \ldots, b_k\}$. We can then define a prime k-tuplet as a sequence of consecutive primes $\{p_1, \ldots, p_k\}$, such that $p_k - p_1 = s(k)$ and $p_i - p_1 = b_i$, $i = 2, \ldots, k$. The definition excludes a finite number (for each k) of dense clusters at the beginning of the prime number sequence; for example, $\{97, 101, 103, 107, 109\}$ satisfies the conditions of the definition of a prime 5-tuplet, but $\{3, 5, 7, 11, 13\}$ does not because all three residues modulo 3 are represented.

The definition is motivated by the Prime k-tuple Conjecture, as stated by Dickson [1] and in a quantitative form by Hardy and Littlewood [2]. The function $s(k)$ has the property that there cannot be more than a finite number of sets of k consecutive primes where the difference between the largest and the smallest prime is less than $s(k)$. On the other hand, the Prime k-tuple Conjecture predicts that the prime k-tuplets we have defined above occur infinitely often for each k and each admissible set $\{b_1, \ldots, b_k\}$.

The simplest case is $s(2) = 2$, corresponding to prime twins. Next, $s(3) = 6$, where there are two types of prime triplets: $\{p, p+2, p+6\}$ and $\{p, p+4, p+6\}$. Then $s(4) = 8$ with just one pattern $\{p, p+2, p+6, p+8\}$ of prime quadruplets, $s(5) = 12$ with two patterns of prime quintuplets, $\{p, p+4, p+6, p+10, p+12\}$ and $\{p, p+2, p+6, p+8, p+12\}$, $s(6) = 16$ with one pattern $\{p, p+4, p+6, p+10, p+12, p+16\}$, and so on.

We are assuming that k is not too large. In general, however, proving that there exists at least one prime k-tuplet for each k seems to be a problem of extreme difficulty, and it has not yet been solved. Let $\rho^*(x)$ be the number of elements in the largest admissible set contained in the interval $[1, x]$. Hensley and Richards

Received by the editor July 24, 1997.
1991 *Mathematics Subject Classification.* Primary 11A41, 11Y11.

[3] have shown that $\rho^*(x)$ exceeds $\pi(x)$ for all sufficiently large x. The Prime k-tuple Conjecture would then imply the existence of infinitely many super-dense prime k-tuplets with more primes in the k-tuplet than there are in $[0, s(k)]$. This is inconsistent with a conjecture of Hardy and Littlewood, which states that for integers $x, y > 2$ we always have $\pi(x+y) \le \pi(x) + \pi(y)$. Gordon and Rodemich [4] examined the behaviour of $\rho^*(x)$ and, in particular, they determined the crossover point at which $\rho^*(x)$ first exceeds $\pi(x)$.

An algorithm for computing $s(k)$

For $k \ge 3$, it is possible to compute $s(k)$ recursively by means of the simple algorithm given below. The notation $p\#$ is that of Caldwell and Dubner [5]; for p prime, $p\#$ is the product of all the primes up to and including p.

Procedure $s(k)$:

Do $S(s, 3, 1)$ for $s = s(k-1)+2, s(k-1)+4, \ldots$ until an admissible set B is found.

Procedure $S(s, q, H)$:

Step 1. Set $U = q\#$, the product of all the primes $\le q$. Set $D = \frac{U}{q}$ and $h = H$.

Step 2. Set $B = \{i\colon i = 0, 2, \ldots, s,\ \gcd(h+i, U) = 1\}$.

Step 3. If B does not contain both 0 and s, go to step 8.

Step 4. If B has less than k elements, go to step 8.

Step 5. If B has more than k elements, do $S(s, q', h)$, where q' is the next prime after q. Then go to step 8.

Step 6. If B has exactly k elements and if for each prime p, $q < p \le k$, all residues modulo p are represented by B, go to step 8.

Step 7. Indicate that B is an admissible set and report $s(k) = s$.

Step 8. Add D to h. If $h < H + U$, go to step 2. Otherwise return.

Starting with $s(2) = 2$ and applying the procedure successively to $k = 3, 4, \ldots, 20$, we obtain Table 1, which shows $s(k)$ and admissible patterns.

Table 1

k	$s(k)$	Number of patterns	Patterns $\{b_1 = 0, b_2, \ldots, b_k = s(k)\}$
2	2	1	$\{0, 2\}$
3	6	2	$\{0, 2, 6\}$, $\{0, 4, 6\}$
4	8	1	$\{0, 2, 6, 8\}$
5	12	2	$\{0, 2, 6, 8, 12\}$, $\{0, 4, 6, 10, 12\}$
6	16	1	$\{0, 4, 6, 10, 12, 16\}$
7	20	2	$\{0, 2, 6, 8, 12, 18, 20\}$, $\{0, 2, 8, 12, 14, 18, 20\}$
8	26	3	$\{0, 2, 6, 12, 14, 20, 24, 26\}$, $\{0, 2, 6, 8, 12, 18, 20, 26\}$, $\{0, 6, 8, 14, 18, 20, 24, 26\}$
9	30	4	$\{0, 4, 6, 10, 16, 18, 24, 28, 30\}$, $\{0, 4, 10, 12, 18, 22, 24, 28, 30\}$, $\{0, 2, 6, 8, 12, 18, 20, 26, 30\}$, $\{0, 2, 6, 12, 14, 20, 24, 26, 30\}$

TABLE 1 (continued)

k	$s(k)$	Number of patterns	Patterns $\{b_1 = 0, b_2, \ldots, b_k = s(k)\}$
10	32	2	$\{0, 2, 6, 8, 12, 18, 20, 26, 30, 32\}$, $\{0, 2, 6, 12, 14, 20, 24, 26, 30, 32\}$
11	36	2	$\{0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 36\}$, $\{0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36\}$
12	42	2	$\{0, 6, 10, 12, 16, 22, 24, 30, 34, 36, 40, 42\}$, $\{0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42\}$
13	48	6	$\{0, 6, 12, 16, 18, 22, 28, 30, 36, 40, 42, 46, 48\}$, $\{0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48\}$, $\{0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 36, 46, 48\}$, $\{0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48\}$, $\{0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48\}$, $\{0, 2, 12, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48\}$
14	50	2	$\{0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50\}$, $\{0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 50\}$
15	56	4	$\{0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56\}$, $\{0, 2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56\}$, $\{0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56\}$, $\{0, 6, 8, 14, 20, 24, 26, 30, 36, 38, 44, 48, 50, 54, 56\}$
16	60	2	$\{0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48, 54, 58, 60\}$, $\{0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60\}$
17	66	4	$\{0, 4, 10, 12, 16, 22, 24, 30, 36, 40, 42, 46, 52, 54, 60, 64, 66\}$, $\{0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48, 54, 58, 60, 66\}$, $\{0, 6, 8, 12, 18, 20, 26, 32, 36, 38, 42, 48, 50, 56, 60, 62, 66\}$, $\{0, 2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56, 62, 66\}$
18	70	2	$\{0, 4, 10, 12, 16, 22, 24, 30, 36, 40, 42, 46, 52, 54, 60, 64, 66, 70\}$, $\{0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48, 54, 58, 60, 66, 70\}$
19	76	4	$\{0, 6, 10, 16, 18, 22, 28, 30, 36, 42, 46, 48, 52, 58, 60, 66, 70, 72, 76\}$, $\{0, 4, 6, 10, 16, 22, 24, 30, 34, 36, 42, 46, 52, 60, 64, 66, 70, 72, 76\}$, $\{0, 4, 6, 10, 12, 16, 24, 30, 34, 40, 42, 46, 52, 54, 60, 66, 70, 72, 76\}$, $\{0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48, 54, 58, 60, 66, 70, 76\}$
20	80	2	$\{0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 50, 56, 62, 66, 68, 72, 78, 80\}$, $\{0, 2, 8, 12, 14, 18, 24, 30, 32, 38, 42, 44, 50, 54, 60, 68, 72, 74, 78, 80\}$

THE LARGEST KNOWN PRIME k-TUPLETS

At this point it is convenient to record the largest prime k-tuplet known to the author (at time of writing), for $k = 2, 3, \ldots, 16$. I am not aware of any examples for $k \geq 17$ other than the easily identifiable ones that occur near the beginning of the prime number sequence. In keeping with similar published lists, all the numbers listed below are true, proven primes. As before, $p\#$ denotes the product of the primes up to p.

$k = 2$: $\{242206083 \times 2^{38880} + b\colon b = -1, 1\}$ (11713 digits, 1995, K.-H. Indlekofer & A. Járai [6])

$k = 3$: $\{437850590(2^{3567} - 2^{1189}) - 6 \times 2^{1189} + b\colon b = -5, -1, 1\}$ (1083 digits, 1996, T. Forbes [7])

$k = 4$: $\{10^{499} + 883750143961 + b\colon b = 0, 2, 6, 8\}$ (500 digits, 1996, Warut Roonguthai [8])

$k = 5$: $\{8947613442 \times 53\# \times 2^{672} + 101 + b\colon b = 0, 2, 6, 8, 12\}$ (232 digits, 1997, A. O. L. Atkin [9])

$k = 6$: $\{82248305245 \times 43\# \times 2^{479} + 16057 + b\colon b = 0, 4, 6, 10, 12, 16\}$ (172 digits, 1997, A. O. L. Atkin [9])

$k = 7$: $\{426955143694213197848463574726328636553002998029907759380111141003679237691 + b\colon b = 0, 2, 6, 8, 12, 18, 20\}$ (75 digits, 1997, A. O. L. Atkin [9])

$k = 8$: $\{582994762600347672560616756460401857268344059297139419451 + b : b = 0, 2, 6, 8, 12, 18, 20, 26\}$ (57 digits, 1996, A. O. L. Atkin [9])

$k = 9$: $\{11456782178002488855779277536193082378054961 + b\colon b = 2, 6, 8, 12, 18, 20, 26, 30, 32\}$ (44 digits, 1996, A. O. L. Atkin [9])

$k = 10$: $\{11456782178002488855779277536193082378054961 + b\colon b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32\}$ (44 digits, 1996, A. O. L. Atkin [9])

$k = 11$: $\{495064300630708278713578451 + b\colon b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36\}$ (27 digits, 1997, A. O. L. Atkin [9])

$k = 12$: $\{495064300630708278713578451 + b\colon b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42\}$ (27 digits, 1997, A. O. L. Atkin [9])

$k = 13$: $\{964013473328959309238999 + b\colon b = 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48\}$ (24 digits, 1997, T. Forbes)

$k = 14$: $\{11319107721272355839 + b\colon b = 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 50\}$ (20 digits, 1997, T. Forbes)

$k = 15$: $\{84244343639633356306067 + b\colon b = 0, 2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56\}$ (23 digits, 1997, T. Forbes)

$k = 16$: $\{1522014304823128379267 + b\colon b = 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60\}$ (22 digits, 1997, T. Forbes)

Primality proofs for the triplets ($k = 3$) can be established by the methods of Brillhart, Lehmer and Selfridge [10]. Writing

$$N = 437850590(2^{3567} - 2^{1189}) - 6 \times 2^{1189},$$

the primality of $N + 1$, $N - 1$ and $N - 5$ follow from the partial factorizations

$$N = 2^{1191} \times 3^2 \times 7^2 \times (\text{composite})$$

and

$$N - 6 = (2^{1189} + 1) \times 2 \times 17 \times 144887 \times (\text{composite}).$$

The number $2^{1189} + 1$ has been completely factored into primes by the Cunningham project [11].

The entries for $k = 5, 6, \ldots, 12$ are unpublished results of Oliver Atkin, and I am very grateful for his permission to include them in this paper.

Guy [12, Section A9] lists a number of prime k-tuplets, including the only two large prime 14-tuplets known at that time, found by Dimitrios Betsis and Sten Säfholm. The entries for $k = 13$ and $k = 14$ in the above list are new.

Apart from $\{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67\}$, $\{17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73\}$ and subsets of the 16-tuplets reported in this paper, only eight further prime 15-tuplets are currently known to the author, namely:

$$\{8985208997951457604337 + b\colon b \in B_{15}\},$$
$$\{2958380122665046736597 + b\colon b \in B'_{15}\},$$
$$\{2088253704394088213987 + b\colon b \in B_{15}\},$$
$$\{1337707385720650557617 + b\colon b \in B_{15}\},$$
$$\{944716030613719714367 + b\colon b \in B'_{15}\},$$
$$\{205700275761622834847 + b\colon b \in B_{15}\},$$
$$\{107862607835977274207 + b\colon b \in B_{15}\},$$
$$\{36351118555624575707 + b\colon b \in B_{15}\}.$$

Here,

$$B_{15} = \{0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56\}$$

and

$$B'_{15} = \{0, 2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56\}.$$

The one beginning 205700275761622834847 was first published in [13].

Only three prime 16-tuplets are known to me, the obvious example $\{13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73\}$, the one given in the main list above, and

(1)
$$\{47710850533373130107 + b\colon b = 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60\},$$

first announced on the Internet (on 20 May 1997) via the NMBRTHRY mailing list.

Prime 16-tuplets

We now give a brief description of the search for prime 16-tuplets that led to the discovery of (1). Recalling that $s(16) = 60$, it is easy to determine that there are just two admissible patterns of primes:

$$\{p + b\colon b = 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60\} \tag{2}$$

and

$$\{p - b\colon b = 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60\}. \tag{3}$$

For our purpose it is natural to combine both (2) and (3) into a single form,

$$\{|p + b|\colon b \in B\}, \tag{4}$$

where

$$B = \{0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60\}$$

is the set of displacements and where p is now allowed to take both positive and negative values. The Prime k-tuple Conjecture implies that each of (2) and (3) occurs with all 16 elements prime for infinitely many values of p.

Let q be a prime greater than 23, and let $Q = q\# = 2 \times 3 \times \cdots \times q$. To find suitable candidates for the first element p of a prime 16-tuplet (4), we consider numbers of the form $gQ+h$, where h satisfies $\gcd(\prod_{b\in B}(h+b),Q)=1$ and g runs from $-G$ to G for some fixed positive integer G.

It is possible to generate the h's very efficiently by means of the Chinese Remainder Theorem. Let $Q=m_0m_1\dots m_r$, where $m_0,m_1,\dots,m_r$ are pairwise coprime. For $i=0,1,\dots,r$, let H_i be the set of residues $j \pmod{m_i}$, $0\le j<m_i$, such that $\gcd(j+b,m_i)=1$ for all $b\in B$. Denote their number by $\varphi_B(m_i)$. Thus φ_B is a generalization of Euler's φ-function. Write

$$C_i=\left\{\left(h\left(\left(\frac{Q}{m_i}\right)^{-1} \bmod m_i\right)\frac{Q}{m_i} \bmod Q\right) : h\in H_i\right\},$$

where $((\frac{Q}{m_i})^{-1} \bmod m_i)$ denotes the unique integer x, $0\le x<m_i$ for which $x\frac{Q}{m_i}\equiv 1 \pmod{m_i}$. Let c_i run through the $\varphi_B(m_i)$ elements of C_i and let

$$h=c_0+c_1+\cdots+c_r.$$

By a straightforward application of the Chinese Remainder Theorem, h runs through the residues $(\bmod\, Q)$ such that $\gcd(h+b,Q)=1$ for all $b\in B$.

From the programmer's point of view, what is so marvellous about this approach is that we can calculate each of the $\varphi_B(m_0)+\varphi_B(m_1)+\cdots+\varphi_B(m_r)$ coefficients $c_0\in C_0$, $c_1\in C_1,\dots,c_r\in C_r$ in advance and store them in an array to be indexed by a kind of $(r+1)$-digit number $[d_0d_1\dots d_r]$. The digit d_i ranges from 0 to $\varphi_B(m_i)-1$ and indexes the numbers c_i in the set C_i. One can imagine a nest of iterative loops, one for each digit, $d_0,d_1,\dots,d_r$, numbered from the innermost to the outermost. We begin with the outer loop and set

$$h_r=c_r \pmod Q.$$

Then at the jth stage, $j=r-1,r-2,\dots,0$, we compute

$$h_j=h_{j-1}+c_j \pmod Q,$$

eventually to yield the final sum

$$h=h_0=h_1+c_0=c_r+c_{r-1}+\cdots+c_0 \pmod Q$$

in the central loop. Hence, assuming $\varphi_B(m_0)$ is not too small, the effort required to generate the next h is essentially just the addition of c_0 to h_1. It is not necessary to reduce the sum modulo Q in the innermost loop if the parameter G is increased by one.

If the search is to be spread over several computers, this structure provides a convenient method of parcelling out ranges by distributing the workload on the basis of the high-order digits of $[d_0d_1\dots d_r]$.

In our search for prime 16-tuplets, the divisors of Q are the composite integer 23# and the primes $29,31,37,\dots,q$ for some suitable q. By a straightforward computation, $\varphi_B(23\#)=160$, $\varphi_B(29)=14$ and $\varphi_B(p)=p-16$ for prime $p\ge 31$. We perform a sieving procedure to eliminate those g's, $-G\le g\le G$, where $\prod_{b\in B}(gQ+h+b)$ is divisible by a prime p, $q<p\le P$ for some fixed prime P, the sieve limit. For each p and for each $b\in B$, we compute

$$g_{h,b}=G-hQ^{-1}-bQ^{-1} \pmod p,$$

where Q^{-1} is the multiplicative inverse of Q modulo p.

We exploit the 32-bit architecture of the computer by processing the h's in batches of 32 at a time. The sieve table is an array of 32-bit words, each bit position in the word corresponding to a specific h. Thus we have 512 $g_{h,b}$'s, 16 for each h. We eliminate all $g_{h,b} + ip$, where i runs from 0 to $[2G/p]$, by flagging the appropriate bits in the sieve table. The survivors, numbers $x = gQ + h$ such that $\gcd(x + b, P!) = 1$ for all $b \in B$, can then go forward to be checked for probable primality by the usual type of test based on the Fermat-Euler theorem.

For large primes, it is quite appropriate to treat all the residues $g_{h,b} \pmod p$ as if they are different. However, for primes $p \le L$, with suitably chosen L, it is more efficient to consolidate the $g_{h,b}$'s into residue classes modulo p; then the sieve requires only p bit-operations for each i, rather than 512. Although occasionally a residue class $(\bmod\, p)$ is empty, it turns out that the penalty for testing this possibility is too severe. It is faster to operate on all p residue classes anyway. Whilst the cut-off parameter L can be determined accurately by experiment, we found that the value which suggests itself, namely 512, seems to be about right.

Let X denote the size of the largest numbers we wish to test. We must choose G and $Q = q\#$ such that $GQ = X$, at least approximately. There is a balance between G and Q. The time taken to perform the sieving operation is approximately a linear function of G, $\alpha G + \beta$, say, where the constant term β represents the fixed overheads of setting up the sieve for a batch of h's. On the one hand, we do not want G to be too big, for then we could add an extra factor q' to Q. This results in $\varphi_B(q')$ times as many sieving operations. To keep the numbers limited by X, we need a corresponding reduction of G by a factor q', and if G is large, the sieve will consequently run nearly q' times as fast. Hence there will be an overall performance improvement by a factor of somewhat less than $q'/\varphi_B(q')$. On the other hand, there is a limit to this process of trading between Q and G. Eventually G will be too small and the overheads term β of the linear function will become significant.

The actual parameters used by the computer program were $G = 8000$ and $q = 53$. Hence $Q = 53\# = 32589158477190044730$ and the range of numbers searched was $\pm 2.6 \times 10^{23}$ approximately. A sieve limit of $P = 997$ was more or less optimal.

Cunningham chains

A Cunningham chain of length k is a finite set of primes $\{p_1, \dots, p_k\}$, where either

$$p_{i+1} = 2p_i + 1, \qquad i = 1, 2, \dots, k-1,$$

or

$$p_{i+1} = 2p_i - 1, \qquad i = 1, 2, \dots, k-1.$$

The subject is discussed in Section A7 of Guy's book [12], in which it is reported that Günter Löh [14] discovered two 12-chains, one of each type, and one 13-chain of the second type.

The computer program used to find the prime 16-tuplets needs only trivial modifications to search for long Cunningham chains of both types. The displacement set is

$$B = \{2^i - 1 \colon i = 0, 1, 2, \dots, 15\};$$

$Q = 2^{15} \times 43\#$, and the divisors of Q for the generation of h's by the Chinese Remainder Theorem are $2^{15}19\#$, together with the primes 23, 29, 31, 37, 41 and 43. The corresponding values of φ_B are $\varphi_B(2^{15}19\#) = 108$, $\varphi_B(23) = 12$, $\varphi_B(29) = 13$, $\varphi_B(31) = 26$, $\varphi_B(37) = 21$, $\varphi_B(41) = 25$ and $\varphi_B(43) = 29$.

So far, at the time of writing, the program has found several chains of length 14: seven chains of the first type (we list just the initial prime)

$$\begin{aligned}
&2330543688171775790 9,\\
&6253105248013318694 9,\\
&8214754592091070880 9,\\
&12254027672386963319 9,\\
&14374829242253283803 9,\\
&27626338263637162763 9,\\
&38593175525034578447 9;
\end{aligned}$$

and five chains of the second type

$$\begin{aligned}
&758133173223699273 1,\\
&5844045539037622488 1,\\
&11409243451760098230 1,\\
&31761016841751714660 1,\\
&38296659075934098840 1.
\end{aligned}$$

Note, added September 1998

During the time that has elapsed since the preparation of the initial version of this paper, the author and others have extended many of the results. In particular, the author has discovered prime 17-tuplets,

$$\{3259125690557440336631 + b\colon b = 0, 6, 8, 12, 18, 20, 26, 32, 36, 38, 42, 48, 50, 56, 60, 62, 66\},$$

several Cunningham chains of 15 elements and one Cunningham chain of 16 elements. The initial prime of the sixteen-element chain is

$$3203000719597029781.$$

Further details will appear in a forthcoming paper.

I would like to thank the referee for drawing my attention to the papers of Dickson [1] and Gordon and Rodemich [4].

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