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ON THE SPECTRUM OF THE ZHANG-ZAGIER HEIGHT

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ABSTRACT. From recent work of Zhang and of Zagier, we know that their height $\mathfrak{H}(\alpha)$ is bounded away from 1 for every algebraic number α different from $0, 1, 1/2 \pm \sqrt{-3}/2$. The study of the related spectrum is especially interesting, for it is linked to Lehmer's problem and to a conjecture of Bogomolov. After recalling some definitions, we show an improvement of the so-called Zhang-Zagier inequality. To achieve this, we need some algebraic numbers of small height. So, in the third section, we describe an algorithm able to find them, and we give an algebraic number with height $1.2875274\ldots$ discovered in this way. This search up to degree 64 suggests that the spectrum of $\mathfrak{H}(\alpha)$ may have a limit point less than 1.292. We prove this fact in the fourth part.

1. INTRODUCTION

Let P be a polynomial in n variables with coefficients in $\mathbb{Z}$. We define the Mahler measure of P as

$$M(P(z_1,\ldots,z_n)) = \exp\left\{\int_0^1\cdots\int_0^1 \log|P(e^{2i\pi t_1},\ldots,e^{2i\pi t_n})|\,dt_1\cdots dt_n\right\}.$$

If P is a one variable polynomial, $P(z) = a_0\prod_{j=1}^d (z-\alpha_j)$, it is well known that

$$M(P) = |a_0|\prod_{j=1}^d \max(1,|\alpha_j|).$$

We denote in this case the absolute Mahler measure of P, i.e., $M(P)^{1/d}$ by $\mathfrak{M}(P)$. For $\alpha \in \overline{\mathbb{Q}}$, $M(\alpha)$ and $\mathfrak{M}(\alpha)$ are the Mahler and the absolute Mahler measure, respectively, of the irreducible polynomial of α with coefficients in $\mathbb{Z}$. The *Zhang-Zagier height* or simply the *height* of α, denoted by $\mathfrak{H}(\alpha)$, is then defined as $\mathfrak{H}(\alpha) = \mathfrak{M}(\alpha)\mathfrak{M}(1-\alpha)$. From results of Zhang and Zagier (cf. [Zh92], [Za93]), we know that if α is an algebraic number different from the roots of $(z^2-z)(z^2-z+1)$,

$$\mathfrak{H}(\alpha) \geqslant \sqrt{\frac{1+\sqrt{5}}{2}} = 1.2720196\ldots. \tag{1}$$

Our work relies on a computer search for polynomials of small height. G. Rhin and C. J. Smyth made a remark that simplifies greatly this search. They pointed out in [RS97] that if $Q(z) = P(z)P(1-z)$,

$$\mathfrak{H}(P) = M(Q)^{1/d} = \mathfrak{M}(Q)^2 = \mathfrak{H}(Q).$$

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So we only need to consider polynomials of even degree which are symmetric under $z \mapsto 1 - z$, i.e., polynomials in

$$X = z(1 - z).$$

Using the results of this search we shall prove the following theorem, where $\Phi_{10}(z)$ represents the 10th cyclotomic polynomial.

Theorem 1. *Let α be an algebraic number different from the roots of $(z^2 - z)(z^2 - z + 1)\Phi_{10}(z)\Phi_{10}(1 - z)$. Then*

$$\mathfrak{H}(\alpha) \geqslant 1.2817770214.$$

The inequality of Zhang-Zagier and a fortiori this last theorem incidentally give an answer to Lehmer's problem for polynomials in X (cf. [B78] for a survey of the question). Note at this point that $M(P(X))$ shall always refer to the Mahler measure with respect to the variable z.

We also studied the spectrum of $\mathfrak{H}(\alpha)$ because it is connected to an important conjecture of Bogomolov, now proved (cf. for example [DP98] and [U98]). Indeed, for certain plane curves, this conjecture asserts the existence of a real $\mu > 1$ which is the smallest limit point of the set of the normalized heights of the algebraic numbers which lie, with their conjugates, on the curve. However, we do not know any value of μ for any curve. A natural example is $x + y = 1$, and in this situation the normalized height coincides with the Zhang-Zagier height, so that a study of $\mathfrak{H}(\alpha)$ enables us to compute an approximate value of μ. Namely, we shall show

Theorem 2. *The smallest limit point μ of $\mathcal{V} = \{\mathfrak{H}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ belongs to $[1.28177702, 1.2916674]$.*

Let us now explain how to improve the lower bound of (1).

2. Proof of Theorem 1

First, we recall the lemma Zagier used in [Za93] in order to prove his theorem. The standard notation $\log^+ |z|$ represents $\max(0, \log |z|)$.

Lemma 1. *Let $z \in \mathbb{C}$. We then have*

$$\log^+ |z| + \log^+ |1 - z| \geqslant C_1 \log |z^2 - z| + C_2 \log |z^2 - z + 1| + C,$$

where $C_1 = \frac{\sqrt{5}-1}{2\sqrt{5}}$, $C_2 = \frac{1}{2\sqrt{5}}$ *and* $C = \frac{1}{2} \log\left(\frac{1+\sqrt{5}}{2}\right)\cdot$

The lemma comes from what we call from now on an *auxiliary function*. Here it is

$$f(z) = \log^+ |z| + \log^+ |1 - z| - C_1 \log |P_1(X)| - C_2 \log |P_2(X)|,$$

where $P_1(X)$ and $P_2(X)$ are simply $z - z^2$ and $z^2 - z + 1$ expressed in terms of X, namely $P_1(X) = X$ and $P_2(X) = 1 - X$. The constants C_1 and C_2 are tuned with the aim of maximizing the minimum of f on $\mathbb{C}$. For these constants, the inequality becomes an equality if and only if z is a root of $\Phi_{10}(z)\Phi_{10}(1 - z)$.

One way to get a better lower bound in (1) is to add a third polynomial to f. The first we tried was of course $\Phi_{10}(z)\Phi_{10}(1 - z)$. Since this product is clearly symmetric under $z \mapsto 1 - z$, it is a polynomial P_3 of the variable X, namely $P_3(X) = X^4 - 2X^3 + 4X^2 - 3X + 1$. We then needed to find the constants C_1, C_2, C_3 which give the largest C such that the auxiliary function

$$\log^+ |z| + \log^+ |1 - z| - C_1 \log |P_1(X)| - C_2 \log |P_2(X)| - C_3 \log |P_3(X)|$$

TABLE 1.

Polynomial P_i	Constant C_i	$\mathfrak{H}(P_i)$
X	$\frac{85682}{350099}$	1
$1-X$	$\frac{70486}{375995}$	1
$X^4-2X^3+4X^2-3X+1$	$\frac{7895}{1046601}$	1.2720196...
$X^8-3X^7+8X^6-16X^5+26X^4-27X^3+17X^2-6X+1$	$\frac{1154}{848811}$	1.2974311...
X^3+X^2-2X+1	$\frac{399}{322591}$	1.3030333...
$X^8-2X^7+4X^6-7X^5+13X^4-16X^3+12X^2-5X+1$	$\frac{620}{672441}$	1.2919397...
$X^{16}-4X^{15}+10X^{14}-19X^{13}+39X^{12}-85X^{11}+179X^{10}-337X^9$ $+554X^8-761X^7+830X^6-691X^5+427X^4-190X^3+58X^2-11X+1$	$\frac{617}{675843}$	1.2944955...
$X^{12}-3X^{11}+8X^{10}-18X^9+36X^8-62X^7+97X^6$ $-123X^5+114X^4-73X^3+31X^2-8X+1$	$\frac{2264}{6600679}$	1.2888421...
$X^{16}-4X^{15}+10X^{14}-17X^{13}+26X^{12}-47X^{11}+119X^{10}-298X^9$ $+592X^8-878X^7+963X^6-780X^5+464X^4-199X^3+59X^2-11X+1$	$\frac{1197}{5000683}$	1.2875274...

is greater than C for all $z \in \mathbb{C}$. We computed them by means of an algorithm of C. J. Smyth, detailed in [Sm81]. Thus we were able to compute an approximation of C, namely $C = \log 1.2789960\ldots$. But the auxiliary function takes this value at some complex numbers which do not appear to be conjugate algebraic numbers. We then realized that no algebraic number is of height $\exp C$. So we were not able to repeat the process. Instead, we decided to complete the last auxiliary function by polynomials with particularly small height. First, we tested the ones V. Flammang found in her thesis (cf. [F94]). Then we carried out our own search for higher degrees by a method that we shall discuss in the next section.

Proof of Theorem 1. There are two steps: first an analytic part and then an arithmetic one. The analytic part copies the ideas of Lemma 1 and therefore relies on the concept of an auxiliary function. We describe further the polynomials involved, their respective heights and the optimal constants found by Smyth's algorithm. Note that we did not find exact values. The C_i's in Table 1 are only approximations.

With these settings, let

$$f_{\mathbf{c}}(z) = \log^+ |z| + \log^+ |1-z| - \sum_{i=1}^{9} C_i \log |P_i(X)|.$$

If d_i denotes $\deg_X P_i$, we remark that

$$1 - \sum_{i=1}^{9} d_i C_i = 0.4931085\ldots. \tag{2}$$

So $f_{\mathbf{c}}(z)$ tends to $+\infty$ as z tends to infinity. The same conclusion holds if z tends to any root of $\prod_{i=1}^{9} P_i(z(1-z))$. Besides, it is easy to see that $f_{\mathbf{c}}$ is harmonic off the two circles $|z| = 1$, $|z-1| = 1$ and away from the roots of $\prod_{i=1}^{9} P_i(z(1-z))$. Therefore, $f_{\mathbf{c}}$ attains its minimum at some points located on $|z| = 1$ and $|z-1| = 1$. As $f_{\mathbf{c}}(z) = f_{\mathbf{c}}(1-z)$, we only search for the minimum on the circle $|z| = 1$ and, for these z, we set $S = |z(1-z)|^2$. We see that the auxiliary function can be expressed in terms of S, i.e., $f_{\mathbf{c}}(z) = g(S)$. So to prove the theorem, we first search the minimum of $g(S)$ on $[0,4]$. The derivative of g vanishes at the roots of a polynomial which is of degree 133 with large integer coefficients. By the command `realroot` of Maple, we exhibited 28 intervals at a precision of 10^{-7}, each one containing a root of $g'(S)$. Here are these intervals.

$[0.260236844, 0.260236852]$	$[0.279423587, 0.279423594]$	$[0.302677564, 0.302677572]$
$[0.306441359, 0.306441367]$	$[0.325068042, 0.325068049]$	$[0.436903372, 0.436903380]$
$[0.448283918, 0.448283926]$	$[0.478844501, 0.478844509]$	$[0.502754539, 0.502754547]$
$[0.524517529, 0.524517536]$	$[0.529897101, 0.529897109]$	$[0.539879933, 0.539879940]$
$[1.929120556, 1.929120563]$	$[1.959240399, 1.959240407]$	$[2.038860343, 2.038860351]$
$[2.117468275, 2.117468283]$	$[2.222416870, 2.222416878]$	$[3.016049951, 3.016049959]$
$[3.055418812, 3.055418819]$	$[3.113223948, 3.113223955]$	$[3.249860741, 3.249860749]$
$[3.400025241, 3.400025249]$	$[3.573429264, 3.573429272]$	$[3.657728590, 3.657728598]$
$[3.785134435, 3.785134443]$	$[3.833153583, 3.833153591]$	$[3.844168827, 3.844168834]$
		$[3.931477770, 3.931477778]$

We then used Sturm sequences, through the command `polsturm` of PARI, to check that we did not omit any root and that the second derivative did not vanish in these intervals. Therefore, $f_{\mathbf{c}}$ keeps the same concavity and thus stays over its tangent near each one of its minimums. So we ended up computing a lower bound for $f_{\mathbf{c}}$, as claimed. Finally,

$$f_{\mathbf{c}}(z) \geqslant \log(1.2817770214)$$

for all $z \in \mathbb{C}$.

Then comes the arithmetic argument. Let α be an algebraic number different from the roots of $\prod_{i=1}^{9} P_i(z(1-z))$ and let $P(z) = a_0 \prod_{j=1}^{d}(z-\alpha_j)$ be its minimal polynomial. By the previous result,

$$\log^+ |\alpha_j| + \log^+ |1-\alpha_j| - \sum_{i=1}^{9} C_i \log |P_i(\alpha_j(1-\alpha_j))| \geqslant \log(1.2817770214)$$

for each conjugate of α. Summing these inequalities for j from 1 to d, one gets

$$\sum_{j=1}^{d} \log^+ |\alpha_j| + \sum_{j=1}^{d} \log^+ |1-\alpha_j| - \sum_{i=1}^{9} C_i \log \prod_{j=1}^{d} |P_i(\alpha_j(1-\alpha_j))| \geqslant d \log(1.2817770214).$$

Observing that

$$a_0^{2d_i} \prod_{j=1}^{d} |P_i(\alpha_j(1-\alpha_j))| = |\operatorname{Res}(P(z), P_i(z(1-z)))|$$

and adding $2\log|a_0|$ on both sides, we obtain

$$2\log|a_0| + \sum_{j=1}^{d}\log^+|\alpha_j| + \sum_{j=1}^{d}\log^+|1-\alpha_j| \geqslant \sum_{i=1}^{9} C_i \log|\operatorname{Res}(P(z), P_i(z(1-z)))|$$
$$+2\Big(1-\sum_{i=1}^{9} d_i C_i\Big)\log|a_0| + d\log(1.2817770214).$$

Now, $|\operatorname{Res}(P(z), P_i(z(1-z)))|$ is a positive integer and $(1-\sum_{i=1}^{9} d_iC_i) > 0$. Thus

$$M(\alpha)M(1-\alpha) \geqslant 1.2817770214^d$$

which completes this proof. □

Remark. As all the P_i's but P_1, P_2, P_3 have a height greater than 1.2817770214, no new isolated point of the spectrum of $\mathfrak{H}(\alpha)$ was discovered.

It is time to explain how we found some of the P_i's.

3. Search for polynomials of small height

Initially, we searched small heights in the hope of improving (1). After a while, we found the problem interesting in itself. We first checked roots of unity ζ_n. Unfortunately, if we consider $(p_k)_{k\in\mathbb{N}}$, the prime numbers sequence, we shall show, in the next section, that $\mathfrak{H}(\zeta_{p_k})$ tends to $1.3813545\ldots$ as k tends to infinity. The best candidate turned out to be ζ_{14} with a height of $1.3097840\ldots$, which is far too large. In the previous section, we said a few words on the work of V. Flammang. More precisely, she made an inventory of all polynomials of height less than 1.3 up to degree 18 in z, i.e., degree 9 in X. Her method consists of computing the height of polynomials whose integers coefficients are bounded by some inequalities and linked to one another by certain linear relationships. However, these bounds are exponential and higher degrees are completely unreachable.

So we conceived a new approach which we hoped would produce more polynomials of small height. Let $P(X) = \sum_{j=0}^{d} a_j X^{d-j}$ be a polynomial. First, we restricted our search to monic polynomials. Indeed, from Theorem 1 and relation (2), the inequality

$$|a_0| \leqslant (1.3/1.2817770214)^{d/0.4931085}$$

asserts that up to degree 24 a polynomial of height less than 1.3 is necessarily monic. Besides, we noticed that interesting polynomials have low resultant, often equal to 1 or -1, with some polynomials with a very small height or belonging to the previous auxiliary function. This is quite normal. If the height of P is low, the auxiliary function must have a value close to its minimum at each root of P, and $\log|\operatorname{Res}(P, P_i)|$ must be as little as possible. Moreover, we can easily show that

$$\log M(P(X)) - C_1\log|\operatorname{Res}(P,X)| - C_2\log|\operatorname{Res}(P,1-X)| \geqslant d\log 1.2817770214.$$

Hence

$$|\operatorname{Res}(P,X)| \leqslant \left(\frac{\mathfrak{H}(P)}{1.2817770214}\right)^{d/C_1} \text{and } |\operatorname{Res}(P,1-X)| \leqslant \left(\frac{\mathfrak{H}(P)}{1.2817770214}\right)^{d/C_2}.$$

For example, for d less than 12, $|\operatorname{Res}(P,X)|$ is 1 or less under the assumption that $\mathfrak{H}(P) < 1.3$. In the same manner, $|\operatorname{Res}(P,1-X)| \leqslant 1$ if $d \leqslant 9$ and $\mathfrak{H}(P) < 1.3$. For any other polynomial P_i of the auxiliary function, the corresponding constant C_i

is too small, so the bound is too large. Thus, we simply ask that $|\operatorname{Res}(P, P_i)| = 1$ a priori. As we all know,

$$\operatorname{Res}(P, P_i) = \operatorname{Res}(P \bmod P_i, P_i).$$

So the main idea is to build P from each remainder R_i of the Euclidean division of P by P_i. Of course, there are, in general, infinitely many polynomials R_i having resultant with P_i equal to ± 1. However, as the Mahler measure of P and the degree of R_i are both bounded above, only a finite subset $\mathcal{R}_i$ of the R_i need be considered. The construction of $\mathcal{R}_i$ relies on the following remark. If $|\operatorname{Res}(P, P_i)| = 1$, then $P(\theta_i)$ is a unit of $\mathbb{K} = \mathbb{Q}(\theta_i)$, where θ_i is a root of P_i. By Dirichlet's theorem, the algebraic number $P(\theta_i)$ can be expressed in terms of the fundamental units and of the roots of unity which belong to $\mathbb{K}$. As we can write any fundamental unit or root of unity of $\mathbb{K}$ as a polynomial in θ_i, the R_i's are simply the product modulo P_i of these polynomials to some power. The commands `bnfinit` and `bnfisunit` of PARI are very useful at this point. The technique used to build P from R_i is a matter of linear algebra. In general, we select two polynomials P_1 and P_2 whose degrees d_1 and d_2 verify $d = d_1 + d_2$. Then, we build P from the remainders R_1 and R_2 belonging respectively to $\mathcal{R}_1$ and $\mathcal{R}_2$ and compute its height, by the method of Graeffe (cf. [DHJ95]).

The algorithm was programmed in PARI. The computations, performed on a Pentium PII at 233Mhz, took from a few minutes to a couple of days, depending on the degree. More than 41000 polynomials were found for degrees in X ranging from

TABLE 2.

$\mathfrak{H}$ / Degree	1.29	1.291	1.292	1.293	1.294	1.296	1.298	Record
10						8	28	1.2945155...
11					1	6	22	1.2939545...
12	2	2	4	7	14	34	87	1.2888421...
13				4	7	51	186	1.2926938...
14			1	6	18	102	265	1.2917134...
15			1	1	3	11	91	1.2914361...
16	1	2	5	18	45	197	430	1.2875274...
17		2	11	33	70	272	369	1.2907680...
18			1	8	20	196	612	1.2913799...
19				1	4	26	51	1.2926006...
20	2	9	49	121	280	1084	2612	1.2893428...
21		4	11	36	109	706	1155	1.2904063...
22			1	5	20	41	46	1.2913747...
23			1	2	10	42	60	1.2917477...
24	2	35	129	440	1351	7991	22056	1.2888365...
25	1	8	32	134	384	1768	2369	1.2893561...
26		1	32	130	374	1722	2411	1.2909655...
27		2	4	12	64	752	3261	1.2901873...
28	1	3	21	122	483	2360	5093	1.2895016...
32		1	5	21	86	402	550	1.2907082...
Total	9	69	308	1101	3343	17771	41754	1.2875274...

10 to 32. Our investigation is not exhaustive. Gaps might even be numerous. But we think that the probability of having missed interesting polynomials is small. The results are gathered in Table 2. In each column, we can see the number of polynomials found of height less than the specified value $\mathfrak{H}$. The column *Record* shows the best height found for each degree. Note the 9 polynomials of height less than 1.29, hence smaller than 1.2903349..., which was the previous record (cf. [F94]). Nevertheless, none of them seems to be the second nontrivial point of the spectrum. The best height found, 1.2875274..., comes from the polynomial

$$X^{16} - 4X^{15} + 10X^{14} - 17X^{13} + 26X^{12} - 47X^{11} + 119X^{10} - 298X^9 \\ + 592X^8 - 878X^7 + 963X^6 - 780X^5 + 464X^4 - 199X^3 + 59X^2 - 11X + 1.$$

We also remark that the spectrum becomes very dense around 1.292. As we shall see in the next section, there is nothing surprising about this.

4. Search for limit points and proof of Theorem 2

We begin with a lemma which turns to be a very good tool to construct limit points.

Lemma 2. *Let P be a polynomial in two variables y and z, such that* $\deg_z P > 0$. *Let ζ_n be $e^{\frac{2i\pi}{n}}$ and assume that for all n and all k, $P(\zeta_n^k, z)$ is not identically zero. We then have*

$$M(P(y,z))^{(1/\deg_z P)} = \lim_{n\to\infty} \mathfrak{M}\left(\prod_{k=1}^{n} P(\zeta_n^k, z)\right).$$

Proof. We write

$$P(y,z) = a_0(y) \prod_{j=1}^{\deg_z P} (z - z_j(y)).$$

We detail the demonstration when $a_0(y)$ does not vanish on the torus. In the case when it does vanish on the torus, we can use a limiting argument. The main idea of the proof is the use of Riemann sums. On the one hand,

$$\log M(P(y,z)) = \int_0^1 \log|a_0(e^{2i\pi t})|\,dt + \sum_{j=1}^{\deg_z P} \int_0^1 \log^+|z_j(e^{2i\pi t})|\,dt. \tag{3}$$

On the other hand, as $a_0(\zeta_n^k) \neq 0$ for all k

$$\log \mathfrak{M}\left(\prod_{k=1}^{n} P(\zeta_n^k, z)\right) = \frac{1}{n \deg_z P} \log M\left(\prod_{k=1}^{n} P(\zeta_n^k, z)\right).$$

As the Mahler measure is multiplicative

$$\log M\left(\prod_{k=1}^{n} P(\zeta_n^k, z)\right) = \sum_{k=1}^{n} \log|a_0(\zeta_n^k)| + \sum_{j=1}^{\deg_z P} \sum_{k=1}^{n} \log^+|z_j(\zeta_n^k)|.$$

One can see Riemann sums which lead to (3) when n tends to infinity. □

As an application, one can cite

$$\lim_{n\to\infty} \mathfrak{M}((z-1)^n - 1) \quad = \quad M(z - 1 - y) = \exp L'(-1, \chi_{-3}) = 1.3813545\ldots.$$

If $(p_k)_{k\in\mathbb{N}}$ represents the prime numbers sequence, then

$$\lim_{k\to\infty}\mathfrak{M}(\zeta_{p_k})\mathfrak{M}(1-\zeta_{p_k})=\exp L'(-1,\chi_{-3}).$$

Put $\mathcal{V}=\{\mathfrak{H}(\alpha)\mid\alpha\in\overline{\mathbb{Q}}\}$. We already know that $\exp L'(-1,\chi_{-3})$ is a limit point of $\mathcal{V}$. Can we obtain other values, possibly lower? We try to answer this question now.

Let $P_m(X)$ $(1\leqslant m\leqslant k)$, $Q_m(X)$ $(1\leqslant m\leqslant \ell+1)$ be polynomials with integer coefficients, such that $\deg Q_{\ell+1}>0$ and

$$\frac{\prod_{m=1}^{k}P_m^{n_m}}{Q_{\ell+1}\prod_{m=1}^{\ell}Q_m^{n_{k+m}}}\neq\pm 1 \tag{4}$$

for all $(n_1,\dots,n_{k+\ell})\in\mathbb{N}^{k+l}$. We put $\chi(s)=e^{2i\pi s}(1-e^{2i\pi s})$ and, for the vector $\mathbf{q}=(q_1,\dots,q_{k+\ell})$ in $\mathbb{R}_+^{k+\ell}$, we define

$$f(\mathbf{q})=\int_0^1\int_0^1\log\left|\left(\prod_{m=1}^{k}P_m^{q_m}\right)(\chi(s))\ -e^{2i\pi t}\left(Q_{\ell+1}\prod_{m=1}^{\ell}Q_m^{q_{k+m}}\right)(\chi(s))\right|\,dsdt.$$

If $\mathbf{q}$ belongs to $\mathbb{Q}_+^{k+l}$, one can always write $q_m=a_m/b$ with a_m and b in $\mathbb{N}$. Besides, the change of variable $t\mapsto t+j/b$ in the previous integral does not affect the value of $f(\mathbf{q})$. Summing these integrals for j from 0 to $b-1$, we obtain

$$bf(\mathbf{q})=\int_0^1\int_0^1\log\left|\left(\prod_{m=1}^{k}P_m^{bq_m}\right)(\chi(s))-e^{2i\pi t}\left(Q_{\ell+1}^b\prod_{m=1}^{\ell}Q_m^{bq_{k+m}}\right)(\chi(s))\right|\,dsdt,$$

having replaced tb by t. Now, as $bq_m\in\mathbb{N}$ we see that

$$\left(\prod_{m=1}^{k}P_m^{bq_m}\right)(X)-y\left(Q_{\ell+1}^b\prod_{m=1}^{\ell}Q_m^{bq_{k+m}}\right)(X) \tag{5}$$

is a polynomial. Hence $bf(\mathbf{q})$ is its logarithmic Mahler measure. The condition $\deg Q_{\ell+1}>0$ asserts that the degree D in the variable z of (5) is positive. More precisely,

$$D=2b\max\left(\sum_{m=1}^{k}q_m\deg P_m,\deg Q_{\ell+1}+\sum_{m=1}^{\ell}q_{k+m}\deg Q_m\right).$$

Therefore, by (4) we can apply Lemma 2 and we obtain

$$M\left(\left(\prod_{m=1}^{k}P_m^{bq_m}\right)(X)-y\left(Q_{\ell+1}^b\prod_{m=1}^{\ell}Q_m^{bq_{k+m}}\right)(X)\right)^{1/D}$$
$$=\lim_{n\to\infty}\mathfrak{M}\left(\left(\prod_{m=1}^{k}P_m^{bq_m}\right)^n(X)-\left(Q_{\ell+1}^b\prod_{m=1}^{\ell}Q_m^{bq_{k+m}}\right)^n(X)\right).$$

Thus, writing

$$h(\mathbf{q})=\exp\left(\frac{2bf(\mathbf{q})}{2b\max(\sum_{m=1}^{k}q_m\deg P_m,\deg Q_{\ell+1}+\sum_{m=1}^{\ell}q_{k+m}\deg Q_m)}\right),$$

we point out that $h(\mathbf{q})$ is the limit when n tends to infinity of the sequence

$$\mathfrak{H}\left(\left(\prod_{m=1}^{k} P_m^{bq_m}\right)^n (X) - \left(Q_{\ell+1}^b \prod_{m=1}^{\ell} Q_m^{bq_{k+m}}\right)^n (X)\right).$$

One can also build such a sequence when the q_m's are irrational, because h does not depend on b and is obviously continuous.

Thanks to this result, we found, after many attempts, a limit point less than 1.2916674. With the above notation, the polynomials are

$$P_1(X)=X, \quad P_2(X)=1-X, \quad P_3(X)=X^4-2X^3+4X^2-3X+1,$$

$$P_4(X)=X^{12}-3X^{11}+8X^{10}-18X^9+36X^8-62X^7+97X^6-123X^5+114X^4-73X^3+31X^2-8X+1,$$

$$P_5(X)=X^{12}-3X^{11}+7X^{10}-14X^9+30X^8-58X^7+96X^6-123X^5+114X^4-73X^3+31X^2-8X+1.$$

As for $Q_1(X)$, it is equal to the product of the following two polynomials of degree 24.

$$\begin{aligned}Q_\alpha(X)=&X^{24}-6X^{23}+24X^{22}-77X^{21}+217X^{20}-546X^{19}+1252X^{18}-2647X^{17}+5195X^{16}\\&-9457X^{15}+15898X^{14}-24521X^{13}+34402X^{12}-43345X^{11}+48207X^{10}-46413X^9\\&+37963X^8-25934X^7+14558X^6-6596X^5+2357X^4-642X^3+126X^2-16X+1,\\Q_\beta(X)=&X^{24}-5X^{23}+16X^{22}-39X^{21}+85X^{20}-180X^{19}+385X^{18}-796X^{17}+1551X^{16}\\&-2907X^{15}+5421X^{14}-10003X^{13}+17368X^{12}-26734X^{11}+34951X^{10}-37880X^9\\&+33603X^8-24203X^7+14041X^6-6486X^5+2342X^4-641X^3+126X^2-16X+1.\end{aligned}$$

In particular $\deg Q_1 > 0$. The respective heights of these polynomials are

$$\begin{aligned}\mathfrak{H}(P_1) &= \mathfrak{H}(P_2) = 1, & \mathfrak{H}(P_3) &= 1{,}272019650\ldots,\\ \mathfrak{H}(P_4) &= 1{,}288842118\ldots, & \mathfrak{H}(P_5) &= 1{,}289442542\ldots,\\ \mathfrak{H}(Q_\alpha) &= 1{,}290471208\ldots, & \mathfrak{H}(Q_\beta) &= 1{,}290478982\ldots.\end{aligned}$$

We checked that condition (4) was fulfilled and we calculated the integral on $[0,1]\times[0,1]$ of

$$\log\left|\left(P_1^{q_1}P_2^{q_2}P_3^{q_3}P_4^{q_4}P_5^{q_5}\right)(\chi(s)) - e^{2i\pi t}Q_1(\chi(s))\right|.$$

The following q_m's were found by successive attempts. The value

$$h(17.9, 12.2, 0.9, 0.35, 0.29) = 1.2916673\ldots \tag{6}$$

was computed to great precision by a Riemann sum, with PARI, from the formula

$$f(q_1,\ldots,q_5) = \log M\left(Q_1(X)\right) + \int_0^1 \log^+\left|\frac{\left(\prod_{m=1}^5 P_m^{q_m}\right)(\chi(s))}{Q_1(\chi(s))}\right| ds.$$

However, this result concerns polynomials, not algebraic numbers. Fortunately, the following three lemmas enable us to deduce a corresponding result for algebraic numbers, i.e., Theorem 2.

Lemma 3. *Let W be a polynomial with integer coefficients. Let us assume $W(X) = \prod_{i=1}^r W_i(X)$ and $\mathfrak{H}(W) \leqslant H$. Then, there exists i such that $\mathfrak{H}(W_i) \leqslant H$.*

Proof. If not, $\mathfrak{H}(W_i)$ would be greater than H for all i. If d_i represents $\deg_X W_i$, we would have $\mathfrak{H}(W) = \prod_{i=1}^r M(W_i(X))^{1/\sum_{i=1}^r d_i} > H$, which is absurd. ∎

Lemma 4. *Let A and B be two relatively prime polynomials. Then*

$$\gcd(A^m - B^m, A^n - B^n) = A^{\gcd(m,n)} - B^{\gcd(m,n)}.$$

Proof. See, for example, Exercise 4.38 of [GKP94], pointing out that the proof is the same for integers or for polynomials. □

Lemma 5. *Let c, d be positive integers, V and W two relatively prime polynomials such that neither X nor $1-X$ divide W. If $U_n(X) = X^{cn}(1-X)^{dn}V(X)^n - W(X)^n$ and if $\mathfrak{H}(U_n)$ tends to r as n tends to infinity, then there exists a sequence $(T_k)_{k\in\mathbb{N}}$ of distinct and irreducible polynomials whose heights are nontrivial and such that $\limsup_{k\to\infty} \mathfrak{H}(T_k) \leqslant r$.*

Proof. Recall that $(p_k)_{k\in\mathbb{N}}$ is the prime numbers sequence. As X and $1-X$ are the only polynomials of trivial height, each factor of U_{p_k} has a height strictly larger than 1. Lemma 3, when applied to

$$\frac{U_{p_k}}{X^c(1-X)^d V(X) - W(X)}$$

for each k, provides an irreducible factor T_k, whose height is at most

$$\mathfrak{H}\left(\frac{U_{p_k}}{X^c(1-X)^d V(X) - W(X)}\right).$$

The assumption

$$\mathfrak{H}(U_n) \underset{n\to\infty}{\longrightarrow} r$$

implies that

$$\mathfrak{H}\left(\frac{U_n}{X^c(1-X)^d V(X) - W(X)}\right) \underset{n\to\infty}{\longrightarrow} r.$$

Thus $\limsup_{k\to\infty} \mathfrak{H}(T_k) \leqslant r$ as claimed. Finally, Lemma 4 ensures that

$$\gcd\left(\frac{U_{p_{k_1}}}{X^c(1-X)^d V(X) - W(X)}, \frac{U_{p_{k_2}}}{X^c(1-X)^d V(X) - W(X)}\right) = 1.$$

In particular the T_k's are distinct. This completes the proof. □

Proof of Theorem 2. For this, we use the previous computations, especially (6). We put

$$c = 1790,\ d = 1220,\ V = P_3^{90} P_4^{35} P_5^{29} \text{ and } W = Q_1^{100}.$$

Then we verify that the conditions of Lemma 5 are fulfilled in order to obtain a sequence of distinct algebraic numbers whose height tends to the value of h at $(17.9, 12.2, 0.9, 0.35, 0.29)$. Thus there exists a limit point of $\mathcal{V}$ smaller than 1.2916674. This result, combined with Theorem 1, enables us to conclude that the smallest limit point of $\mathcal{V}$ belongs to $[1.28177702, 1.2916674]$. □

5. Conclusion

Some signs lead us to believe that 1 and 1.2720196... are the only isolated points of $\mathcal{V}$. Indeed, the first points of the spectrum of $\mathfrak{M}(\alpha)\mathfrak{M}(1/(1-\alpha))\mathfrak{M}(1-1/\alpha)$ have low degree (cf. [D98]) and there exist relationships between them. In fact, the first point is trivial and is the measure of $D_1(X) = 1 - X$, the second point is given by $D_2(X) = D_1(X)^3 - X^2$, the third by $D_3(X) = D_2(X)^2 - X^2 D_1(X)^3$ and the fourth by $D_3(X)^2 - X^2 D_1(X)^3 D_2(X)^2$. No such connections were found for $\mathfrak{H}(\alpha)$. Moreover, the exhaustive search of V. Flammang (cf. [F94]) shows, with our investigation, that no polynomial of degree less than 10 can give rise to the third point of $\mathcal{V}$, if any. Finally, lots of polynomials can be used in the auxiliary function to improve on the Zhang-Zagier inequality. We kept only the best ones, and it is quite strange that among these six new polynomials, none of them appears to be the second nontrivial point of $\mathcal{V}$.

If this speculation was right, the smallest limit point of $\mathcal{V}$ would be less than 1.2875274. Anyway, it seems that it is less than 1.29, according to our search (see Section 3).

We also conjecture that every real greater than 1.2916674 is a limit point of $\mathcal{V}$. To support this, first note that h is a continuous function of the q_m's. We then point out that $h(1,0,0,0,0) = 1.367978\ldots$, so there exist real numbers q_1, q_2, q_3, q_4, q_5 such that $h(q_1, q_2, q_3, q_4, q_5) = r$, for every $r \in [1.2916674, 1.367978]$. For another choice of polynomials, namely, $P_1(X) = 2$ and $Q_1(X) = X^2 - X + 1$, one can reach infinity from 1.3641, with the result that $[1.2916674, +\infty[$ is entirely covered. Obviously, given $r \geqslant 1.2916674$ there exist real numbers $q_{m,r}$ such that the sequence of polynomials

$$U_n(X) = \left(\prod_{m=1}^{k} P_m^{\lfloor n q_{m,r} \rfloor} \right)(X) - Q_1^n(X)$$

satisfies

$$\mathfrak{H}(U_n) \underset{n\to\infty}{\longrightarrow} r.$$

To prove our conjecture, we need only show the existence of factors U_n' of U_n with $\limsup_{n\to\infty} \frac{\deg U_n'}{\deg U_n} = 1$. Although the U_n's are experimentally nearly always irreducible, we were however not able to say anything interesting about their factorization.

To conclude, let us say a few words about a search for polynomials of small Mahler measure. Thanks to Smyth's theorem (cf. [Sm71]), we only needed to consider reciprocal polynomials. It is clear that every reciprocal polynomial of degree $2d$ divided by z^d can be written as a polynomial of degree d in $T = z + 1/z$. So we only investigated polynomials in this new variable T.

The main idea was again to build P from its remainders modulo some P_i, slightly modifying the algorithm described in Section 3. If $\frac{n}{m}$ is not a power of a prime, it is known that $\mathrm{Res}(\Phi_n, \Phi_m) = 1$. In the same way, Silverman pointed out in [Si95] that $\mathrm{Res}(\Phi_n, P) = 1$, for P with a small Mahler measure and for some appropriate n. Hence, the P_i's were chosen among cyclotomic polynomials which were then expressed in terms of T. Unfortunately, the polynomials obtained are totally real. So they have numerous fundamental units, approximately twice as many as for those obtained from the Zhang-Zagier height computations. That is the reason why we were not able to find polynomials in T with degree larger than 19. Because of Boyd's exhaustive computations (cf. [B80], [B89]), we started at degree 11. Our search for degrees ranging from 11 to 19 allowed us to find again the best Mahler measures for these degrees (cf. [M95]), but did not provide any new Mahler measure.

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