MATHEMATICS OF COMPUTATION
Volume 70, Number 233, Pages 389–394
S 0025-5718(00)01219-9
Article electronically published on June 12, 2000

ENUMERATING SOLUTIONS TO $p(a) + q(b) = r(c) + s(d)$

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ABSTRACT. Let p, q, r, s be polynomials with integer coefficients. This paper presents a fast method, using very little temporary storage, to find all small integers (a, b, c, d) satisfying $p(a)+q(b) = r(c)+s(d)$. Numerical results include all small solutions to $a^4+b^4+c^4 = d^4$; all small solutions to $a^4+b^4 = c^4+d^4$; and the smallest positive integer that can be written in 5 ways as a sum of two coprime cubes.

1. INTRODUCTION

Let H be a positive integer. How can one find all positive integers $a, b, c, d \le H$ satisfying $a^3 + 2b^3 + 3c^3 = 4d^3$?

The following method is standard. Sort the set $\{(a^3 + 2b^3, a, b) : a, b \le H\}$ into increasing order in the first component. Similarly sort $\{(4d^3 - 3c^3, c, d) : c, d \le H\}$. Now merge the sorted lists, looking for collisions. The sorting takes time $H^{2+o(1)}$ and space $H^{2+o(1)}$.

It does not seem to be widely known that one can save a factor of H in space. Section 3 explains how to enumerate $\{(a^3 + 2b^3, a, b)\}$ and $\{(4d^3 - 3c^3, c, d)\}$ in order, using $O(H^2)$ heap operations on two heaps of size H. Heaps are reviewed in section 2. The remaining sections of this paper give several numerical examples. See `http://pobox.com/~djb/sortedsums.html` for a UNIX implementation of most of the algorithms discussed here.

A standard improvement is to split the range of $a^3 + 2b^3$ and $4d^3 - 3c^3$ into several (0-adic or p-adic) intervals. For example, one can separately consider each possibility for $4d^3 - 3c^3 \bmod 7$, and skip pairs (a, b) with $a^3 + 2b^3 \bmod 7 \in \{2, 5\}$.

Notes. Lander and Parkin in [11] enumerated solutions to $a^4 + b^4 = c^4 + d^4$ using time $H^{3+o(1)}$ and space $H^{1+o(1)}$.

Ekl in [2] pointed out that the time of the Lander-Parkin method could be reduced to $H^{2+o(1)}$. I made the same observation independently in April 1997, when Yuri Tschinkel asked me about the example described in section 4 below. David W. Wilson made the same observation independently in October 1997, for the example described in section 5 below. The difference between my method, Ekl's method, and the Lander-Parkin method is the difference between a heap, a balanced tree, and an unstructured array.

Received by the editor July 10, 1998 and, in revised form, January 4, 1999.
2000 *Mathematics Subject Classification.* Primary 11Y50; Secondary 11D25, 11D41, 11P05, 11Y16.
The author was supported by the National Science Foundation under grant DMS–9600083.

The use of heaps to enumerate sums in sorted order actually appeared much earlier in another context, namely William S. Brown's algorithm for multiplication of sparse power series. See [9, exercise 5.2.3–29]; compare [9, exercise 5–18].

2. Heaps

A *heap* is a sequence $x_1, x_2, \ldots, x_n$ satisfying $x_{\lfloor k/2 \rfloor} \le x_k$ for $2 \le k \le n$: i.e., $x_1 \le x_2$, $x_1 \le x_3$, $x_2 \le x_4$, $x_2 \le x_5$, $x_3 \le x_6$, $x_3 \le x_7$, etc.

The smallest element of a heap $x_1, x_2, \ldots, x_n$ is x_1. Given y, one can permute $y, x_2, \ldots, x_n$ into a new heap by the following algorithm. First set $j \leftarrow 1$. Then perform the following steps repeatedly: set $k \leftarrow 2j$; stop if $k > n$; set $k \leftarrow k+1$ if $k < n$ and $x_{k+1} < x_k$; stop if $y \le x_k$; exchange y, which is now in the jth position, with x_k; set $j \leftarrow k$. The total number of operations here is $O(\log n)$.

In particular, using $O(\log n)$ operations, one can permute $x_n, x_2, \ldots, x_{n-1}$ into a new heap. By a similar algorithm, also using $O(\log n)$ operations, one can permute $x_1, x_2, \ldots, x_n, y$ into a new heap.

Notes. Heaps were published by Williams in [22]. Floyd in [5] pointed out an algorithm using $O(n)$ operations to permute any sequence of length n into a new heap.

For some practical improvements in heap performance see [9, exercise 5.2.3–18] and [9, exercise 5.2.3–28]. The bottom-up algorithm in [9, exercise 5.2.3–18] is due to Floyd; the "new" algorithms announced many years later in [1] and [21] are the same as Floyd's.

There are many other data structures that support insertion of new elements and removal of the smallest element. Any such structure is called a *priority queue*. Examples include *leftist trees*, as discussed in [9, section 5.2.3]; *loser selection trees*, as discussed in [9, section 5.4.1]; *balanced trees*, as discussed in [9, section 6.2.3]; and *B-trees*, as discussed in [9, section 6.2.4]. See also [10, page 152]. The reader can replace the heap in section 3 with any priority queue. Beware, however, that some "fast" priority queues are several times bigger and slower than heaps; see, for example, section 10 below.

3. Enumerating sums

Fix $p, q \in \mathbf{Z}[x]$. This section explains how to print $\{(p(a)+q(b), a, b) : a, b \le H\}$ in increasing order in the first component, using space $H^{1+o(1)}$.

First build a table of $\{(p(a), a) : a \le H\}$. Sort the table into increasing order in the first component; say $p(a_1) \le p(a_2) \le \cdots$.

Next build a heap containing $\{(p(a_1)+q(b), 1, b) : b \le H\}$. Perform the following operations repeatedly until the heap is empty:

1. Find and remove the smallest element (y, n, b) in the heap.
2. Print (y, a_n, b); by induction $y = p(a_n) + q(b)$ at this point.
3. Insert $(p(a_{n+1}) - p(a_n) + y, n+1, b)$ into the heap if a_{n+1} exists.

Step 1 and step 3 can be combined into a single heap operation.

This algorithm takes time $H^{1+o(1)}$ for initializations, plus $H^{o(1)}$ for each of the H^2 outputs, for a total of $H^{2+o(1)}$. There are at most H elements in the heap at any moment.

Refinements. One can easily save half the heap operations if $p = q$: start with $\{(p(a_n) + p(a_n), n, a_n)\}$; print (y, b, a_n) along with (y, a_n, b) if $a_n \neq b$.

One can speed up the manipulation of y, and in some cases save space, by storing $p(a_2) - p(a_1), p(a_3) - p(a_2), \ldots$ instead of $p(a_2), p(a_3), \ldots$.

One need not bother building tables of $n \mapsto a_n$ and $n \mapsto p(a_n)$ if p is a sufficiently dull function.

Generalizations. Given functions p, q, r, s from finite sets A, B, C, D to an ordered group, one can enumerate $\{(a, b, c, d) \in A \times B \times C \times D : p(a) + q(b) = r(c) + s(d)\}$ by the same algorithm. For example, one can enumerate small solutions (a, b, c, d) to $a^3 + 2b^3 = 3c^3 + 4d^3$ with $a, b, c, d \in \mathbf{Z}[w]/(w^2 + w + 1)$, using lexicographic order on $\mathbf{Z}[w]/(w^2 + w + 1)$. See section 10 for another example.

One can restrict attention to a subset of $A \times B$, simply by skipping to the next suitable a for each b. See sections 9 and 10 for examples.

There are many functions that are not of the form $a, b \mapsto p(a) + q(b)$ but that are nevertheless amenable to sorted enumeration. For example, one can apply the method here to any function f such that $a \mapsto f(a, b)$ is monotone for each b. See section 6 for an example.

4. Example: $a^3 + b^3 = c^3 + d^3$

There are 12137664 solutions (a, b, c, d) to $a^3 + b^3 = c^3 + d^3 > 0$ with $a \neq c$, $a \neq d$, $-10^5 \leq a, b, c, d \leq 10^5$, and $a\mathbf{Z} + b\mathbf{Z} + c\mathbf{Z} + d\mathbf{Z} = \mathbf{Z}$. In other words, there are 12137664 rational points of height at most 10^5 on the surface $x^3 + y^3 + z^3 = 1$ away from the lines on the surface.

This computation took $1.4 \cdot 10^{13}$ cycles on a Pentium II-350. It takes roughly twice as long to do a similar computation for $pa^3 + qb^3 = pc^3 + qd^3$; roughly three times as long for $pa^3 + pb^3 = rc^3 + sd^3$; and roughly four times as long for $pa^3 + qb^3 = rc^3 + sd^3$.

Notes. Peyre and Tschinkel have checked some of my numerical results and some of their theoretical computations against the best available conjecture. See [16]. Heath-Brown in [8] had previously enumerated solutions to $a^3 + b^3 = c^3 + 2d^3$ and $a^3 + b^3 = c^3 + 3d^3$ with $-10^3 \leq a, b, c \leq 10^3$ by a cubic-time method.

In some cases one can save time by using [8, Theorem 1].

5. Example: many equal sums of two positive cubes

The smallest integer that can be written in k ways as a sum of two cubes of positive integers is 1729 for $k = 2$; 87539319 for $k = 3$; 6963472309248 for $k = 4$; and 48988659276962496 for $k = 5$. There are no 6-way integers below 10^{18}. (There are two other 5-way integers below 10^{18}: $391909274215699968 = 8 \cdot 48988659276962496$ and 490593422681271000.)

This computation took $7.9 \cdot 10^{14}$ cycles on an UltraSPARC II-296.

Notes. The answer for $k = 3$ was found by Leech in [14]. The answer for $k = 4$ was found by Rosenstiel, Dardis, and Rosenstiel in [17]. The answer for $k = 5$ was found by David W. Wilson in 1997 and independently by me in 1998. There is an answer for every k; see [19] for the best known bounds.

6. Example: many equal sums of two cubes

The smallest positive integer that can be written in k ways as a sum of two cubes is 91 for $k = 2$; 728 for $k = 3$; 2741256 for $k = 4$; 6017193 for $k = 5$; 1412774811 for $k = 6$; 11302198488 for $k = 7$; and 137513849003496 for $k = 8$. There are no 9-way integers below $2.5 \cdot 10^{17}$. (There are 37 other 8-way integers below $2.5 \cdot 10^{17}$.)

This computation took $9.2 \cdot 10^{14}$ cycles on an UltraSPARC II-296. To keep the heap small, I enumerated pairs (a, b) with $a \geq b/2$ and $1 \leq a^3 + (b-a)^3 \leq 2.5 \cdot 10^{17}$, in order of $a^3 + (b - a)^3$; these conditions imply $1 \leq b \leq 10^6$.

Notes. The answers for $k \in \{5, 6, 7\}$ were found by Randall Rathbun, according to [7, page 141]. The answer for $k = 8$ appears to be new.

7. Example: many equal sums of two coprime cubes

The smallest positive integer that can be written in k ways as a sum of two cubes of coprime integers is 91 for $k = 2$; 3367 for $k = 3$; 16776487 for $k = 4$; and 506433677359393 for $k = 5$. Each of these integers is squarefree. There are no 6-way integers below $2.5 \cdot 10^{17}$. (There is one other 5-way integer, namely 137904678696613339.)

I found these results during the computation described in section 6. A separate computation, skipping pairs (a, b) with a common factor, would have been somewhat faster.

Notes. The answer for $k = 4$ was found by Rathbun, according to [7, page 141]. The answer for $k = 5$ appears to be new.

Silverman proved in [18] that the number of pairs of integers (a, b) satisfying $a^3 + b^3 = n$ is bounded by a particular function of the rank over $\mathbf{Q}$ of the elliptic curve $x^3 + y^3 = n$, if n is cubefree. It is not known how tight Silverman's bound is.

Paul Vojta found that 15170835645 can be written in 3 ways as a sum of two cubes of coprime *positive* integers.

8. Example: $a^4 + b^4 = c^4 + d^4$

There are 516 solutions (a, b, c, d) to $a^4 + b^4 = c^4 + d^4$ with $0 < b \leq a$, $0 < d \leq c$, $c < a \leq 10^6$, and $a\mathbf{Z} + b\mathbf{Z} + c\mathbf{Z} + d\mathbf{Z} = \mathbf{Z}$. This computation took roughly 10^{15} cycles on an UltraSPARC II-296.

The fourth power of 10^6 does not fit into a 64-bit integer. I actually enumerated values of $(a^4 \bmod m) + (b^4 \bmod m) + (0 \text{ or } m)$ greater than or equal to m, where $m = 2^{60} - 93$. Then I checked each collision $a^4 + b^4 \equiv c^4 + d^4 \pmod{m}$ to see whether $a^4 + b^4 = c^4 + d^4$.

Notes. 218 of the 516 solutions were already known: Lander and Parkin in [11] exhaustively found all solutions with $a^4 + b^4 < 7.885 \cdot 10^{15}$; Lander, Parkin, and Selfridge in [13] exhaustively found all solutions with $a^4 + b^4 \leq 5.3 \cdot 10^{16}$; Zajta in [23] found many solutions with $a \leq 10^6$ by various ad-hoc techniques.

9. Example: $a^4+b^4+c^4=d^4$

The only positive solutions (a,b,c,d) to $a^4+b^4+c^4=d^4$ with $d \le 2.1\cdot 10^7$ and $a\mathbf{Z}+b\mathbf{Z}+c\mathbf{Z}+d\mathbf{Z}=\mathbf{Z}$ are permutations of the solutions

$$\begin{gathered}(95800, 414560, 217519, 422481),\\ (1390400, 2767624, 673865, 2813001),\\ (5507880, 8332208, 1705575, 8707481),\\ (5870000, 11289040, 8282543, 12197457),\\ (12552200, 14173720, 4479031, 16003017),\\ (3642840, 7028600, 16281009, 16430513),\\ (2682440, 18796760, 15365639, 20615673).\end{gathered}$$

This computation took $4.5\cdot 10^{15}$ cycles on a Pentium II-350.

I used several p-adic restrictions here. One can permute a,b,c so that $a\in 2\mathbf{Z}$ and $b\in 10\mathbf{Z}$. Then $a\in 8\mathbf{Z}$, $b\in 40\mathbf{Z}$, $d-1\in 8\mathbf{Z}$, and $c\equiv \pm d \pmod{1024}$ by [20, Theorem 1]; also $d\notin 5\mathbf{Z}$. There are roughly $H^2/320$ possibilities for (a,b) and $H^2/10240$ possibilities for (c,d) if $d\le H$. I enumerated sums modulo $2^{60}-93$ as in section 8.

Notes. Euler conjectured that $a^4+b^4+c^4=d^4$ had no positive integer solutions. Ward in [20] proved that there are no solutions with $d\le 10^4$. Lander, Parkin, and Selfridge in [13] proved that there are no solutions with $d\le 2.2\cdot 10^5$. Elkies in [4] proved that there are infinitely many solutions with $a\mathbf{Z}+b\mathbf{Z}+c\mathbf{Z}+d\mathbf{Z}=\mathbf{Z}$, and exhibited two examples. Elkies commented that the smaller example, with $d=20615673$, "seems beyond the range of reasonable exhaustive computer search." Frye in [6] subsequently found the solutions with $d=422481$, and proved that there are no other solutions with $d\le 2\cdot 10^6$. Allan MacLeod subsequently found the solutions with $d=2813001$ by Elkies's method. The solutions with $d\in\{8707481, 12197457, 16003017, 16430513\}$ appear to be new.

For each (c,d) satisfying various p-adic restrictions, Ward factored d^4-c^4 into primes and then found all representations of d^4-c^4 as a sum of squares; the total time of Ward's algorithm is $H^{2+o(1)}$ with modern factoring methods, but the $o(1)$ is fairly large. Lander, Parkin, Selfridge, and Frye instead enumerated possibilities for b, and checked for each b whether $d^4-c^4-b^4$ was a fourth power; Frye estimated that his program used about $H^3/490000$ fourth-power tests to find all solutions with $d\le H$.

10. Example: $a^7+b^7+c^7+d^7=e^7+f^7+g^7+h^7$

The five smallest integers that can be written in 2 ways as sums of four positive seventh powers are 2056364173794800, 12191487610289536, 263214614245734400, 696885239160606459, and 1560510414117060608. There are no other examples below 420^7.

I began this computation by generating a sorted table of $\{a^7+b^7 : a\ge b\}$. Then I enumerated sums $(a^7+b^7)+(c^7+d^7)$ in order, skipping inputs $((a,b),(c,d))$ with $b<c$. Searching up to 155^7, to verify the smallest example, took $1.4\cdot 10^{10}$ cycles (and roughly 340 kilobytes of memory) on an UltraSPARC I-167. Searching up to 420^7 took $1.4\cdot 10^{12}$ cycles.

Notes. All the examples here were found by Ekl in [2] and [3]. However, Ekl needed $1.6 \cdot 10^{11}$ cycles on an HP PRISM-50 (and roughly 8900 kilobytes of memory) to find the first example. Presumably the main reason is that the priority queue in [2] and [3] was a balanced tree, whereas the priority queue here is a heap.

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