MATHEMATICS OF COMPUTATION
Volume 70, Number 234, Pages 471–487
S 0025-5718(00)01259-X
Article electronically published on November 27, 2000

EIGENVALUE AND EIGENFUNCTION ERROR ESTIMATES FOR FINITE ELEMENT FORMULATIONS OF LINEAR HYDROELASTICITY

PAT RYAN

ABSTRACT. Convergence of an approximate method for determining vibrational eigenpairs of an elastic solid containing an incompressible fluid is examined. The field variables are solid displacement and fluid pressure. We show that in suitable Sobolev spaces a variational formulation exists whose solution eigenvalues and eigenfunctions are identified with those of a compact operator. A nonconforming finite element approximation of this variational problem is described and optimal a priori error estimates are obtained for both the eigenvalues and eigenfunctions.

1. INTRODUCTION

Accurate modeling of the interaction between liquid propellants and structures is important to the prediction of dynamic loads in launch vehicles and spacecraft. It is common practice to represent the motion of these systems by superposition of the response of a small number of approximate eigenfunctions, obtained from a finite element representation of an inviscid, incompressible fluid coupled to a linear elastic structure. Previous authors have addressed both mathematical and computational aspects of this problem. The existence of real eigenpairs of a two-field variational formulation, in the presence of a free surface, was established by Berger, Boujot and Ohayon [3] and Boujot [6]. However, the Galerkin finite element discretization of fluid pressure and structural displacement variables results in a nonsymmetric matrix eigenvalue problem, and direct solution of this nonsymmetric problem results in complex eigenvalues. Solution procedures which eliminate the fluid pressure variable at the matrix level by static condensation to the fluid boundary have been presented by Coppolino [8] and the text by Morand and Ohayon [13]. These methods result in a symmetric matrix eigenvalue problem whose solution results in real eigenvalues. The statically condensed fluid representation takes the form of an "added mass matrix". The existence of real eigenpairs of a continuous variational counterpart of the added mass formulation was shown by Bourquin [4], for the special case of a rigid tank with an elastic bottom.

Although existence of solutions and computational procedures have been addressed, the rate of convergence of these approximate methods has not been completely established [13]. Numerical results for typical vehicles show that, at the

Received by the editor February 2, 1999.
2000 *Mathematics Subject Classification.* Primary 65N30, 70J30; Secondary 65N25, 74F10.
Key words and phrases. Hydroelasticity, finite element, eigenvalue, error estimates.
This research was sponsored in part by funding from the United States Air Force.

lowest natural frequencies, the tank wall moves essentially as a rigid body and the motion is predominately fluid slosh. Convergence for slosh mode prediction has been addressed by comparison to analytical solutions for rigid containers with simple geometries [8]. However, as the natural frequency increases, the fluid free surface motion becomes quasi-static and tank wall elastic motion dominates. These latter mode shapes are of greatest importance to structural dynamic loads prediction, yet no relevant analytical solutions exist to verify accuracy.

The contribution of the present work is to provide a quantitative measure of the accuracy of finite element methods for the prediction of dynamic loads in an unrestrained elastic flight vehicle. To focus on this aspect, an atypical model problem is studied: one in which the fluid is completely enclosed in an elastic container. In Section 2 we state the pointwise equilibrium equations and establish the existence of discrete real eigenpairs of a related variational problem. In Section 3 we provide a precise statement of a finite element approximation to this variational eigenvalue problem using simple C^0 elements for both fluid and solid. In Section 4 we establish a priori error estimates for the approximate eigenvalues and eigenfunctions. It is shown that optimal rates of convergence are realized for three-dimensional polyhedral domains typically encountered in engineering practice.

2. The variational eigenvalue problem

An unrestrained three dimensional linear elastic solid encloses an incompressible, inviscid fluid as shown in Figure 1. The system is free of external forces. The equations of pointwise equilibrium are

$$-\nabla \cdot \boldsymbol{\sigma} = \lambda \rho^s \boldsymbol{u} \quad \text{in} \quad \Omega^s, \tag{2.1}$$

$$\triangle p = 0 \quad \text{in} \quad \Omega^f, \tag{2.2}$$

$$\frac{\partial p}{\partial n} = \lambda \rho^f \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma, \tag{2.3}$$

$$\boldsymbol{n} \cdot \boldsymbol{\sigma}_n = p \quad \text{on} \quad \Gamma, \tag{2.4}$$

$$\boldsymbol{\sigma}_n - (\boldsymbol{n} \cdot \boldsymbol{\sigma}_n) n = \boldsymbol{0} \quad \text{on} \quad \Gamma, \tag{2.5}$$

$$\boldsymbol{\sigma}_n = \boldsymbol{0} \quad \text{on} \quad \partial\Omega^s \backslash \Gamma, \tag{2.6}$$

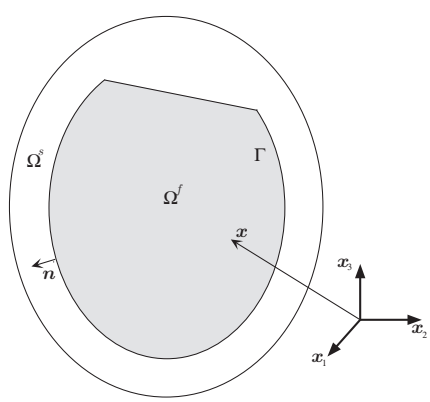

Figure 1. Fluid enclosed in an elastic solid

where λ is the square of the circular frequency of vibration, $\boldsymbol{u}$ is the solid displacement field, $\boldsymbol{n}$ is the unit normal vector on the fluid-solid interface, p is the fluid pressure, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and $\boldsymbol{\sigma}_n = \sigma_{ij} n_j^s$ is the surface traction vector where $\boldsymbol{n}^s$ is the outward normal to Ω^s. The symmetric strain tensor $\boldsymbol{\epsilon}$ is defined by the strain-displacement relation

$$\boldsymbol{\epsilon} = \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{in} \qquad \Omega^s,$$

while the stress-strain relation is

$$\boldsymbol{\sigma} = c_{ijkl}\ \epsilon_{kl} \qquad \text{in} \qquad \Omega^s,$$

where c_{ijkl} are the elastic constants. The fluid density ρ^f is constant while the solid density ρ^s, elastic constants c_{ijkl} and the field variables $\boldsymbol{u}$ and p are functions of the spatial variable $\boldsymbol{x} = (x_1, x_2, x_3)$. In what follows, we use standard Sobolev spaces and notation (see, for example, Aubin [1]). The coupled problem takes place in a simply connected region $\bar{\Omega} \in \mathbb{R}^3$, where $\Omega = \Omega^s \cup \Gamma \cup \Omega^f, \Omega^s \cap \Omega^f = \emptyset$. Consistent with typical engineering models, the fluid is contained in a region Ω^f assumed to be convex, simply connected and polyhedral with unit outward normal vector $\boldsymbol{n}$. The fluid-solid interface $\Gamma = \bar{\Omega}^s \cap \bar{\Omega}^f$ comprises the union of a fixed finite number of facets such that each facet is a smooth two dimensional manifold. As a result, $\boldsymbol{n}$ is locally smooth but globally an element of $L_2(\Gamma)^3$. We denote by γ^s the trace operator on $H^1(\Omega^s)^3$ and by γ^f the trace operator on $H^1(\Omega^f)$. Let

$$\begin{aligned}
Q &= \{q \mid q \in H^1(\Omega^f)/\mathcal{P},\ \triangle q = 0 \text{ in } \Omega^f\},\\
\mathcal{P} &= \{q_0 \mid q_0 \ \text{ is a constant function in } \bar{\Omega}^f\},\\
\partial Q &= \{\xi \mid \xi = \gamma^f q, q \in Q\} && \text{(the trace space of } Q),\\
\partial Q' &= \{\nu \mid \nu \in H^{-\frac{1}{2}}(\Gamma), \langle \nu, q_0\rangle = 0\ \forall q_0 \in \mathcal{P}\} && \text{(the dual space of } \partial Q),\\
H &= \{\mathbf{v} \mid \mathbf{v} = (\boldsymbol{v}, v_n) \in L_2(\Omega^s)^3 \times \partial Q'\},\\
V &= \{\mathbf{v} \mid \mathbf{v} = (\boldsymbol{v}, v_n) \in H^1(\Omega^s)^3 \times L_2(\Gamma)\ \cap\ H, r(\mathbf{v}, \boldsymbol{\Phi}_R) = 0\},
\end{aligned}$$

where $v_n = \boldsymbol{n} \cdot \gamma^s \boldsymbol{v}$ is the normal trace on Γ of a function $\boldsymbol{v} \in H^1(\Omega^s)^3$. We set $\boldsymbol{\Phi}_R = (\boldsymbol{\phi}_R, \boldsymbol{n} \cdot \gamma^s \boldsymbol{\phi}_R)$, where $\boldsymbol{\phi}_R$ is a six-dimensional subspace of $H^1(\Omega^s)^3$ representing zero strain energy states in the solid. The bilinear form $r(\cdot,\cdot)$ is an inner product on H and $\langle \cdot, \cdot \rangle$ is the duality pairing on $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$. Denote by $\|\cdot\|_{1,\Omega^s}, \|\cdot\|_{0,\Omega^s}$ the $H^1(\Omega^s)^3$ and $L_2(\Omega^s)^3$ norms, respectively, and by $\|\cdot\|_{1,\Omega^f}, \|\cdot\|_{0,\Omega^f}$ the $H^1(\Omega^f)$ and $L_2(\Omega^f)$ norms, respectively. Finally, let $\|\cdot\|_{s,\Gamma}$ denote the $H^s(\Gamma)$ norm. A Galerkin weighted residual form of the eigenvalue problem represented by equations (2.1) through (2.6) follows.

Find $\lambda, \mathbf{u}, p \in \mathbb{R} \times V \times Q$ such that

$$\int_{\Omega^s} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\ d\Omega - \langle v_n, \gamma^f p\rangle = \lambda \int_{\Omega^s} \rho^s \boldsymbol{u} \cdot \boldsymbol{v}\ d\Omega \qquad \forall \boldsymbol{v}\ \in V, \tag{2.7}$$

$$\int_{\Omega^f} \nabla p \cdot \nabla q\ d\Omega = \lambda\, \rho^f \langle u_n, \gamma^f q\rangle \qquad \forall q\ \in Q. \tag{2.8}$$

We now develop a problem statement in which (2.7) and (2.8) are combined into a single variational equation.

The elements of $Q \subset H^1(\Omega^f)/\mathcal{P}$ are an equivalence class $[q]$ of harmonic functions $q \in H^1(\Omega^f)$ differing by an arbitrary element of $\mathcal{P}$. It is well known (see, for example, Ciarlet [11]) that, for the norm on $H^1(\Omega^f)/\mathcal{P}$ defined by

$$\|[q]\|_{H^1(\Omega^f)/\mathcal{P}} = \inf_{q_0 \in \mathcal{P}} \|q + q_0\|_{1,\Omega^f} ,$$

there exists a coercivity constant α such that

$$\alpha \, \|[q]\|^2_{H^1(\Omega^f)/\mathcal{P}} \leq \int_{\Omega^f} \nabla q \cdot \nabla q \, d\Omega \quad \forall \ [q] \ \in H^1(\Omega^f)/\mathcal{P} \tag{2.9}$$

and a continuity constant M such that

$$\int_{\Omega^f} \nabla p \cdot \nabla q \, d\Omega \leq M \, \|[p]\|_{H^1(\Omega^f)/\mathcal{P}} \, \|[q]\|_{H^1(\Omega^f)/\mathcal{P}} \quad \forall \ [p],[q] \ \in H^1(\Omega^f)/\mathcal{P}. \tag{2.10}$$

We remark that the bilinear form $\int_{\Omega^f} \nabla p \cdot \nabla q \, d\Omega$ is continuous on $H^1(\Omega^f)$ as well, so there exists a constant C satisfying

$$\|[q]\|_{H^1(\Omega^f)/\mathcal{P}} \leq C \, \|q\|_{1,\Omega^f} . \tag{2.11}$$

As a consequence of (2.9) and (2.10), for any $\lambda \, \rho^f u_n \in \partial Q'$, equation (2.8) has a solution $[p] \in Q$ unique up to an arbitrary element $p_0 \in \mathcal{P}$. Thus we may define an operator $\mathbf{R}^f : \partial Q' \to Q$ by $[p] = p_0 + \lambda \, \rho^f \mathbf{R}^f u_n$. Let $\mathbf{S}^f u_n = \gamma^f \mathbf{R}^f u_n$ be the trace on Γ of $\mathbf{R}^f u_n$, and let $\delta = \frac{\partial}{\partial n}$ denote the Neumann operator $Q \to \partial Q'$. For $[p] \in Q$,

$$\int_{\Omega^f} \nabla p \cdot \nabla q \, d\Omega = \lambda \, \rho^f \int_{\Omega^f} \nabla \mathbf{R}^f u_n \cdot \nabla q \, d\Omega \qquad \forall [q] \in Q, \tag{2.12}$$

$$= \langle \delta q, \gamma^f [p] \rangle \qquad \text{(Green's formula)}, \tag{2.13}$$

$$= \lambda \, \rho^f \langle \delta q, \mathbf{S}^f u_n \rangle \qquad \text{(Definition of } \mathbf{S}^f\text{)}. \tag{2.14}$$

We shall make use of the following

Lemma 2.1. *The operator δ is a surjective map from Q onto $\partial Q'$.*

Proof. From the Green formula, for $[q] \in Q$,

$$\int_{\Omega^f} \nabla q \cdot \nabla \phi = \langle \delta q, \gamma^f \phi \rangle \quad \forall \phi \in H^1(\Omega^f)$$

choosing $\phi = q_0 \in \mathcal{P}$ we get $\langle \delta q, q_0 \rangle = 0$, so δ maps Q into $\partial Q'$. Conversely, given any $\nu \in H^{-\frac{1}{2}}(\Gamma)$ with $\langle \nu, q_0 \rangle = 0$, we can solve

$$\int_{\Omega^f} \nabla q \cdot \nabla \phi = \langle \nu, \gamma^f \phi \rangle \quad \forall \phi \in H^1(\Omega^f),$$

for $[q] \in H^1(\Omega^f)/\mathcal{P}$, which shows that any function $\nu \in H^{-\frac{1}{2}}(\Gamma)$ with $\langle \delta q, q_0 \rangle = 0$ is the image of the Neumann operator acting on some $q \in Q$. □

Now, since (2.12) through (2.14) hold for all $[q] \in Q$, Lemma 2.1 ensures that (2.13) and (2.14) hold for all $\delta q \in \partial Q'$, so in particular they hold for any arbitrary element $v_n \in \partial Q'$ and we can write

$$\langle v_n, \gamma^f [p] \rangle = \lambda \, \rho^f \langle v_n, \mathbf{S}^f u_n \rangle \quad \forall v_n \in \partial Q', \tag{2.15}$$

where we made use of $\langle v_n, \gamma^f p_0\rangle = 0$. We *choose* this v_n to be the normal trace the solid weighting function $\boldsymbol{v}$ in (2.7), so that substituting (2.15) into (2.7) and defining

$$a(\mathbf{u},\mathbf{v}) = \int_{\Omega^s} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\, d\Omega, \tag{2.16}$$

$$r(\mathbf{u},\mathbf{v}) = \int_{\Omega^s} \rho^s \boldsymbol{u}\cdot\boldsymbol{v}\, d\Omega + \rho^f \langle v_n, \mathbf{S}^f u_n\rangle, \tag{2.17}$$

we eliminate the fluid variable from (2.7), (2.8) and obtain the following one-field variational eigenvalue problem.

Find $\lambda, \mathbf{u} \in \mathbb{R}\times V$ such that

$$a(\mathbf{u},\mathbf{v}) = \lambda\, r(\mathbf{u},\mathbf{v}) \qquad \forall \mathbf{v} \in V. \tag{2.18}$$

The operator $\mathbf{S}^f$ is known as the Steklov operator (see e.g., Agoshov [2], Bramble and Osborn [3]). The use of a Steklov operator to formulate a one-field variational eigenvalue problem for fluid-structure interaction was proposed by Bourquin [4].

3. Existence of variational solutions

In this section we show that the eigenvalues and corresponding eigenfunctions of variational problem (2.18) may be identified with those of a positive self-adjoint operator $\mathbf{T}$. This operator is then shown to be compact and therefore its spectrum is a discrete sequence of positive real eigenvalues [1]. We let C denote a constant, not necessarily the same at each occurrence, and establish some preliminary results.

Lemma 3.1. *The duality pairing $\langle v_n, \mathbf{S}^f u_n\rangle$ is an inner product on $\partial Q'$.*

Proof. Let $q = \mathbf{R}^f v_n$ in (2.12) and (2.14). Since δ is surjective, $\mathbf{R}^f$ is a right inverse of δ and

$$\begin{aligned}\int_{\Omega^f} \nabla\mathbf{R}^f u_n \cdot \nabla\mathbf{R}^f v_n\, d\Omega &= \langle \delta\mathbf{R}^f v_n, \mathbf{S}^f u_n\rangle \\ &= \langle v_n, \mathbf{S}^f u_n\rangle\end{aligned} \tag{3.1}$$

which is symmetric by inspection and, since $\mathbf{R}^f$ maps $\partial Q'$ into Q, is positive definite by virtue of equation (2.9). □

We denote by $\|\cdot\|_{\partial Q'}$ the natural norm for the inner product defined by equation (3.1), this norm being equivalent to the usual norm $\|\cdot\|_{-\frac{1}{2},\Gamma}$ for all elements of $\partial Q'$.

From Lemma 3.1 we deduce that the bilinear form $r(\cdot,\cdot)$ is an inner product on H. Rather than the natural (product) norm generated by $r(\cdot,\cdot)$, we will find it convenient to use the norm on H defined by

$$\|\cdot\|_H : H \supset (\boldsymbol{v}, v_n) \to \|\boldsymbol{v}\|_{0,\Omega^s} + \left\|\nabla\mathbf{R}^f v_n\right\|_{0,\Omega^f}.$$

Now

$$\begin{aligned}\|\mathbf{u}\|_H\, \|\mathbf{v}\|_H &= (\|\boldsymbol{u}\|_{0,\Omega^s} + \left\|\nabla\mathbf{R}^f u_n\right\|_{0,\Omega^f})(\|\boldsymbol{v}\|_{0,\Omega^s} \left\|\nabla\mathbf{R}^f v_n\right\|_{0,\Omega^f}) \\ &= \|\boldsymbol{u}\|_{0,\Omega^s}\, \|\boldsymbol{v}\|_{0,\Omega^s} + \left\|\nabla\mathbf{R}^f u_n\right\|_{0,\Omega^f} \left\|\nabla\mathbf{R}^f v_n\right\|_{0,\Omega^f} \\ &\quad + \|\boldsymbol{u}\|_{0,\Omega^s} \left\|\nabla\mathbf{R}^f v_n\right\|_{0,\Omega^f} + \|\boldsymbol{v}\|_{0,\Omega^s} \left\|\nabla\mathbf{R}^f u_n\right\|_{0,\Omega^f} \\ &\geq C\, r(\mathbf{u},\mathbf{v}) \qquad \forall \mathbf{u},\mathbf{v} \in H,\end{aligned} \tag{3.2}$$

so $r(\cdot,\cdot)$ is continuous with respect to $\|\cdot\|_H$. For the space V we may choose the norm defined by

$$\|\cdot\|_V : V \supset (\boldsymbol{v}, v_n) \to \|\boldsymbol{v}\|_{1,\Omega^s},$$

since for all $\boldsymbol{v} \in H^1(\Omega^s)^3, \boldsymbol{v} = \mathbf{0}$ implies that $v_n = 0$, $\|\cdot\|_{1,\Omega^s}$ is a norm on V. Furthermore, the bilinear form $a(\cdot,\cdot)$ is continuous on $H^1(\Omega^s)^3$ and coercive on V by the Korn inequality, so there exist constants β and N such that

$$\beta \|\mathbf{v}\|_V^2 \le a(\mathbf{v},\mathbf{v}) \quad \forall \mathbf{v} \in V \tag{3.3}$$

and

$$a(\mathbf{u},\mathbf{v}) \le N \|\mathbf{u}\|_V \|\mathbf{v}\|_V \quad \forall \mathbf{u},\mathbf{v} \in V. \tag{3.4}$$

Remark 3.2. The mapping $\boldsymbol{n}\cdot\gamma^s$ defines, for every element $\boldsymbol{v}$ in $H^1(\Omega^s)^3$, an element v_n unique in $L_2(\Gamma)$. This is not the case when $\boldsymbol{n}\cdot\gamma^s$ is considered as an operator on $L_2(\Omega^s)^3$. Thus elements of H are ordered pairs of independent functions.

By continuity of the injection $H^s(\Gamma) \hookrightarrow H^{s+\epsilon}(\Gamma), \epsilon > 0$, the Schwarz inequality, continuity of the trace operator γ^s, and the assumption that $\boldsymbol{n} \in L_2(\Gamma)$,

$$\begin{aligned} \|v_n\|_{\partial Q'} &\le C \|v_n\|_{0,\Gamma} \le C\ 1 \cdot \|\gamma^s \boldsymbol{v}\|_{0,\Gamma} \\ &\le C \|\gamma^s \boldsymbol{v}\|_{\frac{1}{2},\Gamma} \le C \|\boldsymbol{v}\|_{1,\Omega^s} = C \|\mathbf{v}\|_V \, \forall \mathbf{v} \in V. \end{aligned} \tag{3.5}$$

Furthermore, for all $\mathbf{v} = (\boldsymbol{v}, v_n) \in V$

$$\begin{aligned} \|\mathbf{v}\|_H &= \|\boldsymbol{v}\|_{0,\Omega^s} + \left\|\nabla \mathbf{R}^f v_n\right\|_{0,\Omega^f} \\ &= \|\boldsymbol{v}\|_{0,\Omega^s} + \|v_n\|_{\partial Q'} && \text{by Lemma 3.1,} \\ &\le C \|\boldsymbol{v}\|_{1,\Omega^s} + \|v_n\|_{\partial Q'} && \text{by continuity of } H^1(\Omega^3) \hookrightarrow L_2(\Omega^3), \\ &\le C \|\mathbf{v}\|_V && \text{by (3.5),} \end{aligned}$$

and, since $V \subset H$, the injection of V into H is continuous.

We now prove

Theorem 3.3. *The solution of variational eigenvalue problem* (2.18) *is a countable sequence of real eigenpairs* $(\lambda_j, \mathbf{u}_j)$ *with* $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \cdots < +\infty$.

Proof. We define operators $\mathbf{T}, \mathbf{T}_* : H \to H$ by

$\mathbf{u} \in H$:

$$\begin{aligned} \mathbf{T}\mathbf{u} &\in V, & a(\mathbf{T}\mathbf{u},\mathbf{v}) &= r(\mathbf{u},\mathbf{v}) & \forall \mathbf{v} &\in V, \\ \mathbf{T}_*\mathbf{u} &\in V, & a(\mathbf{v},\mathbf{T}_*\mathbf{u}) &= r(\mathbf{v},\mathbf{u}) & \forall \mathbf{v} &\in V. \end{aligned}$$

The existence and uniqueness of $\mathbf{T}\mathbf{u}$ and $\mathbf{T}_*\mathbf{u}$ is guaranteed by (3.2), (3.3) and the Lax-Milgram lemma. Now $\mathbf{T}$ is a bounded operator from H into V since

$$\begin{aligned} \beta \|\mathbf{T}\mathbf{u}\|_V^2 &\le a(\mathbf{T}\mathbf{u},\mathbf{T}\mathbf{u}) \\ &= r(\mathbf{u},\mathbf{T}\mathbf{u}) \\ &\le M \|\mathbf{u}\|_H \|\mathbf{T}\mathbf{u}\|_H \\ &\le C \|\mathbf{u}\|_H \|\mathbf{T}\mathbf{u}\|_V . \end{aligned}$$

Because the imbedding of $H^1(\Omega^s)^3$ into $L_2(\Omega^s)^3$ and the imbedding of $L_2(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$ are compact, and the injection of V into H is continuous, the imbedding of V into H is compact. Therefore $\mathbf{T}$ is compact from H into H. Furthermore,

the bilinear forms $a(\cdot,\cdot)$ and $r(\cdot,\cdot)$ are symmetric and positive definite on V and H, respectively, and both are symmetric on $V \cup H$, so $\mathbf{T} = \mathbf{T}_*$ is a compact positive self-adjoint operator on H. Finally, since $(\lambda, \mathbf{u})$ is an eigenpair of equation (2.18) if and only if $(\frac{1}{\lambda}, \mathbf{u})$ is an eigenpair of $\mathbf{T}$, we have shown that the eigenpairs of variational problem (2.18) are a countable sequence of positive real eigenvalues tending to infinity with corresponding real eigenfunctions. □

4. Finite element formulation

We now formulate a matrix eigenvalue problem based on finite element discretization of problem (2.18). Let $\{\tau_h\}$ be a triangulation of $\bar{\Omega}$ such that

$$\bar{\Omega} = \bigcup_{T \in \tau_h} T.$$

Let $\{\tau_h\} = \{\mathcal{F}_h\} \cup \{\mathcal{S}_h\}$, such that

$$\bar{\Omega}^f = \bigcup_{T \in \mathcal{F}_h} T \qquad \text{and} \qquad \bar{\Omega}^s = \bigcup_{T \in \mathcal{S}_h} T.$$

Denote by ∂T the boundary of each region T. Let $\{\Lambda_h\}$ be a triangulation of Γ by the boundaries of the solid elements such that

$$\begin{aligned} \Gamma &= \bigcup_{T \in \mathcal{S}_h} \partial T \cap \Gamma \\ &= \bigcup_{\partial T \in \Lambda_h} \partial T. \end{aligned} \tag{4.1}$$

No assumptions are made with respect to the compatibility of fluid and solid mesh on Γ.

Let $\mathcal{Q}_h^k = \{q^h \mid q^h \in C^0(\bar{\Omega}^f) : \forall T \in \mathcal{F}_h, q^h \mid_T \in P_k(T)\}$ be a space of Lagrange fluid finite elements. Since $\mathcal{Q}^h \subset H^1(\Omega^f) \cap C^0(\bar{\Omega}^f)$, the trace operator γ^f is defined, and, for any $q^h \in Q^h$, $\gamma^f q^h$ coincides with the restriction $q^h \mid_\Gamma$. Let $\bar{N}$ be the number of element nodes in the triangulation of $\bar{\Omega}^f$ and let $S_A(\boldsymbol{x}),\ A = 1, 2, \ldots, \bar{N}$, be the set of shape functions in the fluid and $q_A,\ A = 1, 2, \ldots, \bar{N}$, the corresponding nodal pressure amplitudes. Then $\dim \mathcal{Q}^h = \bar{N}$ and any q^h in $\mathcal{Q}_h^k$ has the form

$$q^h = \begin{bmatrix} S_1 & S_2 & \ldots & S_{\bar{N}} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\bar{M}} \end{Bmatrix} = \mathbf{S}^{\mathbf{T}} \mathbf{q}. \tag{4.2}$$

Similarly, let $\mathcal{V}_h^k = \{\boldsymbol{v}^h \mid \boldsymbol{v}^h \in C^0(\bar{\Omega}^s)^3 : \forall T \in \mathcal{S}_h, v_i \mid_T \in P_k(T), i = 1, 2, 3\}$ be a space of Lagrange solid finite elements. Since $\mathcal{V}^h \subset H^1(\Omega^s)^3 \cap C^0(\bar{\Omega}^s)^3$, the trace operator γ^s is defined, and, for any $\boldsymbol{v}^h \in V^h$, $\gamma^s \boldsymbol{v}^h$ coincides with the restriction $\boldsymbol{v}^h \mid_\Gamma$. Let $\bar{M}$ be the number of element nodes in the triangulation of $\bar{\Omega}^s$ and let $N_A(\boldsymbol{x}),\ A = 1, 2, \ldots, \bar{M}$, be the set of shape functions in the solid and $(v_1\ v_2\ v_3)_A^T,\ A = 1, 2, \ldots, \bar{M}$, the corresponding nodal displacement amplitudes.

Then $\dim \mathcal{V}^h = 3\bar{M}$ and any $\boldsymbol{v}^h$ in $\mathcal{V}_h^k$ may be written

(4.3)

$$\boldsymbol{v}^h = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots & N_{\bar{M}} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_{\bar{M}} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots & 0 & 0 & N_{\bar{M}} \end{bmatrix} \begin{Bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_1 \\ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_2 \\ \vdots \\ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{\bar{M}} \end{Bmatrix} = \mathbf{N^T V}.$$

As described in Section 2, the unit normal vector on Γ is globally $L_2(\Gamma)^3$ but is locally smooth so that on the interior of each $\partial T \subset \Gamma$

$$\boldsymbol{n}(\boldsymbol{x}) = \begin{Bmatrix} n_1(\boldsymbol{x}) \\ n_2(\boldsymbol{x}) \\ n_3(\boldsymbol{x}) \end{Bmatrix}$$

is continuous. Note that $\boldsymbol{n}$ need not be uniquely defined on the vertices between adjacent facets ∂T. Let

$$\begin{aligned} \mathbf{K} &= \int_{\Omega^s} \boldsymbol{\sigma}(\mathbf{N}) : \boldsymbol{\epsilon}(\mathbf{N^T})\ d\Omega, \\ \mathbf{M} &= \int_{\Omega^s} \rho^s \mathbf{N} \cdot \mathbf{N^T}\ d\Omega, \\ \mathbf{A} &= \int_{\Gamma} \mathbf{S}\ \boldsymbol{n}^{\mathbf{T}}\ \mathbf{N^T}\ d\Gamma, \\ \mathbf{B} &= \int_{\Omega^f} \nabla \mathbf{S} \cdot \nabla \mathbf{S^T}\ d\Omega, \end{aligned}$$

so that upon substitution of $\mathcal{V}^h$ for V in equation (2.7) and $\mathcal{Q}^h$ for Q in equation (2.8) we obtain

$$\mathbf{V^T}\left\{\mathbf{KU} - \mathbf{A^T p} - \lambda \mathbf{MU}\right\} = 0, \tag{4.4}$$

$$\mathbf{q^T}\left\{\mathbf{Bp} - \lambda \rho^f \mathbf{AU}\right\} = 0. \tag{4.5}$$

The solution to problem (4.4), (4.5) contains a zero eigenvalue of multiplicity 6, corresponding to zero strain energy states of the solid, and of multiplicity 1 corresponding to a constant pressure state in the fluid. We seek a solution which eliminates the pressure variable while simultaneously eliminating 1 zero eigenvalue. Now $\mathbf{p}$ and $\mathbf{q}$, the vectors of nodal pressure and weighting function amplitudes, are elements of $\mathbb{R}^{\bar{N}}$ for which the standard basis is $(\mathbf{e}_1,\ \mathbf{e}_2, \dots, \mathbf{e}_{\bar{N}})$ so that

$$\mathbf{q} = \sum_{i=1}^{\bar{N}} q_i \mathbf{e}_i. \tag{4.6}$$

The matrix $\mathbf{B}$ is not invertible on this basis because the nullspace of $\mathbf{B}$ is the constant vector $\alpha \mathbf{1}_{\bar{N}}$ and is representable by (4.6). We instead choose the basis

$(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{k-1}, \mathbf{1}, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_{\bar{N}})$ so that

$$\mathbf{q} = q_k \mathbf{1}_{\bar{N}} + \sum_{i=1}^{k-1} q_i \mathbf{e}_i + \sum_{i=k+1}^{\bar{N}} q_i \mathbf{e}_i, \tag{4.7}$$

where q_k is the nodal amplitude of the kth node. Assume (for convenience) that $k = 1$ in (4.7) and partition $\mathbf{p}, \mathbf{q}, \mathbf{B}$ and $\mathbf{A}$ according to

$$\mathbf{p} = p_0 \begin{Bmatrix} 1 \\ \mathbf{1}_{\bar{N}-1} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \tilde{\mathbf{p}} \end{Bmatrix}, \qquad \mathbf{q} = q_0 \begin{Bmatrix} 1 \\ \mathbf{1}_{\bar{N}-1} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \tilde{\mathbf{q}} \end{Bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{b}_{22} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2. \end{bmatrix},$$

where q_0 and p_0 are arbitrary constants. Substituting these expressions into equation (4.5) results in the two equations

$$q_0 \lambda \rho^f \mathbf{1}_{\bar{N}}^{\mathbf{T}} \mathbf{A} \mathbf{U} = 0 \tag{4.8}$$

and

$$\tilde{\mathbf{q}}^{\mathbf{T}} \left[\mathbf{b}_{22} \tilde{\mathbf{p}} - \lambda \rho^f \mathbf{a}_2 \mathbf{U} \right] = 0, \tag{4.9}$$

which must be satisfied independently.

Remark 4.1. All terms involving p_0 vanish identically due to the nullity of $\mathbf{B}$ on the space of constant functions.

For nonzero λ, (4.8) is the finite dimensional counterpart of the constraint equation $\langle u_n, q_0 \rangle = 0$. Let u_l be a scalar component of nodal displacement $\mathbf{U}$ on Γ, chosen such that a unit displacement results in a nonzero change in fluid volume. Partition $\mathbf{A}$ and $\mathbf{U}$ and write equation (4.8) as

$$\mathbf{1}_{\bar{N}}^{\mathbf{T}} \left[\hat{\mathbf{a}}_1 \ \hat{\mathbf{a}}_2 \right] \begin{Bmatrix} u_l \\ \hat{\mathbf{U}} \end{Bmatrix} = 0. \tag{4.10}$$

By solving (4.10) for u_l in terms of $\hat{\mathbf{U}}$ and letting

$$\mathbf{C} = \begin{bmatrix} -\frac{\mathbf{1}_{\bar{N}}^{\mathbf{T}} \hat{\mathbf{a}}_2}{\mathbf{1}_{\bar{N}}^{\mathbf{T}} \hat{\mathbf{a}}_1} \\ \mathbf{I}_{3\bar{M}-1} \end{bmatrix}, \tag{4.11}$$

we define a subspace of $\mathcal{V}_k^h$ consisting of all fluid volume preserving motions by

$$\boldsymbol{u}^h = \mathbf{N}^{\mathbf{T}} \mathbf{C} \hat{\mathbf{U}}. \tag{4.12}$$

Now for all solid nodal displacement vectors $\mathbf{U}$ satisfying $\mathbf{U} = \mathbf{C}\hat{\mathbf{U}}$, we can solve (4.9) for $\tilde{\mathbf{p}}$, so that $\mathbf{p}$ may be expressed in terms of p_0 and $\hat{\mathbf{U}}$ as

$$\mathbf{p} = p_0 \begin{Bmatrix} 1 \\ \mathbf{1}_{\bar{N}-1} \end{Bmatrix} + \lambda \rho^f \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} \mathbf{C} \hat{\mathbf{U}}$$

or

$$\mathbf{p} = \quad \mathbf{p_0} \quad + \quad \lambda \rho^f \bar{\mathbf{B}}^{-1} \mathbf{A} \ \mathbf{C} \ \hat{\mathbf{U}}. \tag{4.13}$$

Since $\hat{\mathbf{U}}$ has a representation

$$\hat{\mathbf{U}} = \sum_{i=1}^{l-1} u_i \mathbf{e}_i + \sum_{i=l+1}^{3\bar{M}} u_i \mathbf{e}_i,$$

the solution of (4.13) for all basis vectors $\mathbf{e}_i$, $i = 1, 2, \ldots, l-1, l+1, \ldots, 3\bar{M}$, defines a transformation from a subspace of $\mathcal{V}^h$ consisting of all fluid volume preserving solid motions into a subspace of $\mathcal{Q}^h$. This subspace is an equivalence class of fluid motions excitable by motion of the solid, differing by an arbitrary constant. Finally, we make the substitutions $\mathbf{V} = \mathbf{C}\hat{\mathbf{V}}, \mathbf{U} = \mathbf{C}\hat{\mathbf{U}}$ and equation (4.13) into (4.4), to obtain the matrix eigenvalue problem

$$\mathbf{C^T K C}\ \hat{\mathbf{U}} = \lambda \left\{ \mathbf{C^T M C} + \rho^f \mathbf{C^T A^T \bar{B}^{-1} A C} \right\} \hat{\mathbf{U}}. \tag{4.14}$$

Remark 4.2. The quantity $\rho^f \mathbf{C^T A^T \bar{B}^{-1} A C}$ is the "added mass matrix".

The solution of equation (4.14) is a set of real eigenvalues λ_{h_j} and eigenfunctions $\boldsymbol{u}_{h_j} = \mathbf{N^T \hat{U}}_j,\ j = 1, 2, \ldots, 3\bar{M} - 1$, orthonormal with respect to the total mass matrix. Corresponding to the nullspace of $\mathbf{K}$ will be 6 rigid body eigenfunctions and associated null eigenvalues. The remaining $3\bar{M} - 7$ eigenvalues will be positive.

5. Error analysis

It remains to show in what sense the solution to the matrix eigenvalue problem (4.14) is an approximation of the variational eigenvalue problem (2.18), and to quantify the error in the approximate solution. We begin by defining the function spaces Q^h and V^h used to approximate Q and V. For arbitrary q_k, the set of functions defined by equations (4.2) and (4.7) is an element of the quotient space

$$Q^h = \{q^h \mid q^h \in \mathcal{Q}_h^k / \mathcal{P}\}. \tag{5.1}$$

Now any element of $\mathcal{Q}_h^k$ satisfies the approximation property

$$\inf_{q^h \in \mathcal{Q}_h^k} \{ \mid q - q^h \mid_{1,\Omega^f} + h_f^{-1} \mid q - q^h \mid_{0,\Omega^f} \} \leq C h_f^m \mid q \mid_{m+1,\Omega^f}, \tag{5.2}$$

where h_f is the fluid mesh diameter, $|\cdot|_{m,\Omega^f}$ is the seminorm corresponding to $\|\cdot\|_{m,\Omega^f}$, $m = \min(k, r-1)$ where k is the order of the finite element polynomials and r is the regularity of the function $q \in H^r(\Omega^f)$.

Now for the solid. Since the linearized rigid body motions $\boldsymbol{\phi}_R$ are elements of $C^0(\Omega^s)^3$, they may be represented exactly by functions in $\mathcal{V}_h^k$. We therefore assume that they are reproduced without error by the solution to (4.14) and restrict our attention to the elastic motions by defining V^h to be the space defined by (4.3) and

$$V^h = \{\boldsymbol{v}^h \mid \boldsymbol{v}^h \in \mathcal{V}_h^k, \int_\Gamma v_n^h\, q_0\, d\Gamma = 0 \quad \forall q_0 \in \mathcal{P}, r_h(\boldsymbol{v}^h, \boldsymbol{\phi}_R) = 0\}, \tag{5.3}$$

where $r_h(\cdot,\cdot)$ is an approximation of $r(\cdot,\cdot)$. Elements of $\mathcal{V}_k^h$ satisfy the approximation property

$$\inf_{\boldsymbol{v}^h \in \mathcal{V}_h^k} \{ \mid \boldsymbol{v} - \boldsymbol{v}^h \mid_{1,\Omega^s} + h_s^{-1} \mid \boldsymbol{v} - \boldsymbol{v}^h \mid_{0,\Omega^s} \} \leq C h_s^m \mid \boldsymbol{v} \mid_{m+1,\Omega^s}, \tag{5.4}$$

where h_s is the solid mesh diameter, $|\cdot|_{m,\Omega^s}$ is the seminorm corresponding to $\|\cdot\|_{m,\Omega^s}$, and m is as defined above with r the regularity of the function $\boldsymbol{v} \in H^r(\Omega^s)^3$.

Remark 5.1. The estimates (5.2), (5.4), hold for nonsmooth functions with $1 \le r < 2$ if we identify q^h and $\boldsymbol{v}^h$ with the local averaging interpolate of q and $\boldsymbol{v}$ described by Brenner and Scott [7] (see also Scott and Zhang [16]).

Remark 5.2. The normal trace of any function $v_n^h = \boldsymbol{n} \cdot \gamma^s \boldsymbol{v}^h$ in V^h is an element of $\partial Q'$ since $v_n^h \in L_2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, and $\int_\Gamma \boldsymbol{n} \cdot \gamma^f \boldsymbol{v}\, d\Gamma = 0$ is the restriction of the constraint equation $\langle v_n, q_0 \rangle = 0$ to $v_n \in L_2(\Gamma)$.

We denote with superscript h an arbitrary element of Q^h or V^h, and use subscript h to denote an element of the solution space. Consider the operator $\mathbf{T}_h : H \to H$ defined by the variational problem:

$\mathbf{u} \in H, \qquad \mathbf{T}_h \mathbf{u} \in V^h$ and

$$\int_{\Omega^s} \boldsymbol{\sigma}(\mathbf{T}_h \boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}^h)\, d\Omega - \langle v_n^h, \gamma^f \phi_h \rangle = \int_{\Omega^s} \rho^s \boldsymbol{u} \cdot \boldsymbol{v}^h\, d\Omega \qquad \forall \boldsymbol{v}^h \in V_h, \tag{5.5}$$

$$\int_{\Omega^f} \nabla \phi_h \cdot \nabla q^h\, d\Omega = \rho^f \langle u_n, \gamma^f q^h \rangle\, d\Gamma \qquad \forall q^h \in Q^h. \tag{5.6}$$

Since $Q^h \subset H^1(\Omega^f)/\mathcal{P}$, the bilinear form $\int_{\Omega^f} \nabla \phi_h \cdot \nabla q^h\, d\Gamma$ satisfies the coercivity and continuity properties (2.9), (2.10) for all $\phi_h, q^h \in Q^h$ and (2.11) is satisfied for all $q^h \in Q^h$. As a result, for any u_n in $\partial Q'$, (5.6) is a matrix equation solvable for an element $[\phi_h] \in Q^h$, unique up to an arbitrary constant $\phi_0 \in \mathcal{P}$. So we can define a finite dimensional operator $\mathbf{R}_h^f : \partial Q' \to Q^h$ by $[\phi_h] = \phi_0 + \rho^f \mathbf{R}_h^f u_n$ analogous to (4.13). Let $\mathbf{S}_h^f = \gamma^f \mathbf{R}_h^f$ be the composition of the trace operator γ^f and $\mathbf{R}_h^f$ and denote by ∂Q^h the restriction of Q^h to Γ, then $\mathbf{S}_h^f : \partial Q' \to \partial Q^h$ and we can write

$$\gamma^f [\phi_h] = [\phi_h] \mid_\Gamma = \phi_0 + \rho^f \mathbf{S}_h^f u_n. \tag{5.7}$$

Substitute equation (5.7) into equation (5.5) and let

$$r_h(\mathbf{u}, \mathbf{v}) = \int_{\Omega^s} \rho^s \boldsymbol{u} \cdot \boldsymbol{v}\, d\Omega + \rho^f \langle v_n, \mathbf{S}_h^f u_n \rangle.$$

We then write a one-field finite dimensional variational problem:

$\mathbf{u} \in H, \qquad \mathbf{T}_h \mathbf{u} \in V^h$ and

$$a(\mathbf{T}_h \mathbf{u}, \mathbf{v}^h) = r_h(\mathbf{u}, \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in V^h. \tag{5.8}$$

Since $a(\cdot,\cdot)$ is positive definite on V^h and $r_h(\cdot,\cdot)$ is continuous on H, the existence and uniqueness of $\mathbf{T}_h \mathbf{u}$ is assured. Now $(\frac{1}{\lambda_h}, \boldsymbol{u}_h)$ is an eigenpair of $\mathbf{T}_h$ if and only if $(\lambda_h, \boldsymbol{u}_h)$ is an eigenpair of the following variational problem.

Find $\lambda_h, \mathbf{u}_h \in \mathbb{R} \times V^h$ such that

$$a(\mathbf{u}_h, \mathbf{v}^h) = \lambda_h r_h(\mathbf{u}_h, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in V^h. \tag{5.9}$$

But equation (5.9) is precisely equation (4.14) restricted to the space of nonzero strain energy states. So we have identified the eigenpairs, corresponding to positive eigenvalues, of the matrix eigenvalue problem with those of $\mathbf{T}_h$.

Remark 5.3. Note that $r_h(\cdot,\cdot) : H \times H \to \mathbb{R}$, differs from $r(\cdot,\cdot)$ by the presence of the finite-dimensional operator $\mathbf{S}_h^f$ in place of $\mathbf{S}^f$, so that $V^h \not\subset V$. Also, since ∇q^h will suffer jumps across adjacent elements $T \in \mathcal{F}_h$, $\triangle q^h$ is not defined in all of Ω^f, and $Q^h \not\subset Q$. Therefore the variational eigenvalue problem (5.9) is a nonconforming approximation of (2.18).

Remark 5.4. Recall that we relied upon the surjectivity of the Neumann operator on Q to eliminate the pressure from equations (2.7) and (2.8). Since $Q^h \not\subset Q$ we do not have surjectivity of the Neumann operator acting on Q^h. Instead, we used the fact that Q^h is finite dimensional to eliminate the pressure variable from equations (5.5) and (5.6).

Now we show that for sufficiently refined meshes the solutions of (4.14) are approximations of those of (2.18). We first establish some preliminary results.

Remark 5.5. Although $Q^h \not\subset Q$, the orthogonality relation

$$\int_{\Omega^f} \nabla(\mathbf{R}^f - \mathbf{R}_h^f)u_n \cdot \nabla q^h \, d\Omega = 0 \quad \forall q^h \in Q^h \tag{5.10}$$

holds, since from the Green formula for $\mathbf{R}^f u_n \in Q,\ q^h \in Q^h$,

$$\int_{\Omega^f} \nabla \mathbf{R}^f u_n \cdot \nabla q^h \, d\Omega = \langle u_n, \gamma^f q^h \rangle,$$

and, by definition of $\mathbf{R}_h^f u_n$

$$\int_{\Gamma} \nabla \mathbf{R}_h^f u_n \cdot \nabla q^h \, d\Omega = \langle u_n, \gamma^f q^h \rangle \quad \forall q^h \in Q^h.$$

We now state the following regularity result, the proof of which is found in Girault and Raviart [10].

Lemma 5.6. *Let Ω be a convex polyhedral region in $\mathbb{R}^3$ with boundary $\partial\Omega$ and let $\boldsymbol{v} \in H^1(\Omega)^3$ satisfy $\int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} \, d\Gamma = 0$, then the solution ϕ of the boundary value problem*

$$\begin{aligned} \triangle\phi &= 0 && \text{in} \quad \Omega, \\ \frac{\partial \phi}{\partial n} &= \boldsymbol{v} \cdot \boldsymbol{n} && \text{on} \quad \partial\Omega, \end{aligned}$$

belongs to $H^2(\Omega)/\mathcal{P}$.

Define

$$\partial V = \{v_n \mid v_n \in \prod_{\partial T \in \Lambda_h} H^{\frac{1}{2}}(\partial T) \quad \forall \boldsymbol{v} \in V \cup V^h\}.$$

We equip ∂V with the mesh-dependent norm

$$\|v_n\|_{\partial V} = \left(\sum_{\partial T \in \Lambda_h} \|v_n\|^2_{\frac{1}{2},\partial T} \right)^{\frac{1}{2}},$$

with Λ_h defined by (4.1). The normal traces $\boldsymbol{n} \cdot \gamma^s \boldsymbol{v}$ of functions $\boldsymbol{v} \in V \cup V^h$ are elements of ∂V. It is clear that Lemma 5.6 implies that the solution $p \in Q$ of (2.8) with Neumann boundary data $v_n \in \partial V$ is an element of $H^2(\Omega^f)$, and that there exists a constant C such that $\|[p]\|_{H^2(\Omega^f)/\mathcal{P}} \leq C \|v_n\|_{\partial V}$. Furthermore, by continuity of the trace operator γ^s, there exists a constant C such that

$$\|v_n\|_{\partial V} \leq C \|\boldsymbol{v}\|_{1,\Omega^s} \quad \forall \boldsymbol{v} \in V \cup V^h. \tag{5.11}$$

Now we establish the following error estimate for the Steklov operator (extended from a result due to Bramble and Osborn [5]).

Lemma 5.7.

$$\left\|\mathbf{S}^f - \mathbf{S}_h^f\right\|_{HG} = \sup_{\xi\in G}\sup_{\eta\in H}\frac{|\,\langle\xi,(\mathbf{S}^f-\mathbf{S}_h^f)\eta\rangle\,|}{\|\eta\|_H\,\|\xi\|_G}$$
$$= C\begin{cases} h_f, & \text{if} \quad H=\partial V, \quad G=\partial Q',\\ h_f, & \text{if} \quad H=\partial Q', \quad G=\partial V.\\ h_f^2, & \text{if} \quad H=\partial V, \quad G=\partial V.\end{cases}$$

Proof. Let $\eta\in\partial Q'$. From equations (2.9) and (2.10) we have

$$\begin{aligned}\left\|[(\mathbf{R}^f-\mathbf{R}_h^f)\eta]\right\|_{H^1(\Omega^f)/\mathcal{P}} &\le \frac{1}{\alpha}\int_{\Omega^f}\nabla(\mathbf{R}^f-\mathbf{R}_h^f)\eta\cdot\nabla(\mathbf{R}^f-\mathbf{R}_h^f)\eta\,d\Omega\\ &= \frac{1}{\alpha}\int_{\Omega^f}\nabla(\mathbf{R}^f-\mathbf{R}_h^f)\eta\cdot\nabla(\mathbf{R}^f\eta-q^h)\,d\Omega\\ &\le \frac{M}{\alpha}\left\|[(\mathbf{R}^f-\mathbf{R}_h^f)\eta]\right\|_{H^1(\Omega^f)/\mathcal{P}}\left\|\mathbf{R}^f\eta-q^h\right\|_{1,\Omega^f}\,\forall q^h\in Q^h\end{aligned}$$

and so

$$\left\|[(\mathbf{R}^f-\mathbf{R}_h^f)\eta]\right\|_{H^1(\Omega^f)/\mathcal{P}} \le \frac{M}{\alpha}\inf_{q^h\in Q^h}\left\|\mathbf{R}^f\eta-q^h\right\|_{1,\Omega^f}. \tag{5.12}$$

For any $\xi,\eta\in\partial Q'$,

$$\begin{aligned}\langle\xi,(\mathbf{S}^f-\mathbf{S}_h^f)\eta\rangle &= \int_{\Omega^f}\nabla(\mathbf{R}^f-\mathbf{R}_h^f)\eta\cdot\nabla\mathbf{R}^f\xi\,d\Omega,\\ &= \int_{\Omega^f}\nabla(\mathbf{R}^f-\mathbf{R}_h^f)\eta\cdot\nabla(\mathbf{R}^f\xi-q^h)\,d\Omega,\\ &\le M\left\|[(\mathbf{R}^f-\mathbf{R}_h^f)\eta]\right\|_{H^1(\Omega^f)/\mathcal{P}}\left\|\mathbf{R}^f\xi-q^h\right\|_{1,\Omega^f}\quad\forall q^h\in Q^h,\end{aligned} \tag{5.13}$$

where we made use of (5.10). Upon substitution of (5.12) into (5.13), we obtain

$$\langle\xi,(\mathbf{S}^f-\mathbf{S}_h^f)\eta\rangle \le C\inf_{\phi^h\in Q^h}\left\|\mathbf{R}^f\eta-\phi^h\right\|_{1,\Omega^f}\inf_{q^h\in Q^h}\left\|\mathbf{R}^f\xi-q^h\right\|_{1,\Omega^f}. \tag{5.14}$$

From Lemma 3.1 we have

$$|\mathbf{R}^f\eta|_{1,\Omega^f} \le C\,\|\eta\|_{\partial Q'} \qquad \forall\eta\in\partial Q', \tag{5.15}$$

and from Lemma 5.6 we have

$$|\mathbf{R}^f\eta|_{2,\Omega^f} \le C\,\|\eta\|_{\partial V} \qquad \forall\eta\in\partial V. \tag{5.16}$$

Now using (5.15), (5.16) and the approximation property (5.2) of Q^h, we obtain

$$\inf_{q^h\in Q^h}\left\|\mathbf{R}^f\xi-q^h\right\|_{1,\Omega^f} \le C\begin{cases}\|\xi\|_{\partial Q'} & \text{for } \xi\in\partial Q',\\ h_f\,\|\xi\|_{\partial V} & \text{for } \xi\in\partial V,\end{cases} \tag{5.17}$$

and the result follows from (5.14) and (5.17). □

For the smoothness of the eigenfunctions in the solid, we are concerned with the regularity of the solution $\boldsymbol{u}$ of the boundary value problem

$$\begin{aligned} -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in} && \Omega^s, \\ \boldsymbol{n}\cdot\boldsymbol{\sigma}_n(\boldsymbol{u}) &= g && \text{on} && \Gamma, \\ \boldsymbol{\sigma}_n - \boldsymbol{n}\cdot\boldsymbol{\sigma}_n(\boldsymbol{u}) &= \mathbf{0} && \text{on} && \Gamma, \\ \boldsymbol{\sigma}_n(\boldsymbol{u}) &= \mathbf{0} && \text{on} && \partial\Omega^s\backslash\Gamma, \end{aligned}$$

with $\boldsymbol{f}\in L_2(\Omega^s)^3$ and $g\in H^{\frac{1}{2}}(\Gamma)$. To our knowledge, the precise regularity of this solution for arbitrary polyhedral domains in $\mathbb{R}^3$ has not been established. Since Ω^s contains conical points and re-entrant vertices on Γ, it is probably unreasonable to expect that the solution will be in $H^2(\Omega^s)^3$ even if the external boundary to Ω^s is convex. We will therefore *assume* that the eigenfunctions $\boldsymbol{u}$, restricted to the solid, belong to $H^r(\Omega^s)^3$ for $1<r\le 2$, where r depends on the smoothness of the boundary $\partial\Omega^s$.

Let $\mathbf{T}_{*h}: H\to H$ be defined by

$\mathbf{u}\ \in H,\quad \mathbf{T}_{*h}\mathbf{u}\ \in V^h$:

$$a(\mathbf{v}^h,\mathbf{T}_{*h}\mathbf{u}) = r_h(\mathbf{v}^h,\mathbf{u})\quad \forall\mathbf{v}^h\ \in V^h. \tag{5.18}$$

Note that $r_h(\mathbf{v}^h,\mathbf{u})\neq r_h(\mathbf{u},\mathbf{v}^h)$ for all $\mathbf{u}\in H,\mathbf{v}^h\in V^h$, so $\mathbf{T}_h\neq\mathbf{T}_{*h}$.

We now prove

Lemma 5.8. $\mathbf{T}_h$ *converges to* $\mathbf{T}$ *in operator norm*

$$\lim_{\substack{h_f\to 0\\ h_s\to 0}} \|\mathbf{T}-\mathbf{T}_h\|_H = 0.$$

Proof. Since $r(\cdot,\cdot)$ is an inner product on H, continuous with respect to $\|\cdot\|_H$, we can define the operator norm on H by

$$\|\mathbf{T}\|_H = \sup_{\substack{\mathbf{u}\in H\\ \|\mathbf{u}\|_H=1}} \|\mathbf{T}\mathbf{u}\|_H = \sup_{\substack{\mathbf{u}\in H\\ \|\mathbf{u}\|_H=1}}\ \sup_{\substack{\mathbf{v}\in H\\ \|\mathbf{v}\|_H=1}} r(\mathbf{T}\mathbf{u},\mathbf{v}).$$

Using continuity of $a(\cdot,\cdot)$, the second Strang lemma [11], and (5.11),

$$\begin{aligned} r((\mathbf{T}-\mathbf{T}_h)\mathbf{u},\mathbf{v}) &= a((\mathbf{T}-\mathbf{T}_h)\mathbf{u},\mathbf{T}_*\mathbf{v}) \\ &= a((\mathbf{T}-\mathbf{T}_h)\mathbf{u},(\mathbf{T}_*-\mathbf{T}_{*h})\mathbf{v}) + a((\mathbf{T}-\mathbf{T}_h)\mathbf{u},\mathbf{T}_{*h}\mathbf{v}) \\ &\le M\,\|(\mathbf{T}-\mathbf{T}_h)\mathbf{u}\|_V\,\|(\mathbf{T}_*-\mathbf{T}_{*h})\mathbf{v}\|_V + r(\mathbf{u},\mathbf{T}_{*h}\mathbf{v}) - r_h(\mathbf{u},\mathbf{T}_{*h}\mathbf{v}) \\ &= M\,\|(\mathbf{T}-\mathbf{T}_h)\mathbf{u}\|_V\,\|(\mathbf{T}_*-\mathbf{T}_{*h})\mathbf{v}\|_V + \langle\mathbf{T}_{*h}v_n,(\mathbf{S}^f-\mathbf{S}^f_h)u_n\rangle \\ &\le C\Big\{\big(\inf_{\mathbf{v}^h\in V^h}\|\mathbf{T}\mathbf{u}-\mathbf{v}^h\|_V + \sup_{\mathbf{w}^h\in V^h}\frac{|\,\langle w^h_n,(\mathbf{S}^f-\mathbf{S}^f_h)u_n\rangle\,|}{\|w^h_n\|_{\partial V}}\big)\cdot \\ &\qquad \big(\inf_{\mathbf{f}^h\in V^h}\|\mathbf{T}_*\mathbf{v}-\mathbf{f}^h\|_V + \sup_{\mathbf{g}^h\in V^h}\frac{|\,\langle g^h_n,(\mathbf{S}^f-\mathbf{S}^f_h)v_n\rangle\,|}{\|g^h_n\|_{\partial V}}\big) \\ &\qquad + |\,\langle\mathbf{T}_{*h}v_n,(\mathbf{S}^f-\mathbf{S}^f_h)u_n)\rangle\,|\Big\}. \end{aligned} \tag{5.19}$$

For $\mathbf{v} \in H,\ v_n \in \partial Q'$, while for $\boldsymbol{v}^h \in V^h,\ v_n^h \in \partial V$, so by inequalities (5.19), (5.2), (5.4), Lemma 5.7, and the assumed regularity of $\mathbf{T}\boldsymbol{u} = \mathbf{T}_*\boldsymbol{u}$, we get

$$\begin{aligned}\|\mathbf{T}-\mathbf{T}_h\|_H &= \sup_{\substack{\mathbf{u}\in H\\ \|\mathbf{u}\|_H=1}} \sup_{\substack{\mathbf{v}\in H\\ \|\mathbf{v}\|_H=1}} r((\mathbf{T}-\mathbf{T}_h)\mathbf{u},\mathbf{v})\\ &\le C\{(h_s^{(r-1)}+h_f)\cdot(h_s^{(r-1)}+h_f)+h_f\}\\ &\to 0 \text{ as } (h_s,h_f)\to(0,0).\end{aligned}$$

□

Let m be the multiplicity of an eigenvalue λ^{-1} of $\mathbf{T}$. As a consequence of Lemma 5.8, exactly m eigenvalues of $\mathbf{T}_h$ converge to λ^{-1}; denote these by $\lambda_{1h}^{-1}, \lambda_{2h}^{-1}, \ldots, \lambda_{mh}^{-1}$. We are now in a position to prove

Theorem 5.9. *Let $(\lambda, \mathbf{u})$ be an eigenpair of (2.18) and let $(\lambda_h, \mathbf{u}_h)$ be an eigenpair of (4.14), then for sufficiently small h_f, h_s the eigenvalue error is*

$$|\ \lambda - \frac{1}{m}\sum_{i=1}^{m}\lambda_{ih}\ | \le C\max(h_s^{2(r-1)},\ h_f^2) \tag{5.20}$$

and the eigenfunction error is

$$\|\mathbf{u}-\mathbf{u}_h\|_H \le C\max(h_s^{2(r-1)},\ h_f^2). \tag{5.21}$$

Proof. Let $X = L_2(\Omega^s)^3 \times L_2(\Gamma)$. Then $H, X, V, V^h, a(\cdot,\cdot), r(\cdot,\cdot), r_h(\cdot,\cdot), \mathbf{T}, \mathbf{T}_*, \mathbf{T}_h$ and $\mathbf{T}_{*h}$ satisfy the hypotheses of the extension of Theorem 4.1 of Mercier, Osborn, Rappaz, and Raviart [12] described in the Appendix, and the eigenvalue error is given by

$$\begin{aligned}|\ \lambda - \frac{1}{m}\sum_{i=1}^{m}\lambda_{ih}\ | \le C\Big\{ &\sup_{\substack{\mathbf{u}\in E\\ \|\mathbf{u}\|_H=1}} \sup_{\substack{\mathbf{v}\in E_*\\ \|\mathbf{v}\|_H=1}} |\ r((\mathbf{T}-\mathbf{T}_h)\mathbf{u},\mathbf{v})\ |\\ &+ \left\|(\mathbf{T}-\mathbf{T}_h)_{/E}\right\|_H \cdot \left\|(\mathbf{T}-\mathbf{T}_h)_{/H}\right\|_H\Big\},\end{aligned} \tag{5.22}$$

where E is the space of generalized eigenfunctions of $\mathbf{T}$ corresponding to λ^{-1}, E_* is the space of generalized eigenfunctions of $\mathbf{T}_*$ corresponding to $\bar{\lambda}^{-1}$,

$$\left\|(\mathbf{T}-\mathbf{T}_h)_{/E}\right\|_H = \sup_{\substack{\mathbf{u}\in E\\ \|\mathbf{u}\|_H=1}} \|(\mathbf{T}-\mathbf{T}_h)\mathbf{u}\|_H$$

and

$$\left\|(\mathbf{T}-\mathbf{T}_h)_{/H}\right\|_H = \sup_{\substack{\mathbf{u}\in H\\ \|\mathbf{u}\|_H=1}} \|(\mathbf{T}-\mathbf{T}_h)\mathbf{u}\|_H\,.$$

For $\mathbf{v} \in E = E_*, v_n \in \partial V$, and evaluating the terms in (5.22) we obtain

$$\begin{aligned}\sup_{\substack{\mathbf{u}\in E\\ \|\mathbf{u}\|_H=1}} \sup_{\substack{\mathbf{v}\in E_*\\ \|\mathbf{v}\|_H=1}} |\ r((\mathbf{T}-\mathbf{T}_h)\mathbf{u},\mathbf{v})\ | &= C\{(h_s^{(r-1)}+h_f^2)\cdot(h_s^{(r-1)}+h_f^2)+h_f^2\}\\ &\le C\max(h_s^{2(r-1)},\ h_f^2),\end{aligned} \tag{5.23}$$

and

$$\left\|(\mathbf{T}-\mathbf{T}_h)_{/E}\right\|_H = \sup_{\substack{\mathbf{u}\in E\\ \|\mathbf{u}\|_H=1}} \sup_{\substack{\mathbf{v}\in H\\ \|\mathbf{v}\|_H=1}} |\, r((\mathbf{T}-\mathbf{T}_h)\mathbf{u},\mathbf{v})\,| \le C\max(h_s^{2(r-1)},\ h_f^2). \tag{5.24}$$

The last term in (5.22) was evaluated in Lemma 5.8

$$\left\|(\mathbf{T}-\mathbf{T}_h)_{/H}\right\|_H \le C\max(h_s^{2(r-1)}, h_f), \tag{5.25}$$

so combining (5.23), (5.24) and (5.25) we obtain (5.20).

For the eigenfunction error, Theorem 1 in [15] is directly applicable, and

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_H &\le \left\|(\mathbf{T}-\mathbf{T}_h)_{/E}\right\|_H\\ &\le C\max(h_s^{2(r-1)},\ h_f^2),\end{aligned} \tag{5.26}$$

which proves (5.21). □

In conclusion, we have demonstrated that using fluid elements of the same mesh diameter as the solid elements yields a convergence rate no worse than would be expected in a "dry" structure. Note that this rate is limited by the smoothness of the eigenfunctions in the solid, and localized refinement of the solid mesh could be used to improve the accuracy of the solution. Although the method and analysis are applicable to incompatible fluid and solid meshes of arbitrary interpolation order, this result shows that no improvement in the rate of convergence would be expected from the use of higher order elements in either solid or fluid. The error estimate is optimal in the sense that it is the best that can be achieved with piecewise linear finite elements.

Appendix

An extension of the proof of Theorem 4.1 in reference [12] is required to produce an error estimate result applicable to the nonconforming approximation (5.9) of the variational eigenvalue problem (2.18). This extension is now described. In what follows, equation numbers denoted by [12]($x.x$), refer to those in reference [12]. In equations [12](4.2), [12](4.5), and [12](4.7), replace $r(\cdot,\cdot)$ with $r_h(\cdot,\cdot)$, where $r_h(\cdot,\cdot)$ is a strongly coercive bilinear form on H, and assume that $r(\cdot,\cdot)$ is an inner product on H. By the Lax-Milgram lemma, there exist invertible operators $B_h, B_{*h} : H \to H$ defined by $r_h(u,v) = r(B_h u, v) = r(u, B_{*h} v)$ for all u, v in H. The finite dimensional operator T_{*h} defined by [12](4.7) is then the $r_h(\cdot,\cdot)$-adjoint of T_h defined by [12](4.6), and similarly for the associated spectral projections P_{*h}, P_h. Since $r(\cdot,\cdot)$ is an inner product, the H-adjoint of P_h, denoted by P_h^*, is related to the $r_h(\cdot,\cdot)$-adjoint P_{*h} by

$$r(P_h u, v) = r(u, P_h^* v) = r(u, B_h^*\ P_{*h}\ (B_h^*)^{-1} v)$$

and similarly for T_h, T_h^* and T_{*h}. With these terms defined, [12](4.14) and [12](4.15) are unchanged, but P_h^* replaces P_{*h} in [12](4.16), and from [12](4.13), [12](4.14),

[12](4.15), and the modified [12](4.16), we have

$$| \lambda - \frac{1}{m}\sum_{i=1}^{m}\lambda_{ih} | \leq C \sup_{\substack{u\in E\\ \|u\|_H=1}} \sup_{\substack{v\in E_*\\ \|v\|_H=1}} \Big\{ | r((T-T_h)u,v) | + \|(T-T_h)u\|_H \cdot \|(P_*-P_h^*)v\|_H \Big\}.$$

Now $\|(P_*-P_h^*)v\|_H \leq C\,\|(T_*-T_h^*)v\|_H\,\forall v \in E_*$, $E_* \subset H$, T_*, T_h^* are the H-adjoints of T, T_h, respectively, and $r(\cdot,\cdot)$ is an inner product on H, so

$$\sup_{\substack{v\in E_*\\ \|v\|_H=1}} \|(P_*-P_h^*)v\|_H \leq C \sup_{\substack{u\in H\\ \|u\|_H=1}} \sup_{\substack{v\in H\\ \|v\|_H=1}} |r((T-T_h)u,v)| = C\left\|(T-T_h)_{/H}\right\|_H,$$

and the eigenvalue error estimate of Theorem 5.9 in the present work follows.

References

1. Jean-Pierre Aubin, *Approximation of Elliptic Boundary Value Problems*, Wiley-Interscience, 1972 MR **57:**18139
2. V. I. Agoshov, *Poincaré-Steklov's Operators and Domain Decomposition Methods in Finite Dimensional Spaces* in Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Roland Glowinski, Gene H. Golub, Gerard A. Meurant, Jacques Periaux, Ed.) SIAM, Philadelphia, 1988, 73-112
3. H. Berger, J. Boujot and R. Ohayon, *On a Spectral Problem in Vibration Mechanics: Computation of Elastic Tanks Partially Filled with Liquids*, Journal of Mathematical Analysis and Applications, 51, (1975), 272-298 MR **52:**7310
4. F. Bourquin, *Component mode synthesis and numerical simulation of the added mass* in Computational methods for fluid structure interaction, (J. M. Crolet and R. Ohayon, Ed.), Harlow, Essex, Eng,:Longman Scientific & Technical, New York, Wiley, 1994, 183-197 MR **97h:**73067
5. J. H. Bramble and J. E. Osborn, *Approximation of Steklov Eigenvalues of Nonselfadjoint Second Order Elliptic Operators*, in The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations (A. K. Aziz, Ed.), Academic Press, New York, 1972, 387-408 MR **55:**4735
6. Jaqueline Boujot, *Mathematical Formulation of Fluid-Structure Interaction Problems*, R.A.I.R.O. Model. Math. Anal. Numer. 21 (1987) 239-260 MR **88j:**73032
7. Susanne C. Brenner, L. Ridgway Scott, *The Mathematical Theory of Finite Elements*, Springer-Verlag, New York, 1991
8. Robert N. Coppolino, *A Numerically Efficient Finite Element Hydroelastic Analysis*, Volume 1, Theory and Results, NASA CR-2662, 1976
9. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, 1985 MR **86m:**35044
10. V. Girault and P. A. Raviart, *Finite Element Methods for the Navier-Stokes Equations*, Springer-Verlag, 1986, 54-56 MR **88b:**65129
11. P. G. Ciarlet, *Basic Error Estimates for Elliptic Problems* in the Handbook of Numerical Analysis, Volume II, Finite Element Methods (Part 1) (P. G. Ciarlet and J. L. Lions, Ed.), North Holland, 1991, 17-351 CMP 91:14
12. B. Mercier, J. Osborn, J. Rappaz, and P. A. Raviart, *Eigenvalue Approximation by Mixed and Hybrid Methods*, Math. Comp.,36 (1981) 427-453. MR **82b:**65108
13. H. J.-P. Morand and R. Ohayon, *Fluid-Structure Interactions*, John Wiley & Sons, 1992 (French) MR **93m:**73031
14. J. Tinsley Oden and Leszek F. Demkowicz, *Applied Functional Analysis*, CRC Press, 1996
15. John E. Osborn, *Spectral Approximation of Compact Operators*, Math. Comp., 29, (1975), 712-725 MR **52:**3998
16. L. Ridgway Scott and Shangyou Zhang, *Finite Element Interpolation of Nonsmooth Functions Satisfying Boundary Conditions*, Math. Comp., 54, (1990), 483-493 MR **90j:**65021

Lockheed Martin Missiles and Space, Sunnyvale, California
E-mail address: pat.ryan@lmco.com