

ON THE STABILITY OF THE L^2 PROJECTION IN $H^1(\Omega)$

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ABSTRACT. We prove the stability in $H^1(\Omega)$ of the L^2 projection onto a family of finite element spaces of conforming piecewise linear functions satisfying certain local mesh conditions. We give explicit formulae to check these conditions for a given finite element mesh in any number of spatial dimensions. In particular, stability of the L^2 projection in $H^1(\Omega)$ holds for locally quasiuniform geometrically refined meshes as long as the volume of neighboring elements does not change too drastically.

1. INTRODUCTION

Let $\{\varphi_k\}_{k=1}^M$ denote the nodal basis for a piecewise linear (continuous) finite element approximation space V_h based on a conforming triangulation (of simplices) $\{\tau_l\}_{l=1}^N$ of a polyhedral domain Ω in \mathbf{R}^n , $n = 1, 2, \dots$. We shall always assume that the triangulation is locally (but not globally) quasiuniform. The L^2 projection Q_h of a given function u onto the finite element space V_h is defined by

$$(1.1) \quad \langle Q_h u, v^h \rangle_{L^2(\Omega)} = \langle u, v^h \rangle_{L^2(\Omega)} \quad \text{for all } v^h \in V_h.$$

Here $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ denotes the inner product on $L^2(\Omega)$. The L^2 projection is obviously bounded on $L^2(\Omega)$; indeed, $\|Q_h u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$ holds for all $u \in L^2(\Omega)$. In this paper we are concerned with the stability of the L^2 projection as a map $Q_h : H^1(\Omega) \rightarrow V_h \subset H^1(\Omega)$. In particular, we will prove the stability estimate

$$(1.2) \quad \|Q_h u\|_{H^1(\Omega)} \leq c \cdot \|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega)$$

under certain conditions on the finite element trial space V_h , specifically, on the underlying triangulation.

The stability of Q_h in $H^1(\Omega)$ is of general interest, in particular, for Galerkin finite or boundary element methods for elliptic and parabolic boundary value problems [4, 6, 13]. For example, the stability estimate (1.2) is needed to analyze the properties of a Neumann series corresponding to a second kind boundary integral equation, see [9].

Using interpolation arguments we get

$$(1.3) \quad \|Q_h u\|_{H^s(\Omega)} \leq c \cdot \|u\|_{H^s(\Omega)} \quad \text{for all } u \in H^s(\Omega), \quad s \in [0, 1].$$

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From (1.3) we can conclude the stability estimate (see for example [7])

$$(1.4) \quad c \cdot \|u^h\|_{H^s(\Omega)} \leq \sup_{v^h \in V_h} \frac{|\langle u^h, v^h \rangle_{L^2(\Omega)}|}{\|v^h\|_{\tilde{H}^{-s}(\Omega)}} \quad \text{for all } u^h \in V_h,$$

where $\tilde{H}^{-s}(\Omega) = (H^s(\Omega))'$ is defined by duality with respect to the L^2 inner product. The estimate (1.4) is essentially needed in the derivation of hybrid coupled finite element domain decomposition methods [1] and for hybrid boundary element methods [11] as well as in the construction of efficient preconditioners in finite and boundary element methods [12].

For globally quasiuniform triangulations, the estimate (1.2) is a direct consequence of a global inverse inequality. In [6], (1.2) was shown for nonuniform triangulations in one and two dimensions satisfying certain mesh conditions. The analysis is based on decay properties of the $L^2(\Omega)$ projection and results in conditions which depend on the global behavior of the mesh.

In this paper we prove the stability estimate (1.2) for arbitrary $n = 1, 2, \dots$ provided that local stability conditions are satisfied. This approach is valid for more general trial spaces, in particular, for trial functions of arbitrary polynomial degree, but for simplicity, we only consider the case of piecewise linear basis functions. In this case, we formulate explicit local mesh conditions which imply (1.2). These conditions can be easily checked for a given finite element mesh, allowing the user to redefine the mesh if necessary.

The remainder of this paper is organized as follows. Some preliminary notation is given in Section 2. In Section 3 we recall from [5, 8] the definition as well as some error and stability estimates for a quasi interpolation operator needed in our analysis. Our main result is formulated in Theorem 4.1. The proof is based on a general stability condition and several technical results given in Section 5. In Section 6 we discuss the stability condition in case of piecewise linear basis functions. Based on the eigenvalue analysis of locally defined weighted Gram matrices, we derive computable criteria for guaranteeing that the stability condition is satisfied.

2. NOTATION

Let

$$(2.1) \quad \mathcal{T}_h = \{\tau_i\}_{i=1}^N, \quad \Delta_l := \int_{\tau_l} dx \quad \text{for } l = 1, \dots, N.$$

As usual, we consider a family of meshes depending on h , the maximum diameter of any simplex. We assume that the triangulations are locally quasiuniform. This means that the diameter of the simplex divided by the diameter of the largest ball contained in the simplex is bounded independently of h for all simplices in all triangulations. Define

$$(2.2) \quad h_l := \Delta_l^{1/n} \quad \text{for } l = 1, \dots, N.$$

Let $\{x_k\}$ the set of all nodes of the mesh \mathcal{T}_h , where x_k is associated to the basis function φ_k ($\varphi_k(x_k) = 1$). We define

$$(2.3) \quad \omega_k := \text{supp } \varphi_k, \quad k = 1, \dots, M.$$

Let $I(k)$ denote the index set of all elements τ_l satisfying $\tau_l \subset \omega_k$. Then we define a local mesh size associated to the basis function φ_k by

$$(2.4) \quad \hat{h}_k := \frac{1}{\#I(k)} \sum_{l \in I(k)} h_l \quad \text{for } k = 1, \dots, M.$$

Here and in the rest of the paper, $\#$ denotes cardinality. Since the mesh \mathcal{T}_h is assumed to be locally quasiuniform, there exists a positive constant $\gamma \geq 1$ not depending on h such that

$$(2.5) \quad \gamma^{-1} \leq \frac{\hat{h}_k}{h_l} \leq \gamma \quad \text{for all } l \in I(k), \quad k = 1, \dots, M.$$

We define $J(l)$ to be the set of indices of the vertices in τ_l . Note that an inverse inequality holds locally [4], i.e.,

$$(2.6) \quad \|v^h\|_{H^1(\tau_l)} \leq c \cdot h_l^{-1} \cdot \|v^h\|_{L^2(\tau_l)} \quad \text{for all } v^h \in V_h, \quad l = 1, \dots, N.$$

Here and in the remainder of the paper, we use c with or without subscript to denote a generic positive constant which is independent of h .

3. QUASI INTERPOLATION

To prove the stability estimate (1.2) we need to use a projection operator P_h which is stable in $H^1(\Omega)$ and which satisfies local error estimates in $L^2(\tau_l)$ valid on all finite elements τ_l for $l = 1, \dots, N$. For this we will use the concept of quasi interpolation operators first introduced by Clement in [5]; see also [8].

We define local trial spaces of piecewise linear continuous functions by

$$(3.1) \quad V_h^k := \{v|_{\omega_k} : v \in V_h\}.$$

Let Q_h^k denote the L^2 projection onto V_h^k . We clearly have

$$(3.2) \quad \begin{aligned} \|Q_h^k u\|_{L^2(\omega_k)} &\leq \|u\|_{L^2(\omega_k)}, \\ \|(I - Q_h^k)u\|_{L^2(\omega_k)} &\leq c \cdot \hat{h}_k \cdot |u|_{H^1(\omega_k)}. \end{aligned}$$

Moreover, since the mesh is assumed to be locally quasiuniform, we have (see, e.g., [3])

$$(3.3) \quad \|Q_h^k u\|_{H^1(\omega_k)} \leq c \cdot \|u\|_{H^1(\omega_k)} \quad \text{for all } u \in H^1(\omega_k),$$

for $k = 1, \dots, M$.

Now we define a quasi interpolation operator by

$$(3.4) \quad (P_h u)(x) = \sum_{k=1}^M (Q_h^k u)(x_k) \cdot \varphi_k(x).$$

It is easy to check that P_h is a projection. Moreover, P_h is stable in $H^1(\Omega)$ and satisfies some local error estimates as asserted in the following lemma.

Lemma 3.1. *Let u be in $H^1(\Omega)$. There exists a positive constant c independent of h such that*

$$(3.5) \quad \|(I - P_h)u\|_{L^2(\tau_l)} \leq c \cdot \sum_{k \in J(l)} \hat{h}_k \cdot |u|_{H^1(\omega_k)} \quad \text{for } l = 1, \dots, N.$$

Moreover,

$$(3.6) \quad \|P_h u\|_{H^1(\Omega)} \leq c \cdot \|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega).$$

Proof. The proof follows the general ideas already given in [5]. Let τ_l be an arbitrary but fixed finite element and let $\tilde{k} \in J(l)$ be a fixed index. For $x \in \tau_l$ we have the representation

$$(P_h u)(x) = (Q_h^{\tilde{k}} u)(x) + \sum_{k \in J(l), k \neq \tilde{k}} \left[(Q_h^k u)(x_k) - (Q_h^{\tilde{k}} u)(x_k) \right] \varphi_k(x).$$

Let $s = 0, 1$. Note that

$$\|\varphi_k\|_{H^s(\tau_l)} \leq c \cdot h_l^{n/2-s}.$$

Then, using (3.2) and (3.3), it follows that

$$\begin{aligned} \|(I - P_h)u\|_{H^s(\tau_l)} &\leq c_1 \cdot \hat{h}_k^{1-s} \cdot |u|_{H^1(\omega_{\tilde{k}})} \\ &\quad + c_2 \cdot h_l^{n/2-s} \sum_{k \in J(l), k \neq \tilde{k}} |(Q_h^k u)(x_k) - (Q_h^{\tilde{k}} u)(x_k)| \end{aligned}$$

Now

$$\|v^h\|_{L^\infty(\tau_l)} \leq c \cdot h_l^{-n/2} \cdot \|v^h\|_{L^2(\tau_l)} \quad \text{for all } v^h \in V_h, \quad l = 1, \dots, N.$$

Thus, (3.2) gives, for $x_k \in \tau_l$,

$$\begin{aligned} |(Q_h^k u)(x_k) - (Q_h^{\tilde{k}} u)(x_k)| &\leq \|Q_h^k u - Q_h^{\tilde{k}} u\|_{L^\infty(\tau_l)} \\ &\leq c \cdot h_l^{-n/2} \cdot \|Q_h^k u - Q_h^{\tilde{k}} u\|_{L^2(\tau_l)} \\ &\leq c \cdot h_l^{-n/2+1} \cdot \{|u|_{H^1(\omega_k)} + |u|_{H^1(\omega_{\tilde{k}})}\}. \end{aligned}$$

Hence,

$$\|(I - P_h)u\|_{H^s(\tau_l)} \leq c \cdot \sum_{k \in J(l)} \hat{h}_k^{1-s} \cdot |u|_{H^1(\omega_k)}$$

for $s = 0, 1$ and $l = 1, \dots, N$. Using this estimate for $s = 0$ gives (3.5), while for $s = 1$ we get (3.6) by summing over all elements. \square

4. MAIN RESULTS

In this section, we will formulate and prove the main result of this paper, the stability estimate (1.2) assuming some appropriate mesh conditions. For this, we define local weights

$$(4.1) \quad \gamma_k := \sqrt{\sum_{l \in I(k)} h_l^{-2} \cdot \|\varphi_k\|_{L^2(\tau_l)}^2} \quad \text{for } k = 1, \dots, M.$$

In addition, for each element τ_l , we define local matrices \mathbf{G}_l , \mathbf{D}_l and \mathbf{H}_l by

$$\mathbf{G}_l[j, i] = \langle \varphi_i^l, \varphi_j^l \rangle_{L^2(\tau_l)}, \quad \mathbf{D}_l = \text{diag} \left(\|\varphi_i^l\|_{L^2(\tau_l)}^2 \right), \quad \mathbf{H}_l = \text{diag} \left(\hat{h}_i^l \right),$$

for $i, j = 1, 2, 3$. Here φ_i^l and φ_j^l are the basis functions corresponding to the i th and j th vertex of the l th element, respectively, while \hat{h}_i^l is the related value of \hat{h} .

Now we are able to formulate local conditions to be used in the remainder of this section, specifically,

$$(4.2) \quad (\mathbf{H}_l^{-1} \mathbf{G}_l \mathbf{H}_l \underline{x}^l, \underline{x}^l) \geq c_0 \cdot (\mathbf{D}_l \underline{x}^l, \underline{x}^l) \quad \text{for all } \underline{x}^l \in \mathbf{R}^{\#J(l)}.$$

Here (\cdot, \cdot) denotes the inner product on $\mathbf{R}^{\#J(l)}$.

Remark 4.1. Let H denote the diagonal matrix,

$$H = \text{diag}(\hat{h}_k).$$

For $\underline{u} \in \mathbf{R}^M$, define $\underline{v} = H\underline{u}$ and $\underline{w} = H^{-1}\underline{u}$. The corresponding finite element functions are given by

$$u^h = \sum_{k=1}^M \underline{u}_k \varphi_k, \quad v^h = \sum_{k=1}^M \underline{v}_k \varphi_k, \quad w^h = \sum_{k=1}^M \underline{w}_k \varphi_k.$$

We note that $(\mathbf{H}_l^{-1} \mathbf{G}_l \mathbf{H}_l \underline{u}^l, \underline{u}^l) = \langle v^h, w^h \rangle_{L^2(\tau_l)}$, where \underline{u}^l denotes the components of \underline{u} associated with the element τ_l . By local quasiuniformity, (4.2) is thus equivalent to

$$(4.3) \quad \|u^h\|_{L^2(\tau_l)}^2 \leq c \langle v^h, w^h \rangle_{L^2(\tau_l)}.$$

If all of the simplices have the same measure then $\mathbf{H}_l^{-1} \mathbf{G}_l \mathbf{H}_l = \mathbf{G}_l$, so

$$\langle v^h, w^h \rangle_{L^2(\tau_l)} = \langle u^h, u^h \rangle_{L^2(\tau_l)}$$

and (4.3) is trivial. The inequality will still hold provided that the measures of neighboring simplices do not vary too much. Explicit local conditions on the mesh for the case of piecewise linear elements are given in Section 6.

The following result is the main theorem of this paper.

Theorem 4.1. *Let condition (4.2) be satisfied. Then the L^2 projection $Q_h : H^1(\Omega) \rightarrow V_h \subset H^1(\Omega)$ is stable. In particular, there exists a positive constant c independent of h such that*

$$(4.4) \quad \|Q_h v\|_{H^1(\Omega)} \leq c \cdot \|v\|_{H^1(\Omega)} \quad \text{for all } v \in H^1(\Omega).$$

The proof of this theorem depends on the following lemma. A similar estimate was used in [2] to construct spectrally equivalent multilevel preconditioners in finite element methods in the case of globally quasiuniform meshes. The proof of the lemma will be given in the next section.

Lemma 4.1. *Let condition (4.2) be satisfied. Then there exists a positive constant c such that*

$$(4.5) \quad \sum_{l=1}^N h_l^{-2} \cdot \|v^h\|_{L^2(\tau_l)}^2 \leq c \cdot \sum_{k=1}^M \left[\frac{\langle v^h, \varphi_k \rangle_{L^2(\Omega)}}{\hat{h}_k \cdot \|\varphi_k\|_{L^2(\Omega)}} \right]^2$$

for all $v^h \in V_h$.

Proof of Theorem 4.1. Using the triangle inequality, (3.6), and (2.6), we get

$$\begin{aligned} \|Q_h v\|_{H^1(\Omega)}^2 &\leq 2 \cdot \left\{ \|P_h v\|_{H^1(\Omega)}^2 + \sum_{l=1}^N \|(Q_h - P_h)v\|_{H^1(\tau_l)}^2 \right\} \\ &\leq c \cdot \left\{ \|v\|_{H^1(\Omega)}^2 + \sum_{l=1}^N h_l^{-2} \cdot \|(Q_h - P_h)v\|_{L^2(\tau_l)}^2 \right\}. \end{aligned}$$

From Lemma 4.1 and the Schwarz inequality it follows that

$$\begin{aligned} \sum_{l=1}^N h_l^{-2} \cdot \|(Q_h - P_h)v\|_{L^2(\tau_l)}^2 &\leq c \cdot \sum_{k=1}^M \left[\frac{\langle (Q_h - P_h)v, \varphi_k \rangle_{L^2(\Omega)}}{\hat{h}_k \cdot \|\varphi_k\|_{L^2(\Omega)}} \right]^2 \\ &= c \cdot \sum_{k=1}^M \left[\frac{\langle (I - P_h)v, \varphi_k \rangle_{L^2(\omega_k)}}{\hat{h}_k \cdot \|\varphi_k\|_{L^2(\Omega)}} \right]^2 \\ &\leq c \cdot \sum_{k=1}^M \hat{h}_k^{-2} \cdot \|(I - P_h)v\|_{L^2(\omega_k)}^2 \end{aligned}$$

Hence, the assertion follows from (3.5). \square

5. PROOF OF LEMMA 4.1

In this section, we prove Lemma 4.1. We start by introducing some notation. We define two additional $M \times M$ diagonal matrices given by

$$(5.6) \quad D_\gamma = \text{diag}(\gamma_k), \quad D_\varphi = \text{diag}(\hat{h}_k \cdot \|\varphi_k\|_{L^2(\Omega)}),$$

where \hat{h}_k and γ_k are defined as in (2.4) and (4.1), respectively. The first step in the proof of Lemma 4.1 involves estimating the inverse of a scaled Gram matrix.

Lemma 5.1. *Let assumption (4.2) be satisfied. Then there exists a positive constant c such that*

$$\|\underline{x}\|_2 \leq c \cdot \|A\underline{x}\|_2 \quad \text{for all } \underline{x} \in \mathbf{R}^M,$$

where A is the scaled Gram matrix defined by

$$(5.7) \quad A = D_\varphi^{-1} G D_\gamma^{-1}$$

and G is the Gram matrix $G_{ij} = \langle \varphi_i, \varphi_j \rangle_{L^2(\Omega)}$.

Proof. Let $\underline{u}, \underline{u}^l, \underline{v}, \underline{w}, u^h, v^h$ and w^h be as in Remark 4.1. Setting $\tilde{G} = H^{-1}GH$ and using (4.2) gives

$$\begin{aligned} (\tilde{G}\underline{u}, \underline{u}) &= (G\underline{v}, \underline{w}) = \langle v^h, w^h \rangle_{L^2(\Omega)} = \sum_{l=1}^N \langle v^h, w^h \rangle_{L^2(\tau_l)} \\ &= \sum_{l=1}^N (\mathbf{H}_l^{-1} \mathbf{G}_l \mathbf{H}_l \underline{u}^l, \underline{u}^l) \geq c_0 \cdot \sum_{l=1}^N (\mathbf{D}_l \underline{u}^l, \underline{u}^l) = c_0 \cdot (D\underline{u}, \underline{u}). \end{aligned}$$

Here D is the diagonal matrix with entries $\|\varphi_k\|_{L^2(\Omega)}^2$. Let $D^{1/2} = \text{diag}(\|\varphi_k\|_{L^2(\Omega)})$. From

$$\begin{aligned} c_0 \cdot \|D^{1/2}\underline{u}\|_2^2 &= c_0 \cdot (D\underline{u}, \underline{u}) \leq (\tilde{G}\underline{u}, \underline{u}) \\ &= (D^{-1/2}\tilde{G}\underline{u}, D^{1/2}\underline{u}) \leq \|D^{-1/2}\tilde{G}\underline{u}\|_2 \|D^{1/2}\underline{u}\|_2, \end{aligned}$$

we conclude that

$$c_0 \cdot \|D^{1/2}\underline{u}\|_2 \leq \|D^{-1/2}\tilde{G}\underline{u}\|_2 \quad \text{for all } \underline{u} \in \mathbf{R}^M.$$

Taking $\tilde{\underline{u}} = D_\gamma \underline{u}$ gives

$$c_0 \cdot \|D^{1/2} D_\gamma^{-1} \tilde{\underline{u}}\|_2 \leq \|D^{-1/2} D_\varphi D_\gamma^{-1} \tilde{G} D_\gamma^{-1} \tilde{\underline{u}}\|_2 = \|D^{-1/2} D_\varphi \tilde{A} \tilde{\underline{u}}\|_2,$$

where $\tilde{A} = D_\varphi^{-1} \tilde{G} D_\gamma^{-1}$. The ratio of the diagonal entries satisfies

$$\frac{D^{1/2}[k, k]}{D_\gamma[k, k]} = \frac{\|\varphi_k\|_{L^2(\Omega)}}{\sqrt{\sum_{l \in I(k)} h_l^{-2} \|\varphi_k\|_{L^2(\tau_l)}^2}} \geq c \cdot \hat{h}_k$$

and

$$\frac{D_\varphi[k, k]}{D^{1/2}[k, k]} = \frac{\hat{h}_k \cdot \|\varphi_k\|_{L^2(\Omega)}}{\|\varphi_k\|_{L^2(\Omega)}} = \hat{h}_k$$

for all $k = 1, \dots, M$. Thus,

$$c \cdot \|H\tilde{\mathbf{u}}\|_2 \leq \|H\tilde{A}\tilde{\mathbf{u}}\|_2 \quad \text{for all } \tilde{\mathbf{u}} \in \mathbf{R}^M.$$

Taking $\underline{\mathbf{x}} = H\tilde{\mathbf{u}}$ above gives

$$c \cdot \|\underline{\mathbf{x}}\|_2 \leq \|H\tilde{A}H^{-1}\underline{\mathbf{x}}\|_2 = \|HD_\varphi^{-1}H^{-1}GH D_\gamma^{-1}H^{-1}\underline{\mathbf{x}}\|_2 = \|A\underline{\mathbf{x}}\|_2$$

for all $\underline{\mathbf{x}} \in \mathbf{R}^M$. This completes the proof of the lemma. \square

We now give the proof of Lemma 4.1.

Proof of Lemma 4.1. Let $\underline{\mathbf{v}} \in \mathbf{R}^M$ and set $v^h = \sum_{k=1}^M \underline{v}_k \varphi_k \in V_h$. Then, the left hand side of (4.5) is bounded by

$$\begin{aligned} \sum_{l=1}^N h_l^{-2} \cdot \|v^h\|_{L^2(\tau_l)}^2 &\leq c \cdot \sum_{l=1}^N h_l^{-2} \sum_{k \in J(l)} \underline{v}_k^2 \cdot \|\varphi_k\|_{L^2(\tau_l)}^2 \\ &= c \cdot \sum_{k=1}^M \underline{v}_k^2 \sum_{l \in I(k)} h_l^{-2} \cdot \|\varphi_k\|_{L^2(\tau_l)}^2 \\ &= c \cdot \sum_{k=1}^M \underline{v}_k^2 \gamma_k^2 = c \cdot \sum_{k=1}^M \underline{x}_k^2 = c \cdot \|\underline{\mathbf{x}}\|_2^2, \end{aligned}$$

where $\underline{x}_k = \gamma_k \underline{v}_k$. The right hand side in (4.5) is

$$\begin{aligned} \sum_{k=1}^M \left[\frac{\langle v^h, \varphi_k \rangle_{L^2(\Omega)}}{\hat{h}_k \|\varphi_k\|_{L^2(\Omega)}} \right]^2 &= \sum_{k=1}^M \left[\sum_{j=1}^M \underline{v}_j \cdot \frac{\langle \varphi_j, \varphi_k \rangle_{L^2(\Omega)}}{\hat{h}_k \|\varphi_k\|_{L^2(\Omega)}} \right]^2 \\ &= \sum_{k=1}^M \left[\sum_{j=1}^M \underline{x}_j \cdot \frac{\langle \varphi_j, \varphi_k \rangle_{L^2(\Omega)}}{\gamma_j \hat{h}_k \|\varphi_k\|_{L^2(\Omega)}} \right]^2 \\ &= \sum_{k=1}^M [(A\underline{\mathbf{x}})_k]^2 = \|A\underline{\mathbf{x}}\|_2^2, \end{aligned}$$

using the matrix definition (5.7). Hence, (4.5) follows from Lemma 5.1. \square

Although we only considered the case of piecewise linear basis functions, the same approach may be used for higher order piecewise polynomial finite element spaces. In addition, the case of $V_h \subset H_0^1(\Omega)$ with basis functions vanishing along the boundary $\partial\Omega$ can be treated with only slight modifications. In this case we consider the index set $I(k)$ only for nodes x_k associated with a basis function $\varphi_k \in V_h$. Then all proofs given above apply. Note that the dimension of the local

matrices for boundary simplices decrease since the rows and columns corresponding to boundary nodes do not appear.

6. FINITE ELEMENT SPACES

The stability estimate in Theorem 4.1 is based on the condition (4.2). If we define the symmetric matrix

$$(6.1) \quad \mathbf{G}_l^S := \frac{1}{2} [\mathbf{H}_l \mathbf{G}_l \mathbf{H}_l^{-1} + \mathbf{H}_l^{-1} \mathbf{G}_l \mathbf{H}_l],$$

then (4.2) is the same as

$$(6.2) \quad (\mathbf{G}_l^S \underline{x}_l, \underline{x}_l) \geq c_0 \cdot (\mathbf{D}_l \underline{x}_l, \underline{x}_l) \quad \text{for all } \underline{x}_l \in \mathbf{R}^{\#J(l)}.$$

Let τ_l in \mathcal{T}_h be an arbitrary element. Note that $\#J(l) = n + 1$. A simple computation shows that

$$(6.3) \quad \mathbf{D}_l = d_n \cdot \Delta_l \cdot I \quad \text{with } d_n = \frac{2}{(n+1)(n+2)},$$

where I is the identity matrix in $n + 1$ dimensions. Moreover,

$$(6.4) \quad \mathbf{G}_l^S = \frac{1}{4} \cdot d_n \cdot \Delta_l \cdot \mathbf{A}_l,$$

where the matrix \mathbf{A}_l is defined by

$$\mathbf{A}_l[i, j] = \begin{cases} 4 & \text{for } i = j, \\ \frac{\hat{h}_i^l}{\hat{h}_j^l} + \frac{\hat{h}_j^l}{\hat{h}_i^l} & \text{for } i \neq j, \end{cases} \quad i, j = 1, \dots, n + 1.$$

Here \hat{h}_j^l is the value of \hat{h} corresponding to the j th vertex of the l th element. Hence, to show (6.2) it is sufficient to consider the eigenvalues $\{\lambda_i\}$ of the matrix \mathbf{A}_l . To this end, we give the following proposition.

Proposition 6.1. *Let $\alpha_1, \dots, \alpha_{n+1}$ be real numbers with $\alpha_i \neq 0$ for $i = 1, \dots, n + 1$, and consider the matrix*

$$\mathbf{A}[i, j] = \begin{cases} 4 & \text{for } i = j, \\ \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} & \text{for } i \neq j, \end{cases} \quad i, j = 1, \dots, n + 1.$$

Then, all eigenvalues of \mathbf{A} are in the set $\{\lambda_+, \lambda_-, 2\}$, where

$$(6.5) \quad \lambda_{\pm} = 3 + n \pm \sqrt{\sum_{i=1}^{n+1} \alpha_i^2 \cdot \sum_{i=1}^{n+1} \alpha_i^{-2}}.$$

Proof. We first consider the case when $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ are not all equal. Let $\mathbf{M}_+ = (\alpha_1, \alpha_2, \dots, \alpha_{n+1})^t$ and $\mathbf{M}_- = (\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_{n+1}^{-1})^t$. Then $\mathbf{A} = 2I + \mathbf{N}$, where $\mathbf{N} = \mathbf{M}_- \cdot \mathbf{M}_+^t + \mathbf{M}_+ \cdot \mathbf{M}_-^t$. Note that the matrix \mathbf{N} is symmetric with range equal to the two dimensional subspace spanned by \mathbf{M}_+ and \mathbf{M}_- . Thus, zero is an eigenvalue of \mathbf{N} with multiplicity $n - 1$. We need only compute the two remaining eigenvalues. By expanding the corresponding eigenvectors in the basis $\{\mathbf{M}_+, \mathbf{M}_-\}$,

it is elementary to see that the remaining two eigenvalues of \mathbf{N} are eigenvalues of the matrix

$$\begin{pmatrix} n+1 & \sum_{i=1}^{n+1} \alpha_i^2 \\ \sum_{i=1}^{n+1} \alpha_i^{-2} & n+1 \end{pmatrix}$$

The proposition immediately follows for this case.

If $\alpha_1 = \alpha_2 = \dots = \alpha_{n+1}$ then \mathbf{M}_+ and \mathbf{M}_- are linearly dependent, so 2 is an eigenvalue of \mathbf{A} with multiplicity n . The remaining eigenvalue is $\lambda_+ = 2n+4$. This completes the proof of the proposition. \square

It follows from (6.4) that the local condition (6.2) is satisfied if all eigenvalues of \mathbf{A}_l are strictly positive and bounded away from zero. By the proposition, this is equivalent to

$$(6.6) \quad 3+n - \sqrt{\sum_{i=1}^{n+1} (\hat{h}_i^l)^{-2} \cdot \sum_{i=1}^{n+1} (\hat{h}_i^l)^2} \geq c$$

with c independent of τ_l . Inequality (6.6) provides a mesh constraint for adaptive triangulations.

Remark 6.1. We can satisfy (6.6) for any $0 < c < 2$ provided that we design a mesh which gives rise to a γ (in (2.5)) sufficiently close to one. It immediately follows from (2.5) that

$$3+n - \sqrt{\sum_{i=1}^{n+1} (\hat{h}_i^l)^{-2} \cdot \sum_{i=1}^{n+1} (\hat{h}_i^l)^2} \geq 2 - (n+1)(\gamma^2 - 1).$$

Thus (6.6) holds if

$$\gamma \leq \sqrt{1 + \frac{2-c}{n+1}}.$$

Thus, a mesh gives rise to an $H^1(\Omega)$ stable L^2 projection provided that the change in measures of neighboring simplices are controlled.

Finally, if a finite element mesh \mathcal{T}_h is given, the mesh condition (6.6) and therefore the stability assumption (4.2) can be checked explicitly (by direct computation). To illustrate the applicability of the mesh condition (6.6), we consider an adaptive finite element mesh for $n = 2$ as shown in Figure 1, generated by an adaptive algorithm described in [10]. In Table 1, we give values for

$$c = \max_{\tau_l} \left(5 - \sqrt{\sum_{i=1}^3 (\hat{h}_i^l)^{-2} \cdot \sum_{i=1}^3 (\hat{h}_i^l)^2} \right)$$

as a function of the refinement level L and the number of finite element nodes M .

TABLE 1. Computational results for c

L	0	1	2	3	4	5	6	7	8	9
M	8	17	28	53	87	155	291	532	1034	2003
c	2.00	1.93	1.59	1.52	1.52	1.66	1.61	1.09	1.49	1.50

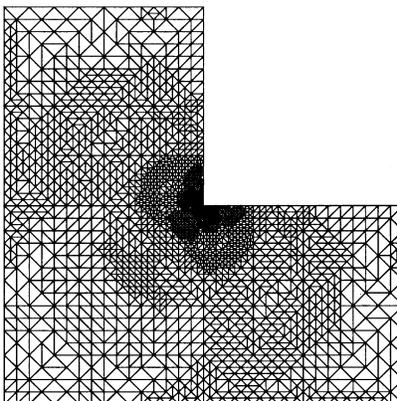


FIGURE 1. Adaptive finite element triangulation

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