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# MERGING THE BRAMBLE-PASCIAK-STEINBACH AND THE CROUZEIX-THOMÉE CRITERION FOR $H^1$-STABILITY OF THE $L^2$-PROJECTION ONTO FINITE ELEMENT SPACES

CARSTEN CARSTENSEN

ABSTRACT. Suppose $\mathcal{S} \subset H^1(\Omega)$ is a finite-dimensional linear space based on a triangulation $\mathcal{T}$ of a domain $\Omega$, and let $\Pi : L^2(\Omega) \to L^2(\Omega)$ denote the $L^2$-projection onto $\mathcal{S}$. Provided the mass matrix of each element $T \in \mathcal{T}$ and the surrounding mesh-sizes obey the inequalities due to Bramble, Pasciak, and Steinbach or that neighboring element-sizes obey the global growth-condition due to Crouzeix and Thomée, $\Pi$ is $H^1$-stable: For all $u \in H^1(\Omega)$ we have $\|\Pi u\|_{H^1(\Omega)} \le C \|u\|_{H^1(\Omega)}$ with a constant $C$ that is independent of, e.g., the dimension of $\mathcal{S}$.

This paper provides a more flexible version of the Bramble-Pasciak-Steinbach criterion for $H^1$-stability on an abstract level. In its general version, (i) the criterion is applicable to *all* kind of finite element spaces and yields, in particular, $H^1$-stability for nonconforming schemes on arbitrary (shape-regular) meshes; (ii) it is *weaker than* (i.e., implied by) *either* the Bramble-Pasciak-Steinbach *or* the Crouzeix-Thomée criterion for regular triangulations into triangles; (iii) it guarantees $H^1$-stability of $\Pi$ a priori for a class of *adaptively-refined* triangulations into right isosceles triangles.

## 1. The $L^2$-projection in a finite element space

Suppose the bounded Lipschitz domain $\Omega$ in $\mathbb{R}^d$ is partitioned into a triangulation $\mathcal{T}$, i.e., $\overline{\Omega} = \bigcup \mathcal{T}$ for a finite set $\mathcal{T}$ of elements $T$ which are closed and whose interiors are Lipschitz domains. The intersection of two distinct elements has zero $d$-dimensional Lebesgue measure. To describe nonconforming finite elements, let $H$ be a closed subset of $H^1(\mathcal{T})$,

$$H_0^1(\Omega) \subseteq H \subset H^1(\mathcal{T}) := \{u \in L^2(\Omega) : \forall T \in \mathcal{T},\, u|_T \in H^1(T)\}, \tag{1}$$

closed with respect to the semi-norm $\|\nabla_{\mathcal{T}} \cdot\|$, where $\|\cdot\|$ denotes the $L^2(\Omega)$-norm and $\nabla_{\mathcal{T}}$ is the $\mathcal{T}$-piecewise action of the gradient $\nabla$ (different from the distributional gradient for discontinuous arguments). For instance, in the conforming setting, the choice of $H = H_0^1(\Omega)$ or $H = H^1(\Omega)$ is a typical example.

Suppose that $\mathcal{S} \subset H$ is an $n$-dimensional subspace with a (not necessarily nodal) basis $(\varphi_1, \varphi_2, \dots, \varphi_n)$, and let $\Pi$ denote the $L^2(\Omega)$-projection defined, for all $u \in H$,

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by

$$\Pi\, u \in \mathcal{S} \quad \text{and} \quad \int_\Omega (u - \Pi\, u)\varphi_j \, dx = 0 \quad \text{for all } j = 1, \dots, n. \tag{2}$$

In this context, the $L^2$-projection $\Pi$ is called $H^1$-stable if there exists a constant $c_1 > 0$ with

$$\|\nabla_{\mathcal{T}}\, \Pi\, u\| \le c_1 \|\nabla_{\mathcal{T}}\, u\| \quad \text{for all } u \in H. \tag{3}$$

Two sets of parameters, the $n$ positive parameters $(d_1, d_2, \dots, d_n)$ and the $\mathcal{T}$-piecewise constant weight $h_{\mathcal{T}}$, defined on $T \in \mathcal{T}$ by $h_T > 0$, will provide the link between the triangulation $\mathcal{T}$ and the discrete space $\mathcal{S}$. Their choice is arbitrary up to the severe restriction of inequality (7) below.

To verify $H^1$-stability of the $L^2$-projection (3) we suppose that there exist a (possibly nonlinear) mapping $P : H \to \mathcal{S}$ and a constant $c_2 > 0$ that satisfy, for all $u \in H$,

$$\|\nabla_{\mathcal{T}}\, P(u)\| + \|h_{\mathcal{T}}^{-1}(u - P(u))\| \le c_2 \|\nabla_{\mathcal{T}}\, u\|. \tag{4}$$

*Remark* 1. In Sections 4, 5, and 6, $h_T$ will be the element-size and $d_\ell$ a measure for the size of supp $\varphi_\ell$.

*Remark* 2. Approximation operators which satisfy (4) for $h_T = \operatorname{diam}(T)$ can be found in [Ca, CF, Cl].

## 2. Mass matrices and two inequalities

To define the mass matrix for a given $T \in \mathcal{T}$, let $\ell(T,1), \ell(T,2), \dots, \ell(T, m(T))$ denote exactly those indices of basis functions whose restrictions $\psi_{T,j} := \varphi_{\ell(T,j)}|_T \in H^1(T)$, $1 \le j \le m(t)$, on $T$ are nonzero. Then the shape functions $(\psi_{T,j} : j = 1, \dots, m(T))$ on $T$ satisfy an inverse inequality (by equivalence of norms),

$$\|\sum_{j=1}^{m(T)} \xi_j \, \nabla \psi_{T,j}\|_{L^2(T)} \le c_3\, h_T^{-1}\, \|\sum_{j=1}^{m(T)} \xi_j\, \psi_{T,j}\|_{L^2(T)} \qquad \text{for all } (\xi_1, \dots, \xi_{m(T)}) \in \mathbb{R}^{m(T)}. \tag{5}$$

The (local) $m(T) \times m(T)$-dimensional mass matrix $M(T)$ and the diagonal matrix $\Lambda(T)$,

$$\Lambda(T)_{jk} = \frac{h_T}{d_{\ell(T,j)}}\, \delta_{jk} \quad \text{and} \quad M(T)_{jk} = \int_T \psi_{T,j}\psi_{T,k}\, dx \quad \text{for all } j,k = 1, \dots, m(T), \tag{6}$$

($\delta_{jk} \in \{0,1\}$ denotes Kronecker's symbol) are supposed to satisfy, for constants $c_4, c_5 > 0$,

$$c_4^2\, x \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x \le x \cdot M(T) x \le c_5\, x \cdot \Lambda(T)^2 M(T) x \quad \text{for all } x \in \mathbb{R}^{m(T)}. \tag{7}$$

*Remark* 3. Inverse estimates [BS, Ci] provide (5) for a size-independent constant $c_3$ if $h_T = \operatorname{diam}(T)$.

*Remark* 4. The first inequality of (7) merely reflects a proper scaling of $d_{\ell(T,j)}$ and $h_T$.

*Remark* 5. The second inequality of (7) implies that $\Lambda(T)^2 M(T)$ has positive definite symmetric part. This is the crucial condition and relates the mass-matrix $M(T)$ to neighboring mesh-sizes.

*Remark* 6. We stress that (7) can *always* be satisfied even with $c_4 = c_5 = 1$ if we let $h_T = d_{\ell(T,j)}$ be equal to a global discretization parameter. For quasi-uniform meshes this implies (7).

*Remark* 7. In the original version [BPS, S], $d_j$ is fixed as the arithmetic mean of all $h_T$ with $T \subset \operatorname{supp}\varphi_j$, where $h_T^d$ is the $d$-dimensional volume of an element $T \in \mathcal{T}$. Then, the Bramble-Pasciak-Steinbach criterion [BPS, (4.2)] implies the crucial second inequality in (7) (and is, in particular situations, equivalent).

## 3. A modified Bramble-Pasciak-Steinbach criterion for $H^1$-stability

Under the present assumptions (1)-(2) and (4)-(7) we have $H^1$-stability of $\Pi$.

**Theorem 1.** *We have* (3) *with* $c_1 = c_2 \max\{1, c_3\, c_5/c_4\}$.

The proof is a review of arguments in [BPS] in an abstract setting, and is included here for completeness. Theorem 1 implies the Bramble-Pasciak-Steinbach criterion [BPS] for a special choice of $h_T$ and $d_j$ (of Remark 7).

*Proof.* Given $u \in H$, define $q_h := P(u) - \Pi\, u = \sum_{\ell=1}^n q_\ell\, \varphi_\ell \in \mathcal{S}$ and $p_h := \sum_{\ell=1}^n q_\ell\, d_\ell^{-2}\, \varphi_\ell \in \mathcal{S}$ so that

$$q_h|_T = \sum_{\ell=1}^n q_\ell\, \varphi_\ell|_T = \sum_{j=1}^{m(T)} \xi_{T,j}\, \psi_{T,j} \quad \text{on } T \in \mathcal{T} \tag{8}$$

for certain coefficient vectors $x_T = (\xi_{T,1}, \dots, \xi_{T,m(T)}) = (q_{\ell(T,1)}, \dots, q_{\ell(T,m(T))})$. The triangle inequality for $\Pi\, u = P(u) - q_h$ and (4)-(5) show that

$$\|\nabla_{\mathcal{T}}\, \Pi u\| \le \|\nabla_{\mathcal{T}}\, P(u)\| + \|\nabla_{\mathcal{T}}\, q_h\| \le c_2\, \|\nabla_{\mathcal{T}}\, u\| + c_3\, \|h_{\mathcal{T}}^{-1}\, q_h\|. \tag{9}$$

According to direct calculations with coefficients from (8), the second inequality in (7) yields

$$\begin{aligned} c_5^{-1}\, \|h_{\mathcal{T}}^{-1}\, q_h\|^2 &= c_5^{-1} \sum_{T\in\mathcal{T}} h_T^{-2}\, x_T \cdot M(T) x_T \le \sum_{T\in\mathcal{T}} h_T^{-2}\, x_T \cdot \Lambda(T)^2 M(T) x_T \\ &= \sum_{T\in\mathcal{T}} \sum_{j=1}^{m(T)} \frac{q_{\ell(T,j)}}{d_{\ell(T,j)}^2} \int_T \varphi_{\ell(T,j)}\, q_h\, dx = \int_\Omega p_h\, q_h\, dx \\ &= \int_\Omega p_h (P(u) - u) dx \le c_2\, \|h_{\mathcal{T}}\, p_h\|\, \|\nabla_{\mathcal{T}}\, u\| \end{aligned} \tag{10}$$

because of (2), Cauchy's inequality, and (4). Similar arguments and (7) lead to

$$\begin{aligned} c_4^2\, \|h_{\mathcal{T}}\, p_h\|^2 &= c_4^2 \sum_{T\in\mathcal{T}} h_T^{-2}\, x_T \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x_T \\ &\le \sum_{T\in\mathcal{T}} h_T^{-2} x_T \cdot M(T) x_T = \|h_{\mathcal{T}}^{-1}\, q_h\|^2. \end{aligned}$$

Utilizing this in (10), we obtain a bound of $\|h_{\mathcal{T}}^{-1}\, q_h\|$, which we need in (9) to see (4). ∎

## 4. Examples for Courant triangles

Suppose $\mathcal{T}$ is a regular triangulation (in the sense of Ciarlet [BS, Ci]) of the bounded Lipschitz domain $\Omega$ in the plane into triangles. Homogeneous Dirichlet conditions may apply on a (relatively closed and possibly empty) boundary part $\Gamma_D$ (matched exactly by edges). Each node $z \in \mathcal{N}$ with nodal basis function $\varphi_z$ involves a positive real number $d_z$ such that $h_T/d_z + d_z/h_T \leq c_6$ for all triangles $T \in \mathcal{T}$ of diameter $h_T$ with vertex $z$. Let $\mathcal{S} := \operatorname{span}\{\varphi_z : z \in \mathcal{K}\}$, where $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$ denotes the set of free nodes, and for the preceding notation identify $(\varphi_z : z \in \mathcal{K})$ and the parameters $(d_z : z \in \mathcal{K})$ with $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $(d_1, d_2, \ldots, d_n)$, respectively.

**Theorem 2.** *Suppose that* $d_z/d_\zeta \leq \kappa < \sqrt{2} + \sqrt{3} \approx 3.1462$ *for all vertices* $z$ *and* $\zeta$ *of some triangle* $T \in \mathcal{T}$*. Then we have* (3).

*Proof.* The mass-matrix of a fixed $T \in \mathcal{T}$ is a multiple of the $3 \times 3$ matrix $M$ with $M_{jk} = 1 + \delta_{jk}$ and $\Lambda(T)$ has diagonal entries $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_j/\lambda_k \leq \kappa$. The eigenvalues of $\Lambda(T)^{-1} A \Lambda(T)^{-1}$ for $A := (\Lambda(T)^2 M + M\Lambda(T)^2)/2$ can be calculated [BPS, S], and their smallest value is $(5 - \mu)$ for $\mu^2 := \sum_{j,k=1}^{3} \lambda_j^2/\lambda_k^2$. A straightforward analysis reveals that $\mu^2 \leq 3 + 2(1 + \kappa^2 + 1/\kappa^2) < 25$, which shows that $A$ is positive definite. Therefore, $(x \cdot Ax)^{1/2}$ defines a norm which is equivalent to $|x|$ in $\mathbb{R}^3$. This and $h_T/d_z \leq c_6$ yield (7). ∎

*Remark* 8. The proof shows that $\sum_{j,k=1}^{3} \lambda_j^2/\lambda_k^2 \leq \nu < 22$ for some constant $\nu$ suffices for (3). Given $d_j$ as in Remark 7, this is the a posteriori criterion of [BPS, S] for two dimensions.

The technical assumption on the artificial, extended triangulation in the following theorem merely reduces the consideration to interior triangles for brevity.

**Theorem 3.** *Suppose* $\mathcal{T} \subset \hat{\mathcal{T}}$ *for some regular triangulation* $\hat{\mathcal{T}}$ *of a Lipschitz domain* $\hat{\Omega} \supset \Omega$ *such that* $\hat{\mathcal{T}}$ *consists of right isosceles triangles only, there are no hanging nodes, and each free node on the boundary is an interior node of* $\hat{\Omega}$*. Then we have* (3).

*Proof.* Theorem 2 yields the assertion if we take

$$d_z = \min\{|z - \zeta| : \zeta \in \mathcal{N},\, \delta(z,\zeta) = 1\},$$

where $\delta(z,\zeta) = 1$ characterizes neighboring vertices $z$ and $\zeta$, i.e., $z, \zeta \in T \cap \mathcal{N}$ for at least one $T \in \mathcal{T}$. Since (up to scaling, transition, and rotation) there are only

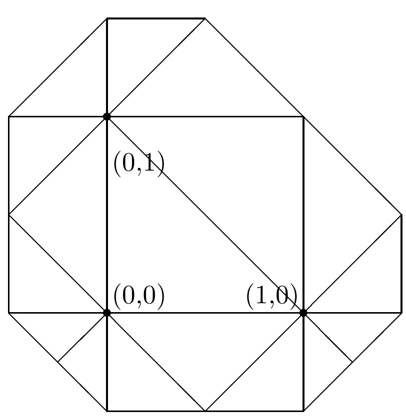


Figure 1: Part of a mesh as a smallest neighborhood of the reference triangle.

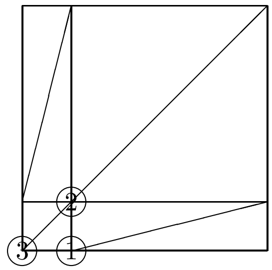


Figure 2: Reference mesh for comparisons in Example 1.

a finite number of possible configurations, it can be checked by a finite number of figures that $d_z/d_\zeta \leq \sqrt{8}$. Figure 1 illustrates a deduction: Suppose $T$ has the vertices $(0,0)$, $(1,0)$, and $(0,1)$. Then, the patch supp $\varphi_z$ of $z = (0,0)$ must include the polygonal domain with vertices $(0,1)$, $(-.5,.5)$, $(-.5,0)$, $(0,-.5)$, $(.5,-.5)$, $(1,0)$. This shows that $1/\sqrt{8} \leq d_{(1,0)} \leq 1$. Similarly, the patch supp $\varphi_{(1,0)}$ must include the polygon $(0,0)$, $(.5,-.5)$, $(1,-.5)$, $(1.5,0)$, $(1.5,3)$, $(1,1)$, $(0,1)$, whence $1/\sqrt{8} \leq d_{(1,0)}, d_{(0,1)} \leq 1$. Consequently, $d_z/d_\zeta \leq \sqrt{8}$ for any choice of two vertices $z$ and $\zeta$ of $T$. □

**Example 1.** Let $\mathcal{T}$ be the mesh of Figure 2 that consists of 8 triangles in a regular pattern that match the square $\Omega := (0,H)^2$ for positive $H = 1+\lambda$, where non-diagonals' lengths are either $\lambda < 1$ or 1. For the nodes 1, 2, and 3 of Figure 2 the choice of $(d_1, d_2, d_3)$ from [BPS], mentioned in Remark 7, is

$$((\lambda + 2\lambda^{1/2})/3, (\lambda^2 + \lambda^{1/2} + 1)/3, \lambda)/\sqrt{2}.$$

The conditions of the Bramble-Pasciak-Steinbach criterion (cf. Remark 8) and those of Theorem 2 are violated for $\lambda < .1349$, which corresponds to an aspect ratio larger than 7.4122 However, Theorem 1 with the parameters from Remark 6 guarantees (3) for any positive $\lambda$ (with a $\lambda$-dependent constant $c_1 = c_1(\lambda)$).

**Example 2.** Take a scaled copy of $\Omega$ and the mesh from Example 1 and extend it by reflection about the $x_1$-axis, the $x_2$-axis, and about the anti-diagonal through the origin to $h(-1,1)^2$; and then extend it $2h$-periodically to the entire plane. The calculations of Example 1 remain valid and we conclude that, for a fixed $\lambda < .1349$, the Bramble-Pasciak-Steinbach criterion is not applicable, but Remark 6 (or the Crouzeix-Thomée criterion) guarantees (3) with an $h$-independent constant $c_1 = c_1(\lambda)$.

The nonconforming Crouzeix-Raviart finite element (cf., e.g., [BS, Ci]) concludes our first series of applications.

**Theorem 4.** *Suppose $T$ is an arbitrary shape-regular triangulation into triangles and $\mathcal{S}$ denotes the $\mathcal{T}$-piecewise affine functions which are continuous at midpoints of edges. Then we have* (3).

*Proof.* The mass-matrices are diagonal, so (7) is a consequence of shape-regularity. The operator $P$ can be chosen exactly as in the conforming case. □

## 5. Weakening of the Crouzeix-Thomée criterion for $H^1$-stability

Part of the Crouzeix-Thomée criterion [CT] is the existence of $c_7$ and $1 \le \kappa := \sqrt{\alpha} < \sqrt{2} + \sqrt{3}$ such that

$$|T_1|/|T_2| \le c_7 \alpha^{l(T_1,T_2)} \quad \text{for all } T_1, T_2 \in \mathcal{T}. \tag{11}$$

Here, $|T_j|$ is the area of $T_j \in \mathcal{T}$ and the neighbor-index $l(T_1, T_2)$ might be defined via a metric $\delta$ on the nodes $\mathcal{N}$: For two distinct nodes $z$ and $\zeta$, $\delta(z,\zeta)$ is the smallest integer $j$ such that there exists a polygon $(z_1, z_2, \dots, z_j)$ of nodes $z_1, \dots, z_j \in \mathcal{N}$ which connects $z = z_1$ with $\zeta = z_j$ along edges, i.e., $\{z_i, z_{i+1}\} \subset \partial T_i$ for some $T_i \in \mathcal{T}$ and all $i = 1, \dots, j-1$; $\delta(z,z) := 0$. For any $T, K \in \mathcal{T}$ and $z \in \mathcal{N}$, let $\delta(z,T) := \min_{\zeta \in T \cap \mathcal{N}} \delta(z,\zeta)$ and $\delta(K,T) = \min_{z \in K \cap \mathcal{N}} \delta(z,T)$. Then, $l(T_1,T_2) = \delta(T_1,T_2) + 1$ if $T_1 \ne T_2$, while $l(T_1,T_2) = 0$ if and only if $T_1 = T_2$.

At first glance, the *local* Bramble-Pasciak-Steinbach and the *global* Crouzeix-Thomée criteria appear incomparable: a large constant $c_7$ prohibits a direct application of (11) in the spirit of Theorems 2 and 3 (as $d_z/d_\zeta \le c_8\,(|T_1|/|T_2|)^{1/2} \le c_7^{1/2} c_8 \kappa \not< \sqrt{2} + \sqrt{3}$ for $\delta(z,\zeta) = 1$). However, all necessities are provided by

$$d_z := \min_{T \in \mathcal{T}} h_T\, \kappa^{\delta(z,T)} \quad \text{for all } z \in \mathcal{N} \quad \text{and} \quad h_T := |T|^{1/2} \quad \text{for all } T \in \mathcal{T}. \tag{12}$$

**Theorem 5.** *Suppose* (11)-(12) *hold for a planar regular triangulation* $\mathcal{T}$. *Then, the conditions of Theorem 2 are satisfied and we have* (3).

*Proof.* Given $z \in K \in \mathcal{T}$, we have $d_z \le h_K$ ($K$ is allowed in the minimization (12), and $\delta(z,K) = 0$) and $l(T,K) - \delta(z,T) \le 1$. With a minimizing $T \in \mathcal{T}$ in (12), (11) shows that

$$h_K/d_z = \frac{h_K}{h_T\, \kappa^{\delta(z,T)}} = \frac{|K|^{1/2}}{|T|^{1/2}} \kappa^{-\delta(z,T)} \le \sqrt{c_7} \kappa^{l(T,K) - \delta(z,T)} \le \sqrt{c_7}\kappa. \tag{13}$$

To bound $d_z/d_\zeta$ for $z, \zeta \in \mathcal{N}$ with $\delta(z,\zeta) = 1$, let $K \in \mathcal{T}$ satisfy $d_\zeta = h_K\, \kappa^{\delta(z,K)}$. The definition (12) and $\delta(z,K) - \delta(\zeta,K) \le 1$ show that

$$d_z/d_\zeta = \frac{d_z}{h_K\, \kappa^{\delta(\zeta,K)}} \le \frac{h_K\, \kappa^{\delta(z,K)}}{h_K\, \kappa^{\delta(\zeta,K)}} = \kappa^{\delta(z,K) - \delta(\zeta,K)} \le \kappa. \quad \square \tag{14}$$

**Example 3.** There exists an adaptively-refined mesh [CV, Figure 1] of right isosceles triangles where the modified Bramble-Pasciak-Steinbach criterion guarantees $H^1$-stability (cf. [S] or Theorem 3) while the Crouzeix-Thomée criterion is not applicable.

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Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel Ludewig-Meyn-Str. 4, D-24098 Kiel, Germany

*E-mail address*: `cc@numerik.uni-kiel.de`