MERGING THE BRAMBLE-PASCIAK-STEINBACH
AND THE CROUZEIX-THOMÉE CRITERION
FOR $H^1$-STABILITY OF THE $L^2$-PROJECTION
ONTO FINITE ELEMENT SPACES

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Abstract. Suppose $S \subset H^1(\Omega)$ is a finite-dimensional linear space based on
a triangulation $T$ of a domain $\Omega$, and let $\Pi : L^2(\Omega) \to L^2(\Omega)$ denote the
$L^2$-projection onto $S$. Provided the mass matrix of each element $T \in T$ and
the surrounding mesh-sizes obey the inequalities due to Bramble, Pasciak, and
Steinbach or that neighboring element-sizes obey the global growth-condition
due to Crouzeix and Thomée, $\Pi$ is $H^1$-stable: For all $u \in H^1(\Omega)$ we have
$\|\Pi u\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)}$ with a constant $C$ that is independent of, e.g.,
the dimension of $S$.

This paper provides a more flexible version of the Bramble-Pasciak-
Steinbach criterion for $H^1$-stability on an abstract level. In its general ver-
sion, (i) the criterion is applicable to all kind of finite element spaces and
yields, in particular, $H^1$-stability for nonconforming schemes on arbitrary
(shape-regular) meshes; (ii) it is weaker than (i.e., implied by) either the
Bramble-Pasciak-Steinbach or the Crouzeix-Thomée criterion for regular tri-
angulations into triangles; (iii) it guarantees $H^1$-stability of $\Pi$ a priori for a
class of adaptively-refined triangulations into right isosceles triangles.

1. The $L^2$-projection in a finite element space

Suppose the bounded Lipschitz domain $\Omega$ in $\mathbb{R}^d$ is partitioned into a triangulation
$T$, i.e., $\Omega = \bigcup T$ for a finite set $T$ of elements $T$ which are closed and whose
interiors are Lipschitz domains. The intersection of two distinct elements has zero
d-dimensional Lebesgue measure. To describe nonconforming finite elements, let $H$
be a closed subset of $H^1(T)$,

$H_0^1(\Omega) \subseteq H \subseteq H^1(T) := \{u \in L^2(\Omega) : \forall T \in T, u|_T \in H^1(T)\},$

closed with respect to the semi-norm $\|\nabla_T \cdot\|$, where $\|\cdot\|$ denotes the $L^2(\Omega)$-norm
and $\nabla_T$ is the $T$-piecewise action of the gradient $\nabla$ (different from the distributional
gradient for discontinuous arguments). For instance, in the conforming setting, the
choice of $H = H_0^1(\Omega)$ or $H = H^1(\Omega)$ is a typical example.

Suppose that $S \subset H$ is an $n$-dimensional subspace with a (not necessarily nodal)
basis $(\varphi_1, \varphi_2, \ldots, \varphi_n)$, and let $\Pi$ denote the $L^2(\Omega)$-projection defined, for all $u \in H$,
by
\begin{equation}
\Pi u \in \mathcal{S} \quad \text{and} \quad \int_{\Omega} (u - \Pi u) \varphi_j \, dx = 0 \quad \text{for all } j = 1, \ldots, n.
\end{equation}
In this context, the $L^2$-projection $\Pi$ is called $H^1$-stable if there exists a constant $c_1 > 0$ with
\begin{equation}
\| \nabla_{T} \Pi u \| \leq c_1 \| \nabla_{T} u \| \quad \text{for all } u \in H.
\end{equation}
Two sets of parameters, the $n$ positive parameters $(d_1, d_2, \ldots, d_n)$ and the $T$-piecewise constant weight $h_T$, defined on $T \in T$ by $h_T > 0$, will provide the link between the triangulation $T$ and the discrete space $\mathcal{S}$. Their choice is arbitrary up to the severe restriction of inequality (7) below.

To verify $H^1$-stability of the $L^2$-projection (3) we suppose that there exist a (possibly nonlinear) mapping $P : H \to \mathcal{S}$ and a constant $c_2 > 0$ that satisfy, for all $u \in H$,
\begin{equation}
\| \nabla_{T} P(u) \| + \| h_T^{-1} (u - P(u)) \| \leq c_2 \| \nabla_{T} u \|.
\end{equation}

**Remark 1.** In Sections 4, 5, and 6, $h_T$ will be the element-size and $d_\ell$ a measure for the size of supp $\varphi_\ell$.

**Remark 2.** Approximation operators which satisfy (4) for $h_T = \text{diam} (T)$ can be found in [Ca, CF, Cl].

## 2. Mass matrices and two inequalities

To define the mass matrix for a given $T \in T$, let $\ell(T, 1), \ell(T, 2), \ldots, \ell(T, m(T))$ denote exactly those indices of basis functions whose restrictions $\psi_{T,j} := \varphi_{\ell(T,j)}|T \in H^1(T)$, $1 \leq j \leq m(t)$, on $T$ are nonzero. Then the shape functions $(\psi_{T,j} : j = 1, \ldots, m(T))$ on $T$ satisfy an inverse inequality (by equivalence of norms),
\begin{equation}
\sum_{j=1}^{m(T)} \xi_j \nabla \psi_{T,j} \|_{L^2(T)} \leq c_3 h_T^{-1} \sum_{j=1}^{m(T)} \xi_j \psi_{T,j} \|_{L^2(T)}
\end{equation}
for all $(\xi_1, \ldots, \xi_{m(T)}) \in \mathbb{R}^{m(T)}$.

The (local) $m(T) \times m(T)$-dimensional mass matrix $M(T)$ and the diagonal matrix $\Lambda(T)$,
\begin{equation}
\Lambda(T)_{jk} = \frac{h_T}{d_\ell(T,j)} \delta_{jk} \quad \text{and} \quad M(T)_{jk} = \int_T \psi_{T,j} \psi_{T,k} \, dx \quad \text{for all } j, k = 1, \ldots, m(T),
\end{equation}
\((\delta_{jk} \in \{0, 1\} \text{ denotes Kronecker's symbol})\) are supposed to satisfy, for constants $c_4, c_5 > 0$,
\begin{equation}
c_4 x \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x \leq x \cdot M(T)x \leq c_5 x \cdot \Lambda(T)^2 M(T)x \quad \text{for all } x \in \mathbb{R}^{m(T)}.
\end{equation}

**Remark 3.** Inverse estimates [BS, Ci] provide (5) for a size-independent constant $c_3$ if $h_T = \text{diam} (T)$.

**Remark 4.** The first inequality of (7) merely reflects a proper scaling of $d_\ell(T,j)$ and $h_T$. 
Remark 5. The second inequality of (7) implies that \(\Lambda(T)^2 M(T)\) has positive definite symmetric part. This is the crucial condition and relates the mass-matrix \(M(T)\) to neighboring mesh-sizes.

Remark 6. We stress that (7) can always be satisfied even with \(c_4 = c_5 = 1\) if we let \(h_T = d_{T;j}\) be equal to a global discretization parameter. For quasi-uniform meshes this implies (7).

Remark 7. In the original version [BPS], \(d_j\) is fixed as the arithmetic mean of all \(h_T\) with \(T \subset \text{supp} \varphi_j\), where \(h_T^d\) is the \(d\)-dimensional volume of an element \(T \in T\). Then, the Bramble-Pasciak-Steinbach criterion [BPS] (4.2) implies the crucial second inequality in (7) (and is, in particular situations, equivalent).

3. A modified Bramble-Pasciak-Steinbach criterion for \(H^1\)-stability

Under the present assumptions (1)-(2) and (4)-(7) we have \(H^1\)-stability of \(\Pi\).

**Theorem 1.** We have (3) with \(c_1 = c_2 \max\{1, c_3 c_5/c_4\}\).

The proof is a review of arguments in [BPS] in an abstract setting, and is included here for completeness. Theorem 1 implies the Bramble-Pasciak-Steinbach criterion [BPS] for a special choice of \(h_T\) and \(d_j\) (of Remark 7).

**Proof.** Given \(u \in H\), define \(q_h := P(u) - \Pi u = \sum_{\ell=1}^n q_\ell \varphi_\ell \in S\) and \(p_h := \sum_{\ell=1}^n q_\ell d_\ell^{-2} \varphi_\ell \in S\) so that

\[
q_h|_T = \sum_{\ell=1}^n q_\ell \varphi_\ell|_T = \sum_{j=1}^{m(T)} \xi_{T,j} \psi_{T,j}, \quad \text{on } T \in T
\]

for certain coefficient vectors \(x_T = (\xi_{T,1}, \ldots, \xi_{T,m(T)}) = (q_{\ell(T,1)}, \ldots, q_{\ell(T,m(T))})\).

The triangle inequality for \(\Pi u = P(u) - q_h\) and (1)-(5) show that

\[
\|\nabla_T \Pi u\| \leq \|\nabla_T P(u)\| + \|\nabla_T q_h\| \leq c_2 \|\nabla_T u\| + c_3 \|h_T^{-1} q_h\|
\]

According to direct calculations with coefficients from (8), the second inequality in (7) yields

\[
c_5^{-1} \|h_T^{-1} q_h\|^2 = c_5^{-1} \sum_{T \in T} h_T^{-2} x_T : M(T) x_T = \sum_{T \in T} h_T^{-2} x_T : \Lambda(T)^2 M(T) x_T
\]

\[
= \sum_{T \in T} \sum_{j=1}^{m(T)} \frac{q_{\ell(T,j)}}{d_{\ell(T,j)}} \int_T \varphi_{\ell(T,j)} q_h \, dx = \int_\Omega p_h q_h \, dx
\]

\[
= \int_\Omega p_h (P(u) - u) \, dx \leq c_2 \|h_T p_h\| \|\nabla_T u\|
\]

because of (2), Cauchy’s inequality, and (3). Similar arguments and (4) lead to

\[
c_4^2 \|h_T p_h\|^2 = c_4^2 \sum_{T \in T} h_T^{-2} x_T : \Lambda(T)^2 M(T) \Lambda(T)^2 x_T
\]

\[
\leq \sum_{T \in T} h_T^{-2} x_T : M(T) x_T = \|h_T^{-1} q_h\|^2.
\]

Utilizing this in (11), we obtain a bound of \(\|h_T^{-1} q_h\|\), which we need in (12) to see (1).
4. Examples for Courant triangles

Suppose $T$ is a regular triangulation (in the sense of Ciarlet [BS, Ci]) of the bounded Lipschitz domain $\Omega$ in the plane into triangles. Homogeneous Dirichlet conditions may apply on a (relatively closed and possibly empty) boundary part $\Gamma_D$ (matched exactly by edges). Each node $z \in N^\ast$ with nodal basis function $\varphi_z$ involves a positive real number $d_z$ such that $h_T/d_z + d_z/h_T \leq c_0$ for all triangles $T \in T$ of diameter $h_T$ with vertex $z$. Let $S := \text{span} \{ \varphi_z : z \in K \}$, where $K := N^\ast \setminus \Gamma_D$ denotes the set of free nodes, and for the preceding notation identify $(\varphi_z : z \in K)$ and the parameters $(d_z : z \in K)$ with $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $(d_1, d_2, \ldots, d_n)$, respectively.

**Theorem 2.** Suppose that $d_z/d_\zeta \leq \kappa < \sqrt{2} + \sqrt{3} \approx 3.1462$ for all vertices $z$ and $\zeta$ of some triangle $T \in T$. Then we have \eqref{eq:3}.

**Proof.** The mass-matrix of a fixed $T \in T$ is a multiple of the $3 \times 3$ matrix $M$ with $M_{jk} = 1 + \delta_{jk}$ and $\Lambda(T)$ has diagonal entries $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_j/\lambda_k \leq \kappa$. The eigenvalues of $\Lambda(T)^{-1} A \Lambda(T)^{-1}$ for $A := (\Lambda(T)^2 M + M \Lambda(T)^2)/2$ can be calculated [BPS, S], and their smallest value is $(5 - \mu)$ for $\mu^2 := \sum_{j,k=1}^{3} \lambda_j^2/\lambda_k^2$. A straightforward analysis reveals that $\mu^2 \leq 3 + 2(1 + \kappa^2 + 1/\kappa^2) < 25$, which shows that $A$ is positive definite. Therefore, $(x \cdot Ax)^{1/2}$ defines a norm which is equivalent to $|x|$ in $\mathbb{R}^3$. This and $h_T/d_z \leq c_0$ yield \eqref{eq:7}.

**Remark 8.** The proof shows that $\sum_{j,k=1}^{3} \lambda_j^2/\lambda_k^2 \leq \nu < 22$ for some constant $\nu$ suffices for \eqref{eq:3}. Given $d_j$ as in Remark 7 this is the a posteriori criterion of [BPS, S] for two dimensions.

The technical assumption on the artificial, extended triangulation in the following theorem merely reduces the consideration to interior triangles for brevity.

**Theorem 3.** Suppose $T \subset \hat{T}$ for some regular triangulation $\hat{T}$ of a Lipschitz domain $\hat{\Omega} \supset \Omega$ such that $\hat{T}$ consists of right isosceles triangles only, there are no hanging nodes, and each free node on the boundary is an interior node of $\hat{\Omega}$. Then we have \eqref{eq:3}.

**Proof.** Theorem 2 yields the assertion if we take $d_z = \min\{|z - \zeta| : \zeta \in N, \delta(z, \zeta) = 1\}$, where $\delta(z, \zeta) = 1$ characterizes neighboring vertices $z$ and $\zeta$, i.e., $z, \zeta \in T \cap N$ for at least one $T \in T$. Since (up to scaling, transition, and rotation) there are only

![Figure 1: Part of a mesh as a smallest neighborhood of the reference triangle.](image)
Figure 2: Reference mesh for comparisons in Example 1.

...a finite number of possible configurations, it can be checked by a finite number of figures that \( d_z / d_\zeta \leq \sqrt{8} \). Figure 1 illustrates a deduction: Suppose \( T \) has the vertices \((0,0), (1,0), \) and \((0,1)\). Then, the patch \( \text{supp} \varphi_z \) of \( z = (0,0) \) must include the polygonal domain with vertices \((0,1), (-.5,.5), (-.5,0), (0,-.5), (.5,-.5), (1,0)\). This shows that \( 1/\sqrt{8} \leq d_{(1,0)} \leq 1 \). Similarly, the patch \( \text{supp} \varphi_{(1,0)} \) must include the polygon \((0,0), (.5,-.5), (1,-.5), (1.5,0), (1.5,3), (1,1), (0,1)\), whence \( 1/\sqrt{8} \leq d_{(1,0)}, d_{(0,1)} \leq 1 \). Consequently, \( d_z / d_\zeta \leq \sqrt{8} \) for any choice of two vertices \( z \) and \( \zeta \) of \( T \).

**Example 1.** Let \( T \) be the mesh of Figure 2 that consists of 8 triangles in a regular pattern that match the square \( \Omega := (0,H^2) \) for positive \( H = 1 + \lambda \), where non-diagonals’ lengths are either \( \lambda < 1 \) or 1. For the nodes 1, 2, and 3 of Figure 2 the choice of \((d_1, d_2, d_3)\) from [BPS], mentioned in Remark 7, is

\[
((\lambda + 2\lambda^{1/2})/3, (\lambda^2 + \lambda^{1/2} + 1)/3, \lambda)/\sqrt{2}.
\]

The conditions of the Bramble-Pasciak-Steinbach criterion (cf. Remark 8) and those of Theorem 2 are violated for \( \lambda \approx .1349 \), which corresponds to an aspect ratio larger than \( 7.4122 \). However, Theorem 1 with the parameters from Remark 6 guarantees (3) with an \( h \)-independent constant \( c_1 = c_1(\lambda) \).

**Example 2.** Take a scaled copy of \( \Omega \) and the mesh from Example 1 and extend it by reflection about the \( x_1 \)-axis, the \( x_2 \)-axis, and about the anti-diagonal through the origin to \( h(-1,1)^2 \); and then extend it \( 2h \)-periodically to the entire plane. The conditions of Example 1 remain valid and we conclude that, for a fixed \( \lambda < .1349 \), the Bramble-Pasciak-Steinbach criterion is not applicable, but Remark 6 (or the Crouzeix-Thomée criterion) guarantees (3) with an \( h \)-independent constant \( c_1 = c_1(\lambda) \).

The calculations of Example 1 remain valid and we conclude that, for a fixed \( \lambda < .1349 \), the Bramble-Pasciak-Steinbach criterion is not applicable, but Remark 6 (or the Crouzeix-Thomée criterion) guarantees (3) with an \( h \)-independent constant \( c_1 = c_1(\lambda) \).

The nonconforming Crouzeix-Raviart finite element (cf., e.g., [BS, Ci]) concludes our first series of applications.

**Theorem 4.** Suppose \( T \) is an arbitrary shape-regular triangulation into triangles and \( S \) denotes the \( T \)-piecewise affine functions which are continuous at midpoints of edges. Then we have (3).

**Proof.** The mass-matrices are diagonal, so (3) is a consequence of shape-regularity. The operator \( P \) can be chosen exactly as in the conforming case.
5. Weakening of the Crouzeix-Thomée criterion for $H^1$-stability

Part of the Crouzeix-Thomée criterion \[\text{CT}\] is the existence of $c_7$ and $1 \leq \kappa := \sqrt{\alpha < 2 + \sqrt{3}}$ such that

\begin{equation}
|T_1| / |T_2| \leq c_7 \alpha^{l(T_1,T_2)} \quad \text{for all } T_1, T_2 \in \mathcal{T}.
\end{equation}

Here, $|T_j|$ is the area of $T_j \in \mathcal{T}$ and the neighbor-index $l(T_1, T_2)$ might be defined via a metric $\delta$ on the nodes $\mathcal{N}$. For two distinct nodes $z$ and $\zeta$, $\delta(z, \zeta)$ is the smallest integer $j$ such that there exists a polygon $(z_1, z_2, \ldots, z_j)$ of nodes $z_1, \ldots, z_j \in \mathcal{N}$ which connects $z = z_1$ with $\zeta = z_j$ along edges, i.e., $\{z_i, z_{i+1}\} \subset \mathcal{T}_i$ for some $T_i \in \mathcal{T}$ and all $i = 1, \ldots, j - 1$; $\delta(z, z) := 0$. For any $T, K \in \mathcal{T}$ and $z \in \mathcal{N}$, let $\delta(z, T) := \min_{\zeta \in T \cap \mathcal{N}} \delta(z, \zeta)$ and $\delta(K, T) = \min_{z \in K \cap \mathcal{N}} \delta(z, T)$. Then, $l(T_1, T_2) = \delta(T_1, T_2) + 1$ if $T_1 \neq T_2$, while $l(T_1, T_2) = 0$ if and only if $T_1 = T_2$.

At first glance, the local Bramble-Pasciak-Steinbach and the global Crouzeix-Thomée criteria appear incomparable: a large constant $c_7$ prohibits a direct application of \[\text{(11)}\] in the spirit of Theorems 2 and 3 (as $d_z / d_\zeta \leq c_8 (|T_1| / |T_2|)^{1/2} \leq c_7^{1/2} c_8 \kappa < \sqrt{2} + \sqrt{3}$ for $\delta(z, \zeta) = 1$). However, all necessities are provided by

\begin{equation}
d_z := \min_{T \in \mathcal{T}} h_T \kappa^{\delta(z, T)} \quad \text{for all } z \in \mathcal{N} \quad \text{and} \quad h_T := |T|^{1/2} \quad \text{for all } T \in \mathcal{T}.
\end{equation}

**Theorem 5.** Suppose \[\text{(11)}\] and \[\text{(12)}\] hold for a planar regular triangulation $\mathcal{T}$. Then, the conditions of Theorem 2 are satisfied and we have \[\text{(3)}\].

**Proof.** Given $z \in K \in \mathcal{T}$, we have $d_z \leq h_K$ ($K$ is allowed in the minimization \[\text{(12)}\]), and $\delta(z, K) = 0$ and $l(T, K) - \delta(z, T) \leq 1$. With a minimizing $T \in \mathcal{T}$ in \[\text{(12)}\], \[\text{(11)}\] shows that

\begin{equation}h_K / d_z = \frac{h_K}{h_T} \kappa^{\delta(z, T)} = \frac{|K|^{1/2}}{|T|^{1/2}} \kappa^{\delta(z, T)} \leq \sqrt{c_7} \alpha^{l(T, K) - \delta(z, T)} \leq \sqrt{c_7} \kappa.
\end{equation}

To bound $d_z / d_\zeta$ for $z, \zeta \in \mathcal{N}$ with $\delta(z, \zeta) = 1$, let $K \in \mathcal{T}$ satisfy $d_\zeta = h_K \kappa^{\delta(z, K)}$. The definition \[\text{(12)}\] and $\delta(z, K) - \delta(\zeta, K) \leq 1$ show that

\begin{equation}d_z / d_\zeta = \frac{d_z}{h_K \kappa^{\delta(z, K)}} \leq \frac{h_K \kappa^{\delta(z, K)}}{h_K \kappa^{\delta(\zeta, K)}} = \frac{\kappa^{\delta(z, K) - \delta(\zeta, K)}}{\kappa} \leq \kappa.
\end{equation}

**Example 3.** There exists an adaptively-refined mesh \[\text{CV}\] Figure 1] of right isosceles triangles where the modified Bramble-Pasciak-Steinbach criterion guarantees $H^1$-stability (cf. \[\text{S}\] or Theorem 3) while the Crouzeix-Thomée criterion is not applicable.

**References**


H$^1$-STABILITY OF THE L$^2$-PROJECTION IN FEM


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