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# ESTIMATES OF $\theta(x;k,l)$ FOR LARGE VALUES OF $x$

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Abstract. We extend a result of Ramaré and Rumely, 1996, about the Chebyshev function $\theta$ in arithmetic progressions. We find a map $\varepsilon(x)$ such that $\mid \theta(x;k,l) - x/\varphi(k) \mid < x\varepsilon(x)$ and $\varepsilon(x) = O\left(\frac{1}{\ln^a x}\right) \quad (\forall a > 0)$, whereas $\varepsilon(x)$ is a constant. Now we are able to show that, for $x \geqslant 1531$,

$$\mid \theta(x;3,l) - x/2 \mid < 0.262 \frac{x}{\ln x}$$

and, for $x \geqslant 151$,

$$\pi(x;3,l) > \frac{x}{2\ln x}.$$

## 1. Introduction

Let $R = 9.645908801$ and $X = \sqrt{\frac{\ln x}{R}}$. Rosser [6] and Schoenfeld [7, Th. 11 p. 342] showed that, for $x \geqslant 101$,

$$\mid \theta(x) - x \mid, \ \mid \psi(x) - x \mid < x\varepsilon(x),$$

where

$$\varepsilon(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} \exp(-X).$$

We adapt their work to the case of arithmetic progressions. Let us recall the usual notations for nonnegative real $x$:

$$\theta(x;k,l) = \sum_{\substack{p \equiv l \bmod k \\ p \leqslant x}} \ln p, \quad \text{where } p \text{ is a prime number,}$$

$$\psi(x;k,l) = \sum_{\substack{n \equiv l \bmod k \\ n \leqslant x}} \Lambda(n), \quad \text{where } \Lambda \text{ is Von Mangold's function,}$$

and $\varphi$ is Euler's function. We show, for $x \geqslant x_0(k)$ where $x_0(k)$ can be easily computed, that

$$\mid \theta(x;k,l) - x/\varphi(k) \mid, \ \mid \psi(x;k,l) - x/\varphi(k) \mid < x\varepsilon(x),$$

where

$$\varepsilon(x) = 3\sqrt{\frac{k}{\varphi(k)C_1(k)}} X^{1/2} \exp(-X)$$

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for an explicit constant $C_1(k)$. We apply the above results for $k = 3$. For small values, we use Ramaré and Rumely's results [3]. We show that for $x \geqslant 1531$,

$$| \theta(x; 3, l) - x/2 | < 0.262 \frac{x}{\ln x}. \tag{1}$$

If we assume that the Generalized Riemann Hypothesis is true, then we can show that, for $x > 1$ and $k \leqslant 432$,

$$| \psi(x; k, l) - x/\varphi(k) | < \frac{1}{4\pi} \sqrt{x} \ln^2 x.$$

Let us define, as usual, $\pi(x)$ the number of primes not greater than $x$. In 1962, Rosser and Schoenfeld ([5, p. 69]) found a lower bound for $\pi(x)$:

$$\pi(x) > \frac{x}{\ln x} \quad \text{for } x \geqslant 17. \tag{2}$$

Letting

$$\pi(x; k, l) = \sum_{p \leqslant x,\ p \equiv l \bmod k} 1,$$

we show an analogous result in the case of arithmetic progression with $k = 3$ and $l = 1$ or 2,

$$\pi(x; 3, l) > \frac{x}{2 \ln x} \quad \text{for } x \geqslant 151.$$

This result, inferred from (1), implies (2) and cannot be proved with Ramaré and Rumely's results.

The method used for $k = 3$ can also be applied for other fixed integers $k$.

## 2. Preliminary lemmas

*Notations.* We will always denote by $\rho$ a nontrivial zero of Dirichlet's function $L$, that is to say a zero such that $0 < \Re\rho < 1$. We write $\rho = \beta + i\gamma$. Let $\wp(\chi)$ be the set of the zeros $\rho$ of the function $L(s, \chi)$, with $0 < \beta < 1$.

For a positive real $H$, following Ramaré and Rumely, we say that GRH($k$,$H$) holds[1] if, for all $\chi$ modulo $k$, all the nontrivial zeros of $L(s, \chi)$ with $|\gamma| \leqslant H$ are such that $\beta = 1/2$.

As in Rosser and Schoenfeld (in [6, 7] where the case $k = 1$ is studied), we must know the distribution of $L(s, \chi)$'s zeros; namely, find a real $H$ such that GRH($k, H$) is satisfied and is a zero-free region.

### 2.1. Zero-free region.

**Theorem 1** (Ramaré and Rumely [3]). *If $\chi$ is a character with conductor $k$, $H \geqslant 1000$, and $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ with $|\gamma| \geqslant H$, then there exists a computable constant $C_1(\chi, H)$ such that*

$$1 - \beta \geqslant \frac{1}{R \ln(k|\gamma|/C_1(\chi, H))}.$$

[1] Note that our GRH is an acronym for the usual Generalized Riemann Hypothesis.

**Examples.** Some examples, extracted from [3, p. 409], appear in the following table.

| $k$ | $H_k$ | $C_1(\chi, H_k)$ |
|---|---|---|
| 1 | 545000000 | 38.31 |
| 3 | 10000 | 20.92 |
| 420 | 2500 | 56.59 |

*Proof.* See Theorem 3.6.3 of Ramaré and Rumely [3, p. 409]. □

*Remark.* For $k \geqslant 1$ and $H_k \geqslant 1000$, $C_1(\chi, H) \geqslant C_1(\chi_0, 1000) \geqslant 9.14$.

As $C_1(\chi, H)$ could be large, we limit $C_1(\chi, H)$ up to $32\pi$ to make some computations. So we have in our hypothesis

$$9.14 \leqslant C_1(\chi, H) \leqslant 32\pi.$$

From now on,

$$C_1(k) = \min(\min_{\chi \bmod k} C_1(\chi, H_\chi), 32\pi). \tag{3}$$

2.2. **GRH($k, H$) and** $N(T,\chi)$**.**

**Lemma 1** (McCurley [1])**.** *Let $C_2 = 0.9185$ and $C_3 = 5.512$. Write $F(y,\chi) = \frac{y}{\pi}\ln\left(\frac{ky}{2\pi e}\right)$ and $R(y,\chi) = C_2\ln(ky) + C_3$. If $\chi$ is a character of Dirichlet with conductor $k$, if $T \geqslant 1$ is a real number, and if $N(T,\chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s,\chi)$ in the rectangle $0 < \beta < 1$, $|\gamma| \leqslant T$, then*

$$|N(T,\chi) - F(T,\chi)| \leqslant R(T,\chi).$$

**Lemma 2** (deduced from [3, Theorem 2.1.1, p. 399] and [9])**.**

- *GRH*$(1, H)$ *is true for* $H = 5.45 \times 10^8$.
- *GRH*$(k, H)$ *is true for* $H = 10000$ *and* $k \leqslant 13$.
- *GRH*$(k, 2500)$ *is true for sets*
  - $E_1 = \{k \leqslant 72\}$,
  - $E_2 = \{k \leqslant 112, k \text{ not prime}\}$,
  - $E_3 = \{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}$.

2.3. **Estimates of** $|\psi(x;k,l) - x/\varphi(k)|$ **using properties of zeros of** $L(s,\chi)$**.** As in Ramaré and Rumely, we remove the zeros with $\beta = 0$ and we consider only primitive $L$-series by adding small terms. Here we take the version stated in [3, Theorem 4.3.1] which is deduced from [1].

**Theorem 2** (McCurley [1])**.** *Let $x > 2$ be a real number, $m$ and $k$ two positive integers, $\delta$ a real number such that $0 < \delta < \frac{x-2}{mx}$, and $T$ a positive real. Let*

$$A(m,\delta) = \frac{1}{\delta^m}\sum_{j=0}^{m}\binom{m}{j}(1+j\delta)^{m+1}. \tag{4}$$

*Assume GRH$(k,1)$. Then*

$$\begin{aligned}\frac{\varphi(k)}{x}\max_{1\leqslant y\leqslant x}|\psi(y;k,l)-\frac{y}{\varphi(k)}| &< A(m,\delta)\sum_{\chi}\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}}\frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|}\\ &\quad+\left(1+\frac{m\delta}{2}\right)\sum_{\chi}\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant T}}\frac{x^{\beta-1}}{|\rho|}+\frac{m\delta}{2}+\tilde{R}/x,\end{aligned}$$

*where* $\sum_\chi$ *denotes the summation over all characters modulo* $k$, $\tilde{R}=\varphi(k)[(f(k)+0.5)\ln x+4\ln k+13.4]$ *and* $f(k)=\sum_{p|k}\frac{1}{p-1}$.

2.4. **One more explicit form of estimates.** The next lemma can be found in [3] with the difference that the authors assumed GRH($k$,$H$) but in fact they used only GRH($k$,1). Since we must apply it with $T>H$, we repeat the proof.

**Lemma 3.** *Let* $\chi$ *be a character modulo* $k$*. Assume GRH*$(k,1)$*. Then, for any* $T\geqslant 1$*, we have*

$$\sum_{\substack{|\gamma|\leqslant T\\\rho\in\wp(\chi)}}\frac{1}{|\rho|}\leqslant\tilde{E}(T)$$

*with* $\tilde{E}(T)=\frac{1}{2\pi}\ln^2(T)+\frac{\ln\left(\frac{k}{2\pi}\right)}{\pi}\ln(T)+C_2+2\left(\frac{1}{\pi}\ln\left(\frac{k}{2\pi e}\right)+C_2\ln k+C_3\right)$.

*Proof.* For $|\gamma|\leqslant 1$, we have GRH$(k,1)$ and so

$$\sum_{\substack{|\gamma|\leqslant 1\\\rho\in\wp(\chi)}}\frac{1}{|\rho|}\leqslant\sum_{\substack{|\gamma|\leqslant 1\\\rho\in\wp(\chi)}}\frac{1}{|1/2+i\gamma|}\leqslant 2N(1,\chi).$$

For $|\gamma|>1$,

$$\sum_{\substack{1<|\gamma|\leqslant T\\\rho\in\wp(\chi)}}\frac{1}{|\rho|}\leqslant\int_1^T\frac{dN(t,\chi)}{t}=\int_1^T\frac{N(t,\chi)}{t^2}dt+\frac{N(T,\chi)}{T}-\frac{N(1,\chi)}{1}.$$

Thus,

$$\sum_{\substack{|\gamma|\leqslant T\\\rho\in\wp(\chi)}}\frac{1}{|\rho|}\leqslant\int_1^T\frac{N(t,\chi)}{t^2}dt+\frac{N(T,\chi)}{T}+N(1,\chi).$$

We conclude by Lemma 1 that

$$\begin{aligned}\int_1^T\frac{N(t,\chi)}{t^2}dt &\leqslant \int_1^T\frac{F(t,\chi)+R(t,\chi)}{t^2}dt\\ &= \frac{1}{\pi}\int_1^T\frac{\ln(kt/(2\pi e))}{t}dt+C_2\int_1^T\frac{\ln(kt)}{t^2}dt+C_3\int_1^T\frac{1}{t^2}dt\\ &= \frac{1}{\pi}\left[\frac{1}{2}\ln^2\left(\frac{kT}{2\pi e}\right)\right]_1^T\\ &\quad+C_2\left\{\left[-\frac{\ln(kt)}{t}\right]_1^T+\int_1^T\frac{1}{t^2}dt\right\}+C_3\left[-1/t\right]_1^T\\ &= \frac{1}{2\pi}\ln^2 T+\frac{1}{\pi}\ln\left(\frac{k}{2\pi e}\right)\ln T+C_2\left(-\frac{\ln(kT)}{T}+\ln k-\frac{1}{T}+1\right)\\ &\quad+C_3(1-1/T).\end{aligned}$$

In the same way, we have an upper bound of

$$\frac{N(T,\chi)}{T} \quad \text{with } \frac{F(T,\chi)+R(T,\chi)}{T}$$

and

$$N(1,\chi) \quad \text{with } F(1,\chi)+R(1,\chi).$$

Finally, we obtain

$$\sum_{\substack{|\gamma|\leqslant T\\ \rho\in\wp(\chi)}} \frac{1}{|\rho|} \leqslant \frac{1}{2\pi}\ln^2(T) + \frac{\ln\left(\frac{k}{2\pi e}\right)}{\pi}\ln(T)$$
$$+ C_2 + 2\left(\frac{1}{\pi}\ln\left(\frac{k}{2\pi}\right) + C_2\ln k + C_3\right) - \frac{C_2}{T}.$$

□

Using the facts that

- if $\rho$ is a zero of $L(s,\chi)$ then $\overline{\rho}$ is zero of $L(s,\overline{\chi})$,
- these zeros are symmetrical with to the line $\Re(z)=1/2$,

we obtain Lemma 4 by examining the proof of [3, Lemma 4.1.3].

**Lemma 4** ([3])**.** *Let*

$$\phi_m(t) = \frac{1}{|t|^{m+1}}\exp\left(\frac{-\ln x}{R\ln(k|t|/C_1(k))}\right) \tag{5}$$

*with* $R = 9.645908801$*. Let* $T \geqslant H$*. We have*

$$\sum_{\substack{|\gamma|\geqslant T\\ \rho\in\wp(\chi)}} \frac{x^\beta}{|\gamma|^{m+1}} + \sum_{\substack{|\gamma|\geqslant T\\ \rho\in\wp(\overline{\chi})}} \frac{x^\beta}{|\gamma|^{m+1}} \leqslant x\sum_{\substack{|\gamma|\geqslant T\\ \rho\in\wp(\chi)}} \phi_m(\gamma) + \sqrt{x}\sum_{\substack{|\gamma|\geqslant T\\ \rho\in\wp(\chi)}} \frac{1}{|\gamma|^{m+1}}.$$

Let us rewrite Lemma 7 of [6] to adapt it to the new functions $F(y,\chi)$ and $R(y,\chi)$ which we use.

**Lemma 5.** *Write* $N(y)=N(y,\chi)$*,* $F(y)=F(y,\chi)$*, and* $R(y)=R(y,\chi)$*. Let* $1<U\leqslant V$ *and* $\phi(y)$ *be a positive and differentiable function for* $U\leqslant y\leqslant V$*. Let* $(W-y)\phi'(y)\geqslant 0$ *for* $U<y<V$*, where* $W$ *does not necessarily belong to* $[U,V]$*. Let* $Y$ *be that one of the numbers* $U,V,W$ *which is not numerically the least or greatest (or is the repeated one, if two among* $U$*,* $V$*,* $W$ *are equal). Take* $j=0$ *or* $1$*, accordingly as* $W<V$ *or* $W\geqslant V$*. Then*

$$\sum_{U<|\gamma|\leqslant V} \phi(|\gamma|) \leqslant \frac{1}{\pi}\int_U^V \phi(y)\ln\left(\frac{ky}{2\pi}\right)dy + (-1)^j C_2\int_U^V \frac{\phi(y)}{y}dy + B_j(Y,U,V),$$

*where*

$$\begin{aligned}
B_0(Y,U,V) &= 2R(Y)\phi(Y) + \{N(V)-F(V)-R(V)\}\phi(V)\\
&\quad -\{N(U)-F(U)+R(U)\}\phi(U),\\
B_1(Y,U,V) &= \{N(V)-F(V)+R(V)\}\phi(V) - \{N(U)-F(U)+R(U)\}\phi(U).
\end{aligned}$$

*Proof.* We have

$$\begin{aligned}\sum_{U<|\gamma|\leqslant V}\phi(|\gamma|) &= \int_U^V \phi(y)dN(y)\\ &= -\int_U^V N(y)\phi'(y)dy + N(V)\phi(V) - N(U)\phi(U).\end{aligned}$$

- $j=1$. We have $W>V$ and so $Y=\min(V,W)=V$. According to Theorem 1, $N(y)\geqslant F(y)-R(y)$.

$$\begin{aligned}\sum_{U<|\gamma|\leqslant V}\phi(|\gamma|) \quad &\leqslant \quad [(N(y)-F(y)+R(y))\phi(y)]_U^V + \frac{1}{\pi}\int_U^V \ln\left(\frac{ky}{2\pi}\right)\phi(y)dy\\ &\quad -\int_U^V R'(y)\phi(y)dy\end{aligned}$$

because $F'(y)=\frac{1}{\pi}\left(\ln\left(\frac{ky}{2\pi e}\right)+1\right)=\frac{1}{\pi}\ln\left(\frac{ky}{2\pi}\right)$. Moreover,

$$-\int_U^V R'(y)\phi(y)dy = -C_2\int_U^V \frac{\phi(y)}{y}dy.$$

- $j=0$. We have $V>W$. Take $Y=\max(U,W)$. Split the integral at $Y$. Then $-\phi'(y)\leqslant 0$ for $y\in[U,Y]$ and $-\phi'(y)\geqslant 0$ for $y\in[Y,V]$. Replacing $N(y)$ by $F(y)-R(y)$ in the first part and by $F(y)+R(y)$ in the second part, we obtain

$$\begin{aligned}\sum_{U<|\gamma|\leqslant V}\phi(|\gamma|) \quad &\leqslant \quad \frac{1}{\pi}\int_U^V \ln\left(\frac{ky}{2\pi}\right)\phi(y)dy + \int_Y^V R'(y)\phi(y)dy - \int_U^Y R'(y)\phi(y)dy\\ &\quad +B_0(Y,U,V).\end{aligned}$$

Moreover,

$$\int_Y^V R'(y)\phi(y)dy \leqslant (-1)^j C_2\int_U^V \frac{\phi(y)}{y}dy$$

and

$$-\int_U^Y R'(y)\phi(y)dy\leqslant 0.$$

□

We want to apply Lemma 5 with $\phi=\phi_m$ defined by (5) and with $W=W_m$ being the root of $\phi_m'$. Let

$$X=\sqrt{\frac{\ln x}{R}} \tag{6}$$

and, for $m\geqslant 0$,

$$W_m=\frac{C_1(k)}{k}\exp(X/\sqrt{m+1}). \tag{7}$$

**Corollary 1** (Corollary from Lemma 5)**.** *Under the hypothesis of Lemma 5, if moreover* $\frac{2\pi}{ke}\leqslant U$*, then*

$$\sum_{U<|\gamma|\leqslant V}\phi(|\gamma|)\leqslant \left\{1/\pi+(-1)^j q(Y)\right\}\int_U^V \phi(y)\ln(ky/2\pi)dy + B_j(Y,U,V),$$

*where* $q(y)=\frac{C_2}{y\ln\left(\frac{ky}{2\pi}\right)}$.

*Proof.* The map $y \mapsto 1/(y\ln(ky/2\pi))$ is decreasing if $y \geqslant 2\pi/(ke)$.

- Case ($j=0$), then $Y=\max(U,W)$.

$$\sum_{U<|\gamma|\leqslant V} \phi(|\gamma|) < B_0(Y,U,V) + \frac{1}{\pi}\int_U^V \phi(y)\ln\left(\frac{ky}{2\pi}\right)dy + \int_Y^V R'(y)\phi(y)dy.$$

$$\begin{aligned}\int_Y^V R'(y)\phi(y)dy &= C_2\int_Y^V \frac{\phi(y)}{y}dy = C_2\int_Y^V \frac{\phi(y)\ln(ky/2\pi)}{y\ln(ky/2\pi)}dy \\ &\leqslant \frac{C_2}{Y\ln(kY/2\pi)}\int_Y^V \phi(y)\ln(ky/2\pi)dy.\end{aligned}$$

- Case ($j=1$), then $Y=V$.

$$-\int_U^V R'(y)\phi(y)dy \leqslant -\frac{C_2}{V\ln(kV/2\pi)}\int_U^V \phi(y)\ln(ky/2\pi)dy.$$

$\square$

**Theorem 3.** *Let $k \geqslant 1$ an integer, $H \geqslant 1000$ a real number. Assume $GRH(k,H)$. Let $x_0 > 2$ be a real number, $m$ a positive integer, and $\delta$ a real number such that $0<\delta<(x_0-2)/(mx_0)$ and let $Y$ be defined as in Lemma 5. We write*

$$\tilde{A}_H = \frac{1}{\pi}\int_H^\infty \phi_m(y)\ln\left(\frac{ky}{2\pi}\right)dy + C_2\int_H^\infty \frac{\phi_m(y)}{y}dy, \tag{8}$$

$$\tilde{B}_H = B_0(Y,H,\infty), \tag{9}$$

$$\tilde{C}_H = \frac{1}{m\pi H^m}\left(\ln\left(\frac{kH}{2\pi}\right)+1/m\right), \tag{10}$$

$$\tilde{D}_H = \left(2C_2\ln(kH)+2C_3+\frac{C_2}{m+1}\right)/H^{m+1}. \tag{11}$$

*Then for all $x \geqslant x_0$, we have*

$$\begin{aligned}\frac{\varphi(k)}{x}\max_{1\leqslant y\leqslant x}|\psi(y;k,l)-\frac{y}{\varphi(k)}| &\leqslant A(m,\delta)\frac{\varphi(k)}{2}\left(\tilde{A}_H+\tilde{B}_H+(\tilde{C}_H+\tilde{D}_H)/\sqrt{x}\right)\\ &\quad+\left(1+\frac{m\delta}{2}\right)\varphi(k)\tilde{E}(H)/\sqrt{x}+\frac{m\delta}{2}+\tilde{R}/x.\end{aligned}$$

*Remark.* We find a version of Theorem 4.3.2 of [3] where $x_0$ is replaced by $x$ in $\tilde{A}$ and $\tilde{B}$.

*Proof.* According to Theorem 2,

$$\begin{aligned}\frac{\varphi(k)}{x}\max_{1\leqslant y\leqslant x}|\psi(y;k,l)-\frac{y}{\varphi(k)}| &< A(m,\delta)\sum_\chi\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>H}}\frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|}\\ &\quad+\left(1+\frac{m\delta}{2}\right)\sum_\chi\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant H}}\frac{x^{\beta-1}}{|\rho|}+\frac{m\delta}{2}+\tilde{R}/x.\end{aligned}$$

We separately examine the different parts:

- We have
$$\sum_{\chi}\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>H}}\frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|}\leqslant\sum_{\chi}\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>H}}\frac{x^{\beta-1}}{|\gamma|^{m+1}}.$$
By Lemma 4,
$$\begin{aligned}\sum_{\chi}\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>H}}\frac{x^{\beta-1}}{|\gamma|^{m+1}} &= \sum_{\chi}\frac{1}{2}\left(\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>H}}\frac{x^{\beta-1}}{|\gamma|^{m+1}}+\sum_{\substack{\rho\in\wp(\overline{\chi})\\|\gamma|>H}}\frac{x^{\beta-1}}{|\gamma|^{m+1}}\right)\\ &\leqslant \frac{1}{2}\sum_{\chi}\left(\sum_{\substack{|\gamma|\geqslant H\\\rho\in\wp(\chi)}}\phi_m(\gamma)+\frac{1}{\sqrt{x}}\sum_{\substack{|\gamma|\geqslant H\\\rho\in\wp(\chi)}}\frac{1}{|\gamma|^{m+1}}\right).\end{aligned}$$
Using Lemma 5 with $U=H$, $V=\infty$, $\phi=\phi_m$, and $W=W_m$,
$$\sum_{\substack{|\gamma|\geqslant H\\\rho\in\wp(\chi)}}\phi_m(\gamma)\leqslant\tilde{A}_H+\tilde{B}_H.$$
Integration by parts gives
$$\sum_{\substack{|\gamma|\geqslant H\\\rho\in\wp(\chi)}}\frac{1}{|\gamma|^{m+1}}\leqslant\tilde{C}_H+\tilde{D}_H.$$
- By GRH$(k,H)$ we have $\beta=1/2$ for all $|\gamma|\leqslant H$, and by Lemma 3,
$$\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant H}}\frac{x^{\beta-1}}{|\rho|}\leqslant\tilde{E}(H)/\sqrt{x}.$$

□

2.5. **The leading term ($\tilde{A}_H$).** To obtain an upper bound for the leading term, we proceed like Rosser and Schoenfeld with upper bounds on the integrals. The next three lemmas are issued directly from [6, p. 251-255].

**Lemma 6** (Functions of incomplete Bessel type)**.** *Let*
$$K_\nu(z,u)=\frac{1}{2}\int_u^\infty t^{\nu-1}H^z(t)dt,$$
*where* $z>0, u\geqslant 0$, *and*
$$H^z(t)=\{H(t)\}^z=\exp\{-\frac{z}{2}(t+1/t)\}.$$
*Further, write* $K_\nu(z,0)=K_\nu(z)$. *Then*

$$K_1(z) \leqslant \sqrt{\frac{\pi}{2z}}\exp(-z)\left(1+\frac{3}{8z}\right), \tag{12}$$

$$K_2(z) \leqslant \sqrt{\frac{\pi}{2z}}\exp(-z)\left(1+\frac{15}{8z}+\frac{105}{128z^2}\right). \tag{13}$$

**Lemma 7.**
$$K_\nu(z,x)+K_{-\nu}(z,x)=K_\nu(z).$$
*Hence,* $K_\nu(z,x)\leqslant K_\nu(z)$ $(\nu\geqslant 0)$.

**Lemma 8.** *Let*

$$Q_\nu(z,x) = \frac{x^{\nu+1}}{z(x^2-1)} \exp\{-z(x+1/x)/2\}.$$

*If $z>0$ and $x>1$, then*

$$K_1(z,x) < Q_1(z,x)$$

*and*

$$K_2(z,x) < (x+2/z)Q_1(z,x).$$

The term $\tilde{A}_H$ can be expressed using incomplete Bessel functions.

**Lemma 9.** *Let $X$ be defined by* (6). *Let $z_m = 2X\sqrt{m} = 2\sqrt{\frac{m\ln x}{R}}$ and $U_m = \frac{2m}{z_m}\ln\left(\frac{kH}{C_1(k)}\right) = \sqrt{\frac{Rm}{\ln x}}\ln\left(\frac{kH}{C_1(k)}\right)$.*

$$\begin{aligned}\tilde{A}_H &= \frac{2}{\pi}\frac{\ln x}{Rm}\left(\frac{k}{C_1(k)}\right)^m K_2(z_m,U_m)\\ &+\frac{2}{\pi}\ln\left(\frac{C_1(k)}{2\pi}\right)\sqrt{\frac{\ln x}{Rm}}\left(\frac{k}{C_1(k)}\right)^m K_1(z_m,U_m)\\ &+2C_2\sqrt{\frac{\ln x}{R(m+1)}}\left(\frac{k}{C_1(k)}\right)^{m+1} K_1(z_{m+1},U_{m+1}).\end{aligned}$$

*Proof.* This is by straightforward algebraic manipulation; for example, we write

$$I = \int_H^\infty \frac{C_2}{y^{m+1}}\exp\left(\frac{-\ln x}{R\ln(ky/C_1(k))}\right)\frac{dy}{y}.$$

Changing variables:

$$\begin{aligned} t &= \sqrt{\frac{R(m+1)}{\ln x}}\ln\left(\frac{ky}{C_1(k)}\right),\\ dt &= \sqrt{\frac{R(m+1)}{\ln x}}\frac{dy}{y}.\end{aligned}$$

Now

$$\begin{aligned}\exp\left(\frac{-\ln x}{R\ln(ky/C_1(k))}\right) &= \exp\left(\frac{-\ln x}{Rt/\sqrt{\frac{R(m+1)}{\ln x}}}\right)\\ &= \exp\left(\sqrt{\frac{(m+1)\ln x}{R}}\frac{1}{t}\right) = \exp\left(\frac{-z_{m+1}}{2}\frac{1}{t}\right)\end{aligned}$$

and

$$\frac{1}{y^{m+1}} = \left(\frac{k}{C_1(k)}\right)^{m+1}\exp\left(-\frac{(m+1)t}{\sqrt{\frac{R(m+1)}{\ln x}}}\right) = \left(\frac{k}{C_1(k)}\right)^{m+1}\exp\left(-t\frac{z_{m+1}}{2}\right).$$

Consequently,

$$I = \int_{U_{m+1}}^\infty C_2\sqrt{\frac{\ln x}{R(m+1)}}\left(\frac{k}{C_1(k)}\right)^{m+1}\exp\left(\frac{-z_{m+1}}{2}(t+1/t)\right).$$

$\square$

2.6. **Study of $f(k)$ which appears in the expression of $\tilde{R}$.** Remember that $f(k) = \sum_{p|k} \frac{1}{p-1}$.

**Lemma 10.** *For an integer $k \geqslant 1$,*

$$f(k) \leqslant \frac{\ln k}{\ln 2}.$$

*Proof.* We prove by recursion that

$$f(k) \leqslant \frac{\ln k}{\ln 2}.$$

For $k = 1$, it is obvious. For $k = 2$, $f(k) = 1 \leqslant \frac{\ln 2}{\ln 2}$. Assume $f(k) \leqslant \frac{\ln k}{\ln 2}$ holds for $k \leqslant n$. Find an upper bound for $f(n+1)$. If $(n+1)$ is prime, then $f(n+1) = 1/n \leqslant \ln n / \ln 2$. If $(n+1)$ is not prime, then there exists $p \leqslant n$, which divides $n$. If $p = 2$ and $2^\alpha \parallel n+1$,

$$\begin{aligned} f(n+1) &= f\left(\frac{n+1}{2^\alpha} \cdot 2^\alpha\right) = f\left(\frac{n+1}{2^\alpha}\right) + f(2) \\ &= 1 + f\left(\frac{n+1}{2^\alpha}\right) \leqslant \frac{\ln(n+1)}{\ln 2} + 1 - \frac{\ln 2}{\ln 2} \\ &\leqslant \frac{\ln(n+1)}{\ln 2}. \end{aligned}$$

If $p > 2$ and $p^\alpha \parallel n+1$,

$$\begin{aligned} f(n+1) &= f\left(\frac{n+1}{p^\alpha} \cdot p^\alpha\right) = f\left(\frac{n+1}{p^\alpha}\right) + f(p) \\ &= \frac{1}{p-1} + f\left(\frac{n+1}{p^\alpha}\right) \leqslant \frac{\ln(n+1)}{\ln 2} + \frac{1}{p-1} - \frac{\ln p}{\ln 2} \\ &\leqslant \frac{\ln(n+1)}{\ln 2} \qquad \text{because } \frac{1}{p-1} - \frac{\ln p}{\ln 2} < 0 \text{ for } p > 2. \end{aligned}$$

□

## 3. The method with $m = 1$

**Theorem 4.** *Let $k$ be an integer, $H \geqslant 1250$, and $H \geqslant k$. Assume GRH$(k, H)$. Let $C_1(k)$ defined by (3). Let $x > 1$. Write $X = \sqrt{\frac{\ln x}{R}}$ and*

$$\varepsilon(x) = 2\sqrt{\frac{k\varphi(k)}{C_1(k)\sqrt{\pi}}}\left(1 + \frac{1}{2X}(15/16 + \ln(C_1(k)/(2\pi)))\right) X^{3/4} \exp(-X).$$

*If $\varepsilon(x) \leqslant 0.2$ and $X \geqslant \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$, then*

$$\max_{1 \leqslant y \leqslant x} \mid \psi(y; k, l) - y/\varphi(k) \mid \leqslant x\varepsilon(x)/\varphi(k).$$

*Proof.* Take $m = 1$ in Theorem 3. Assuming $X \geqslant \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$, then $W_1 \geqslant H$. In this situation, $Y = W_1$ and $\tilde{B}_H < 2R(W_1)\phi_1(W_1)$. For $y > 1$, $R(y)/\ln y$ is decreasing; hence,

$$\begin{aligned}
\tilde{B}_H &< 2R(W_1)\phi_1(W_1) < 2\frac{R(H)}{\ln H}\phi_1(W_1)\ln W_1 \\
&= 2\frac{R(H)}{\ln H}\left(\frac{X}{\sqrt{2}} + \ln\left(\frac{C_1(k)}{k}\right)\right)\phi_1(W_1) \\
&= 2\frac{R(H)}{\ln H}\left(\frac{X}{\sqrt{2}} + \ln\left(\frac{C_1(k)}{k}\right)\right)(k/C_1(k))^2\exp(-2\sqrt{2}X).
\end{aligned}$$

Inserting the upper bounds (12) and (13) into the bound for $\tilde{A}_H$ in Lemma 9,

$$\begin{aligned}
\tilde{A}_H < 2\left(\frac{k}{C_1(k)}\right)\Bigg[&\sqrt{\frac{\pi}{4X}}\exp(-2X)\left(1+\frac{15}{16X}+\frac{105}{512X^2}\right)X^2/\pi \\
&+\frac{1}{\pi}\ln\frac{C_1(k)}{2\pi}X\sqrt{\frac{\pi}{4X}}\exp(-2X)\left(1+\frac{3}{16X}\right) \\
&+C_2\frac{kX}{C_1(k)\sqrt{2}}\sqrt{\frac{\pi}{4\sqrt{2}X}}\exp(-2\sqrt{2}X)\left(1+\frac{3}{16\sqrt{2}X}\right)\Bigg].
\end{aligned}$$

Put

$$F_1 := \frac{1}{\sqrt{\pi}}\frac{k}{C_1(k)}X^{3/2}\exp(-2X)\left[1+\left(\frac{15}{16}+\ln\frac{C_1(k)}{2\pi}\right)\frac{1}{2X}\right]^2.$$

In Lemma 11 below it is shown that

$$\tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R}\frac{2}{x\varphi(k)} < F_1.$$

We must choose $\delta$ to minimize

$$\frac{A(1,\delta)}{2}\varphi(k)F_1 + \delta/2.$$

Write $f = \varphi(k)F_1$. As $A_1(\delta) = (\delta^2 + 2\delta + 2)/\delta$, we must minimize $g(\delta) = (\delta/2 + 1 + 1/\delta)f + \delta/2$. The minimum value here is at $\delta = \sqrt{\frac{2f}{1+f}}$, and the value there is $g(\sqrt{\frac{2f}{1+f}}) = f + \sqrt{2f(1+f)}$.

It is a simple matter to prove that for $0 \leqslant f \leqslant 0.202$,

$$f + \sqrt{2f(1+f)} < 2\sqrt{f}.$$

As $X \geqslant X_0 := \sqrt{2}\ln\left(\frac{kH}{C_1(k)}\right)$, then $x_0 \geqslant \exp(122.5)$, and it is obvious that $\delta$ meets the hypothesis $0 < \delta < (x_0-2)/x_0$ in Theorem 3 since

$$0 < \delta < \sqrt{2}\sqrt{f} < 0.6357 < \frac{x_0}{x_0-2}.$$

□

**Lemma 11.**

$$\tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R}\frac{2}{x\varphi(k)} < F_1.$$

*Proof.* First we prove that $\tilde{A}_H + \tilde{B}_H < F_1$:

$$\begin{aligned}
F_1 &= \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} e^{-2X} \Big(1 + (15/16 + \ln(C_1(k)/2\pi))/X \\
&\quad + (225/1024 + \frac{15}{32}\ln(C_1(k)/2\pi) + \frac{1}{4}\ln^2(C_1(k)/2\pi))/X^2\Big), \\
\tilde{A}_H &< \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} e^{-2X} \left(1 + \frac{15}{16X} + \frac{105}{512X^2} + \ln\left(\frac{C_1(k)}{2\pi}\right)\left(\frac{1}{X} + \frac{3}{16X^2}\right)\right. \\
&\quad \left. + C_2 \frac{k\pi}{C_1(k)\sqrt{2\sqrt{2}}} \exp(-2(\sqrt{2}-1)X)(1/X + 3/(16\sqrt{2}X^2))\right), \\
\tilde{B}_H &< \frac{k}{\sqrt{\pi}C_1(k)} X^{3/2} \exp(-2X)\exp(-2(\sqrt{2}-1)X) \\
&\quad \times \left[\frac{2k\sqrt{\pi}}{C_1(k)\ln H}(C_2\ln(kH) + C_3)\left(\frac{1}{\sqrt{2X}} + \frac{1}{X\sqrt{X}}\ln(C_1(k)/k)\right)\right].
\end{aligned}$$

This yields $F_1 - \tilde{A}_H - \tilde{B}_H > 0$ if

$$\begin{aligned}
F_2 := \frac{1}{X^2}&\left(\frac{15}{1024} + \frac{9}{32}\ln\left(\frac{C_1(k)}{2\pi}\right) + \frac{1}{4}\ln^2\left(\frac{C_1(k)}{2\pi}\right)\right) \\
&> \frac{C_2\sqrt{\pi}k}{C_1(k)} \exp(-2(\sqrt{2}-1)X)\frac{1}{\sqrt{2X}} \\
&\quad \times \left[\sqrt{\frac{\pi}{2\sqrt{2}}}\left(\sqrt{\frac{2}{X}}\frac{3}{16X^{3/2}}\right)\right. \\
&\qquad \left. + 2\left(1 + \frac{\ln k + C_3/C_2}{\ln H}\right)\left(1 + \frac{\sqrt{2}}{X}\ln\frac{C_1(k)}{k}\right)\right].
\end{aligned}$$

This holds if we can show that

$$F_2 > \frac{C_2 k\sqrt{\pi}}{C_1(k)} \exp(-2(\sqrt{2}-1)X)\frac{1}{\sqrt{2X}} \cdot 16.9,$$

since $C_1(k) \leqslant 32\pi$, $H \geqslant 1250$, $X \geqslant \sqrt{2}\ln(1250/32\pi)$, and $k \leqslant H$.

It remains to be proved that

$$\frac{\sqrt{2}C_1(k)}{kC_2\sqrt{\pi}\cdot 16.9}(15/1024 + \cdots) > X^{3/2}\exp(-2(\sqrt{2}-1)X).$$

But for $X \geqslant X_0 := \sqrt{2}\ln\left(\frac{kH}{C_1(k)}\right)$,

$$\begin{aligned}
X^{3/2}\exp(-2(\sqrt{2}-1)X) &< X_0^{3/2}\left(\frac{kH}{C_1(k)}\right)^{-(1+a)} \\
&= \frac{1}{k}\cdot 2^{3/4}\left(\frac{C_1(k)}{H}\right)^{1+a}\left(\frac{\ln^{3/2}(kH/C_1(k))}{k^a}\right),
\end{aligned}$$

where $a = 2\sqrt{2}(\sqrt{2}-1) - 1 \approx 0.17157$. The map $k \mapsto \frac{\ln^{3/2}(kH/C_1(k))}{k^a}$ reaches its maximum for $k = e^{\frac{3}{2a}}\frac{C_1(k)}{H}$. Hence

$$X^{3/2}\exp(-2(\sqrt{2}-1)X) < \frac{C_1(k)}{kH}2^{3/4}\left(\frac{3}{2a}\right)^{3/2}/e^{3/2}.$$

We must compare

$$\frac{\sqrt{2}}{C_2\sqrt{\pi}\cdot 16.9}\,(15/1024+\cdots)\;\text{ with }\;\frac{2^{3/4}(\frac{3}{2a})^{3/2}}{He^{3/2}}.$$

Since $C_1(k)\geqslant 9.14$ (see the remark above (3)) and $C_2=0.9185$, it remains to be proved that

$$0.007976>\frac{2^{3/4}(\frac{3}{2a})^{3/2}}{He^{3/2}}(\approx 0.00776),$$

which is true since $H\geqslant 1250$.

We show below that the remaining terms $(\tilde{C}_H+\tilde{D}_H+3\tilde{E}(H))/\sqrt{x}+\tilde{R}\frac{2}{x\varphi(k)}$ are negligible.

• We will find an upper bound for $A(1,\delta)\frac{\varphi(k)}{2}(\tilde{C}_H+\tilde{D}_H)+\frac{3}{2}\varphi(k)\frac{\tilde{E}(H)}{\sqrt{x}}+\tilde{R}/x$. We assume that $X\geqslant\sqrt{2}\ln\left(\frac{kH}{C_1(k)}\right)$; hence, $X\geqslant X_0:=\sqrt{2}\ln\left(\frac{1250}{32\pi}\right)\approx 3.5644$. It is straightforward but tedious to check that

$$\text{Rest }:=\tilde{C}_H+\tilde{D}_H+3\tilde{E}(H)+\frac{2\tilde{R}}{\varphi(k)\sqrt{x}}\quad\leqslant\quad\begin{cases}1250(\ln H\ln k)^2 & \text{if } k\neq 1,\\ 1250(\ln H)^2 & \text{if } k=1.\end{cases}$$

Let us consider the case $k\neq 1$. As $X\geqslant\sqrt{2}\ln\left(\frac{kH}{C_1(k)}\right)$,

$$\exp\left(\frac{X}{\sqrt{2}}\right)\geqslant\frac{kH}{C_1(k)}.$$

This yields

$$\begin{aligned}\text{Rest }\leqslant 1250(\ln H\ln k)^2 &\leqslant 1250\frac{(\ln H\ln k)^2}{\left(\frac{kH}{C_1(k)}\right)^2}\exp(X\sqrt{2})\\ &\leqslant 1250C_1^2(k)\frac{1}{e^2}\left(\frac{\ln 1250}{1250}\right)^2\exp(X\sqrt{2})\\ &\leqslant K\exp(X\sqrt{2})\quad\text{because } C_1(k)\leqslant 32\pi,\end{aligned}$$

where $K:=55.65$. Now compare

$$\frac{K\exp(X\sqrt{2})}{\sqrt{x}}=K\exp(X\sqrt{2}-RX^2/2)$$

with the term involving $1/X^2$ in $F_1$

$$\frac{1}{X^2}\times\frac{k}{C_1(k)\sqrt{\pi}}X^{3/2}\exp(-2X).$$

We may compute $c$ such that

$$\begin{aligned}K\exp(X\sqrt{2}-RX^2/2)\quad&\leqslant\quad c\times\frac{1}{X^2}\times\frac{k}{C_1(k)\sqrt{\pi}}X^{3/2}\exp(-2X)\\ \Leftrightarrow c\quad&\geqslant\quad K\sqrt{32\pi\sqrt{\pi}}\exp(X\sqrt{2}-RX^2/2+2X)\frac{X^2}{X^{3/2}}\\ \Leftrightarrow c\quad&\geqslant\quad 0.7\cdot 10^{-18}\quad\text{for } X\geqslant X_0.\end{aligned}$$

Thus, the rest is negligible and absorbed by rounding up the constants. □

## 4. The method with $m = 2$

**Lemma 12.** *Let $A(m,\delta)$ be defined as in formula (4). Write*

$$R_m(\delta) = \left(1 + (1+\delta)^{m+1}\right)^m.$$

*Then*

$$A(m,\delta) \leqslant \frac{R_m(\delta)}{\delta^m}.$$

*Proof.* The proof appears in [4, p. 222]. □

**Theorem 5.** *Let an integer $k \geqslant 1$. Remember that $R = 9.645908801$. Let $H \geqslant 1000$. Assume GRH$(k,H)$. Let $C_1(k)$ be defined by (3). Let $X_0$, $X_1$, $X_2$, and $X_3$ be such that*

$$\frac{e^{X_0}}{\sqrt{X_0}} = H\sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}, \quad \frac{e^{X_1}}{X_1} = 10\varphi(k),$$

$$X_2 = kC_1(k)/(2\pi\varphi(k)), \quad X_3 = \frac{2k\pi e}{C_1(k)\varphi(k)}.$$

*Let $X_4 := \max(10, X_0, X_1, X_2, X_3)$. Write*

$$\varepsilon(X) = 3\sqrt{\frac{k}{\varphi(k)C_1(k)}}X^{1/2}\exp(-X).$$

*Then for all real $x$ such that $X = \sqrt{\frac{\ln x}{R}} \geqslant X_4$, we have*

$$\max_{1\leqslant y\leqslant x} |\psi(y;k,l) - y/\varphi(k)| < x\varepsilon\left(\sqrt{\frac{\ln x}{R}}\right),$$

$$\max_{1\leqslant y\leqslant x} |\theta(y;k,l) - y/\varphi(k)| < x\varepsilon\left(\sqrt{\frac{\ln x}{R}}\right).$$

**Corollary 2.** *With the notations and the hypothesis of Theorem 5, let $X_5 \geqslant X_4$ and $c := \varepsilon(X_5)$. For $x \geqslant \exp(RX_5^2)$, we have*

$$|\psi(x;k,l) - x/\varphi(k)|, \quad |\theta(x;k,l) - x/\varphi(k)| < cx.$$

*Proof.* The idea is to judiciously split the integral into two parts, and bound each part optimally, using an $m = 0$ estimate in the first part and an $m = 2$ estimate in the second part.

We want to split the integral at $T$, where $T$ will optimally be chosen later. We take $T$ in the same form as $W_m$ (formula (7)):

$$T := \frac{C_1(k)}{k}\exp(\nu X), \tag{14}$$

where $\nu$ is a parameter.

Assume that $T \geqslant H$ and $1/\sqrt{m+1} \leqslant \nu \leqslant 1$. Hence $W_m \leqslant T \leqslant W_0$. This last hypothesis is needed to apply Corollary 1.

We use Theorem 2 and split the sums at $T$:

$$A(m,\delta)\sum_\chi \sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}} \frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|} + \left(1 + \frac{m\delta}{2}\right)\sum_\chi \sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant T}} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \frac{\tilde{R}}{x}.$$

Define

$$\tilde{A}_1 := \sum_{\chi} \sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant T}} \frac{x^{\beta-1}}{|\rho|},$$

$$\tilde{A}_2 := \sum_{\chi} \sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}} \frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|}.$$

Bounding the term $\tilde{A}_1$, we get

$$\begin{aligned}
\tilde{A}_1 &= \sum_{\chi}\left(\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant H}} \frac{x^{\beta-1}}{|\rho|} + \sum_{\substack{\rho\in\wp(\chi)\\H<|\gamma|\leqslant T}} \frac{x^{\beta-1}}{|\rho|}\right)\\
&= \frac{1}{x}\sum_{\chi}\left(\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant H}} \frac{\sqrt{x}}{|\rho|} + \sum_{\substack{\rho\in\wp(\chi)\\H<|\gamma|\leqslant T}} \frac{x^{\beta}}{|\rho|}\right) \text{ by GRH}(k,H)\\
&= \frac{1}{\sqrt{x}}\sum_{\chi}\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|\leqslant H}} \frac{1}{|\rho|} + \frac{1}{2x}\sum_{\chi}\left(\sum_{\substack{\rho\in\wp(\chi)\\H<|\gamma|\leqslant T}} \frac{x^{\beta}}{|\rho|} + \sum_{\substack{\rho\in\wp(\overline{\chi})\\H<|\gamma|\leqslant T}} \frac{x^{\beta}}{|\rho|}\right)\\
&\leqslant \frac{1}{\sqrt{x}}\varphi(k)\tilde{E}(H) + \frac{1}{2x}\sum_{\chi}\left(\sum_{\substack{\rho\in\wp(\chi)\\H\leqslant|\gamma|\leqslant T}} x\phi_0(\gamma) + \sqrt{x}\sum_{\substack{\rho\in\wp(\chi)\\H\leqslant|\gamma|\leqslant T}} \frac{1}{|\gamma|}\right)\\
&\qquad \text{by Lemmas 3 and 4}\\
&\leqslant \varphi(k)\tilde{E}(T)/\sqrt{x} + \frac{1}{2}\sum_{\chi}\sum_{\substack{\rho\in\wp(\chi)\\H\leqslant|\gamma|\leqslant T}} \phi_0(\gamma).
\end{aligned}$$

Apply Corollary 1 ($j = 1, m = 0$) for the interval $[H,T]$ with $\phi = \phi_0$ and $W = W_0$

$$\sum_{\substack{\rho\in\wp(\chi)\\H\leqslant|\gamma|\leqslant T}} \phi_0(\gamma) = \{1/\pi - q(T)\}\int_H^T \phi_0(y)\ln(ky/2\pi)dy + B_1(T,H,T).$$

Moreover, $B_1(T,H,T) < 2R(T)\phi_0(T)$.

We want to find an upper bound for

$$I_1 := \frac{1}{\pi}\int_H^T \phi_0(y)\ln\left(\frac{ky}{2\pi}\right)dy.$$

Write $V'' = X^2/\ln\left(\frac{kT}{C_1(k)}\right) = X/\nu = Y'' + 2X - \nu X$, where $Y'' := X(1-\nu)^2/\nu$. Write $U'' = X^2/\ln\left(\frac{kH}{C_1(k)}\right)$ and $\Gamma(\alpha,x) = \int_x^\infty e^{-u}u^{\alpha-1}du$. Now

$$\begin{aligned}
\int_H^T \ln\left(\frac{ky}{2\pi}\right)\phi_0(y)dy &= \int_H^T \ln\left(\frac{ky}{2\pi}\right)\exp\left(-X^2/\ln\left(\frac{ky}{C_1(k)}\right)\right)\frac{dy}{y}\\
&= X^4\left\{\Gamma(-2,V'') - \Gamma(-2,U'')\right\}\\
&\quad + X^2\ln\left(\frac{C_1(k)}{2\pi}\right)\left\{\Gamma(-1,V'') - \Gamma(-1,U'')\right\}
\end{aligned}$$

by making the change of variables $y = \frac{C_1(k)}{k}\exp(X^2/u)$. Now if $\alpha \leqslant 1$ and $x > 0$, then $\Gamma(\alpha, x) \leqslant x^{\alpha-1}\int_x^\infty e^{-t}dt = x^{\alpha-1}e^{-x}$. Hence,

$$\int_H^T \ln\left(\frac{ky}{2\pi}\right)\phi_0(y)dy \leqslant X^4 V''^{-3}e^{-V''} + X^2\ln\left(\frac{C_1(k)}{2\pi}\right)V''^{-2}e^{-V''}.$$

This yields

$$\begin{aligned} I_1 &\leqslant \frac{1}{\pi}X^2\left(X^2V''^{-3} + \ln\left(\frac{C_1(k)}{2\pi}\right)V''^{-2}\right)e^{-V''} \\ &= \frac{1}{\pi}e^{-Y''}e^{-2X}\left(\frac{kT}{C_1(k)}\right)\left(\frac{X^4}{(X/\nu)^3} + \frac{dX^2}{(X/\nu)^2}\right) \\ &= \frac{1}{\pi}e^{-Y''}e^{-2X}\left(\frac{kT}{C_1(k)}\right)XG_0, \end{aligned}$$

where $d := \ln\left(\frac{C_1(k)}{2\pi}\right)$ and $G_0 := \nu^2(\nu + d/X)$. With the help of Corollary 1, we write

$$\tilde{A}_1 \leqslant \varphi(k)\tilde{E}(T)/\sqrt{x} + \frac{\varphi(k)}{2}\left\{\frac{1}{\pi}e^{-Y''}e^{-2X}\left(\frac{kT}{C_1(k)}\right)XG_0 + 2R(T)\phi_0(T)\right\}.$$

Bounding the term $\tilde{A}_2$, we get

$$\begin{aligned} \tilde{A}_2 &= \frac{1}{x}\sum_\chi \sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}} \frac{x^\beta}{|\rho(\rho+1)\cdots(\rho+m)|} \\ &= \frac{1}{2x}\sum_\chi\left(\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}}\frac{x^\beta}{|\rho(\rho+1)\cdots(\rho+m)|} + \sum_{\substack{\rho\in\wp(\overline{\chi})\\|\gamma|>T}}\frac{x^\beta}{|\rho(\rho+1)\cdots(\rho+m)|}\right) \\ &\leqslant \frac{1}{2x}\sum_\chi\left(\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}}\frac{x^\beta}{|\gamma|^{m+1}} + \sum_{\substack{\rho\in\wp(\overline{\chi})\\|\gamma|>T}}\frac{x^\beta}{|\gamma|^{m+1}}\right) \\ &= \frac{1}{2x}\sum_\chi\left(x\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}}\phi_m(\gamma) + \sqrt{x}\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}}\frac{1}{|\gamma|^{m+1}}\right) \end{aligned}$$

by Lemma 4.

By using Corollary 1 ($j = 0$) on $[U, V] = [T, \infty)$,

$$\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}}\phi_m(\gamma) \leqslant \{1/\pi + q(T)\}\int_T^\infty \phi_m(y)\ln\left(\frac{ky}{2\pi}\right)dy + B_0(T, T, \infty).$$

We have

$$B_0(T, T, \infty) < 2R(T)\phi_m(T).$$

Moreover,

$$\sum_{\substack{\rho\in\wp(\chi)\\|\gamma|>T}}\frac{1}{|\gamma|^{m+1}} \leqslant \tilde{C}_T + \tilde{D}_T.$$

Let us study more precisely

$$\begin{aligned} I_2 &:= \int_T^\infty \phi_m(y)\ln\left(\frac{ky}{2\pi}\right)dy \\ &= \frac{z_m^2}{2m^2}\left(\frac{k}{C_1(k)}\right)^m\left(K_2(z_m,U_m)+\frac{2md}{z_m}K_1(z_m,U_m)\right), \end{aligned}$$

where $d=\ln\left(\frac{C_1(k)}{2\pi}\right)$ and $U' := U_m = \frac{2m}{z_m}\ln\left(\frac{kT}{C_1(k)}\right) = \nu\sqrt{m}$. Now, by writing $z=z_m$ and using Lemma 8,

$$\begin{aligned} K_2(z,U')+\frac{2dm}{z}K_1(z,U') &< (U'+2/z+2dm/z)Q_1(z,U') \\ &\leqslant \sqrt{m}\left(\nu+\frac{1+dm}{mX}\right)\frac{U'^2}{z(U'^2-1)}e^{-\frac{z}{2}(U'+1/U')}. \end{aligned}$$

But $\frac{z}{2}(U'+1/U') = X\sqrt{m}(\nu\sqrt{m}+1/(\nu\sqrt{m})) = m\nu X + X/\nu = m\nu X + (Y''+2X-\nu X)$, where $Y''=X(1-\nu)^2/\nu$. Hence

$$K_2(z,U')+\frac{2dm}{z}K_1(z,U') < G_1e^{-Y''}\frac{m}{2(m-1)}X^{-1}e^{-2X}\left(\frac{kT}{C_1(k)}\right)^{-(m-1)},$$

where $G_1 := \frac{m-1}{m}\frac{U'^2}{U'^2-1}\left(\nu+\frac{1+dm}{mX}\right)$ because

$$e^{\nu X(m-1)} = \left(\frac{kT}{C_1(k)}\right)^{m-1}$$

and $\frac{\sqrt{m}}{z}=\frac{1}{2}X^{-1}$. This yields

$$I_2 = \int_T^\infty \phi_m(y)\ln(ky/2\pi)dy < \frac{G_1e^{-Y''}}{m-1}\frac{k}{C_1(k)}Xe^{-2X}T^{-(m-1)}$$

Let $G_2 := \frac{R_m(\delta)}{2^m}(1+\pi q(T))$. So, by using Lemma 12,

$$\begin{aligned} &A(m,\delta)\frac{\varphi(k)}{2}(1/\pi+q(T))\int_T^\infty \phi_m(y)\ln(ky/2\pi)dy \\ &\quad< \left(\frac{2}{\delta}\right)^m\frac{\varphi(k)}{2}\left\{\frac{G_2}{\pi}\frac{kG_1e^{-Y''}}{(m-1)C_1(k)}Xe^{-2X}T^{-(m-1)}\right\}. \end{aligned}$$

The results above yield

$$\begin{aligned} &(1+m\delta/2)\tilde{A}_1+A(m,\delta)\tilde{A}_2 \\ &< \frac{XG_2e^{-2X}e^{-Y''}\varphi(k)}{2\pi}\left(\frac{k}{C_1(k)}\right)\left\{\frac{G_1}{m-1}T^{-(m-1)}\left(\frac{2}{\delta}\right)^m+G_0T\right\}+r \end{aligned} \tag{15}$$

because $1+m\delta/2 < R_m(\delta)/2^m < G_2$, with

$$\begin{aligned} r &= \varphi(k)(1+m\delta/2)R(T)\phi_0(T)+A(m,\delta)\varphi(k)R(T)\phi_m(T) \\ &+ \frac{\varphi(k)}{\sqrt{x}}((1+m\delta/2)\tilde{E}(T)+A(m,\delta)(\tilde{C}_T+\tilde{D}_T)/2). \end{aligned}$$

Suppose $G_0/G_1$ were independent of $\nu$; then the expression between braces in (15) would be minimized for

$$T = (G_1/G_0)^{1/m}\cdot\frac{2}{\delta}. \tag{16}$$

With this choice,

$$\frac{G_1}{m-1}T^{-(m-1)}\left(\frac{2}{\delta}\right)^m + G_0T = \frac{m}{m-1}G_1^{1/m}G_0^{1-1/m}\frac{2}{\delta},$$

and we obtain ($G_2 > 1$)

$$\begin{aligned}\varepsilon_1 &:= (1+m\delta/2)\tilde{A}_1 + A(m,\delta)\tilde{A}_2 + \frac{1}{2}m\delta + \frac{\tilde{R}}{x}\\ &< \frac{1}{2}mG_2\left\{Xe^{-2X}e^{-Y''}\frac{2k\varphi(k)}{\delta(m-1)\pi C_1(k)}G_1^{1/m}G_0^{1-1/m} + \delta\right\} + r + \frac{\tilde{R}}{x}.\end{aligned}$$

The expression between braces can be minimized by choosing

$$\delta = \left\{G_0^{1-1/m}G_1^{1/m}e^{-Y''}\frac{2k\varphi(k)}{(m-1)\pi C_1(k)}\right\}^{1/2}X^{1/2}e^{-X}. \tag{17}$$

Hence, we write (by replacing the above value of $\delta$ in (16))

$$T = \left(\frac{G_1}{G_0}\right)^{1/2m}\left(\frac{2C_1(k)}{k\varphi(k)}(m-1)\pi e^{Y''}/G_0\right)^{1/2}X^{-1/2}e^X \tag{18}$$

and

$$\varepsilon_1 < G_2\left(G_0^{1-1/m}G_1^{1/m}e^{-Y''}\frac{2k\varphi(k)}{\pi C_1(k)}\right)^{1/2}\frac{m}{\sqrt{m-1}}X^{1/2}e^{-X} + r + \frac{\tilde{R}}{x}. \tag{19}$$

The value $m = 2$ minimizes the expression $\frac{m}{\sqrt{m-1}}$. For the remainder of the argument, we fix $m = 2$.

We now have two definitions for $T$. On the one hand (equation (18)),

$$T = \left(\frac{G_1}{G_0^3}\right)^{1/4}e^{Y''/2}\sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}}X^{-1/2}e^X$$

with $Y'' = X(1-\nu)^2/\nu$, and on the other hand (equation (14))

$$T = \frac{C_1(k)}{k}\exp(\nu X).$$

These two equations are compatible if and only if there exists $\nu$ such that $f(\nu) = 1$, where

$$f(\nu) = \frac{C_1(k)\varphi(k)}{2\pi k}\left(\frac{G_0^3}{G_1}\right)^{1/2}Xe^{-X(1-\nu)^2/\nu}e^{-2X(1-\nu)}.$$

Here we have $m = 2$ and our assumption $1/\sqrt{m+1} \leqslant \nu \leqslant 1$ gives $1/\sqrt{3} \leqslant \nu \leqslant 1$. Note that

$$\begin{aligned}G_0 &= \nu^2(\nu + d/X),\\ G_1 &= \frac{m-1}{m}\frac{U'^2}{U'^2-1}\left(\nu + \frac{1+dm}{mX}\right) = \frac{\nu^2}{2\nu^2-1}\left(\nu + \frac{1+2d}{2X}\right).\end{aligned}$$

It is easy to check that on the interval $1/\sqrt{2} \leqslant \nu \leqslant 1$, $G_0^3/G_1$ is increasing, and hence, $f(\nu)$ is strictly increasing. Moreover, $\lim_{\nu\to(1/\sqrt{2})^+} f(\nu) = 0$ and $f(1) > 1$

(for all $X \geqslant \frac{2\pi k}{C_1(k)\varphi(k)}$). So there exists a unique $\nu \in ]1/\sqrt{2}, 1[$ such that $f(\nu) = 1$. For $1/\sqrt{2} < \nu < 1$, we have ($m = 2$)

$$H(\nu) := \frac{G_0^3}{G_1} = \frac{[\nu^2(\nu + d/X)]^3}{\frac{\nu^2}{2\nu^2-1}(\nu + \frac{1+2d}{2X})} < (\nu + d/X)^2.$$

Write, for $X \geqslant X_3 := \frac{2\pi k e}{C_1(k)\varphi(k)}$,

$$\nu_0 = 1 - \frac{1}{2X} \ln\left(\frac{C_1(k)\varphi(k)X}{2k\pi}\right). \tag{20}$$

Let us study $H(\nu_0)$:

$$\begin{aligned} H(\nu_0) < 1 \quad &\text{if} \quad \nu_0 + d/X \leqslant 1,\\ \text{equivalently} \qquad & 1 - \frac{1}{2X}\ln\left(\frac{C_1(k)\varphi(k)X}{2\pi k}\right) + \frac{\ln(C_1(k)/2\pi)}{X} \leqslant 1,\\ \text{which holds if} \qquad & X \geqslant X_2 := \frac{kC_1(k)}{2\pi\varphi(k)}. \end{aligned}$$

As

$$f(\nu) = \frac{C_1(k)\varphi(k)}{2k\pi}\left(\frac{G_0^3}{G_1}\right)^{1/2} X \exp(-X(1-\nu)^2/\nu)\exp(-2X(1-\nu)),$$

replacing $\nu_0$ by (20), we obtain

$$f(\nu_0) = \left(\frac{G_0^3}{G_1}\right)^{1/2} \exp\left(-\ln^2\left(\frac{C_1(k)\varphi(k)X}{2k\pi}\right)/(4\nu_0 X)\right).$$

Assume that $\nu_0 > 0$, then, for $X \geqslant X_2$, $f(\nu_0) < 1 = f(\nu)$ and hence $\nu_0 < \nu$. We will require $X \geqslant X_2$.

The assumption $T \geqslant H$ holds if $T \geqslant \frac{C_1(k)}{k}\exp(\nu_0 X) \geqslant H$. Using (20), rewrite $\frac{C_1(k)}{k}\exp(\nu_0 X) = \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} e^{X - \frac{1}{2}\ln X}$. Let $X_0$ satisfy

$$e^{X_0 - \frac{1}{2}\ln X_0} = H\sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}.$$

We have $T \geqslant H$ provided that $X \geqslant X_0$. We will require $X \geqslant X_0$.

For $X \geqslant X_3 = \frac{2k\pi e}{C_1(k)\varphi(k)}$, $\nu_0$ is an increasing function of $X$. We will require that $X \geqslant \max(X_3, 10)$. Then since $C_1(k) \leqslant 32\pi$ and $X \geqslant 10$, we have

$$\nu_0 > 0.7462413 \quad \text{and} \quad \nu_0 < \nu < 1.$$

The assumption $\nu > 1/\sqrt{2}$ is satisfied.

We want to evaluate

$$K := G_2(\sqrt{G_0 G_1} e^{-Y''})^{1/2}, \tag{21}$$

which appears in (19). Again using $C_1(k) \leqslant 32\pi$ and $X \geqslant 10$, we find

$$\begin{aligned} G_0 G_1 &< (1 + d/X)\frac{\nu_0}{2\nu_0^2 - 1}\left(\nu_0 + \frac{1+2d}{2X}\right)\\ &< 8.995. \end{aligned}$$

The following results will be needed in later computations.

1. Since $X \geqslant X_0$ and $\exp(X)/\sqrt{X}$ is increasing for $X \geqslant 1/2$,
$$\sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}X^{1/2}\exp(-X) \leqslant \frac{1}{H}.$$
2. Since $G_0G_1 < 9$,
$$\begin{aligned}\delta &= 2\sqrt[4]{G_0G_1}\exp(-Y''/2)\sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}X^{1/2}e^{-X}\\ &\leqslant 2\sqrt{3}/H.\end{aligned}$$
In particular, for $H \geqslant 1000$, we have $\delta \leqslant 0.00347$.
3.
$$G_2 = \frac{R_2(\delta)}{2^2}(1+\pi q(T)) < (1+3.012\cdot\delta/2)^2(1+\pi q(T)),$$
because
$$\begin{aligned}\frac{R_2(\delta)}{2^2} &= \left\{\frac{(1+\delta)^3+1}{2}\right\}^2\\ &= \left\{1+\frac{1}{2}\delta(3+3\delta+\delta^2)\right\}^2 < \left(1+\frac{3.012}{2}\delta\right)^2\end{aligned}$$
since $1+\delta+\delta^2/3 < 1.0035$.
4. Since $T \geqslant H$,
$$\begin{aligned}q(T) &= \frac{C_2}{T\ln(kT/2\pi)}\\ &\leqslant \frac{C_2}{H\ln(kH/2\pi)}.\end{aligned}$$
But $\exp(-Y''/2) \leqslant 1$ and $H \geqslant 1000$, so this yields
$$\begin{aligned}K &< (8.995)^{1/4}G_2\\ &< (8.995)^{1/4}\left(1+\frac{\pi C_2}{1000\ln(1000/(2\pi))}\right)\times\left(1+\frac{3.012}{2}\frac{2\sqrt{3}}{1000}\right)^2\\ &< 1.751.\end{aligned}$$

Inserting this upper bound of $K$ (see formula (21) in (19), we obtain
$$\begin{aligned}\varepsilon_1 &< 2\sqrt{\frac{2}{\pi}}K\sqrt{\frac{k\varphi(k)}{C_1(k)}}X^{1/2}\exp(-X)+r+\frac{\tilde{R}}{x}\\ &< 2.7941\sqrt{\frac{k\varphi(k)}{C_1(k)}}X^{1/2}\exp(-X)+r+\frac{\tilde{R}}{x}. \qquad (22)\end{aligned}$$

Now we want to bound $r$ and $\frac{\tilde{R}}{x}$.

• An upper bound for $\varphi(k)(1+\delta)R(T)\phi_0(T)$ and $\varphi(k)A(2,\delta)R(T)\phi_2(T)$. Recall that
$$\begin{aligned}R(T) &= C_2\ln(kT)+C_3,\\ \phi_0(T) &= \frac{1}{T}\exp\left(-X^2/\ln(kT/C_1(k))\right),\\ \phi_m(T) &= \phi_0(T)T^{-m}.\end{aligned}$$

Now

$$\phi_0(T)=\frac{1}{T}\exp(-X^2/(\nu X))=\frac{1}{T}\exp(-\frac{1}{\nu}X)\leqslant\frac{1}{T}\exp(-X)$$

and

$$\frac{1}{T}=X^{1/2}\exp(-X)\sqrt{\frac{k\varphi(k)}{C_1(k)}}\left(\frac{G_0}{2\pi e^{Y''}}\right)^{1/2}\left(\frac{G_0}{G_1}\right)^{1/4},$$

hence

$$\begin{aligned}R(T)\phi_0(T)&\leqslant\frac{C_2\ln(kT)+C_3}{T}\exp(-X)\\&\leqslant\sqrt{X}e^{-X}\sqrt{\frac{k\varphi(k)}{C_1(k)}}\left[(C_2\ln(kT)+C_3)\left(\frac{G_0}{2\pi e^{Y''}}\right)^{1/2}\left(\frac{G_0}{G_1}\right)^{1/4}e^{-X}\right].\end{aligned}$$

But

$$\begin{aligned}G_0&\leqslant 1+\frac{\ln(C_1(k)/2\pi)}{X},\\ \frac{G_0}{G_1}&\leqslant 2\nu^2-1<1\quad(m=2),\\ \exp(Y'')&\geqslant 1,\\ \ln(kT)&=\nu X+\ln(C_1(k))\leqslant X+\ln(C_1(k))\leqslant X+\ln(32\pi).\end{aligned}$$

So, since $X\geqslant 10$ and $C_1(k)\leqslant 32\pi$,

$$\begin{aligned}&(1+\delta)\varphi(k)\left[(C_2\ln(kT)+C_3)\left(\frac{G_0}{2\pi e^{Y''}}\right)^{1/2}\left(\frac{G_0}{G_1}\right)^{1/4}\exp(-X)\right]\\&\leqslant\varphi(k)\left(1+\frac{2\sqrt{3}}{1000}\right)[C_2(X+\ln 32\pi)+C_3]\sqrt{\frac{1+\ln 16/10}{2\pi}}\exp(-X)\\&\leqslant 0.857\varphi(k)X\exp(-X).\end{aligned}$$

Furthermore, if $X_1$ is defined by $\exp(X_1)/X_1=10\varphi(k)$, and if we require that $X\geqslant X_1$, then this term is bounded by 0.0857. Hence, under the hypotheses on $X$ in Theorem 5, an upper bound for $\varphi(k)(1+\delta)R(T)\phi_0(T)$ is

$$0.09\cdot\sqrt{\frac{k\varphi(k)}{C_1(k)}}X^{1/2}\exp(-X).$$

Next, by (16)

$$\delta T=2\sqrt{\frac{G_1}{G_0}}.$$

Hence, by Lemma 12

$$A(2,\delta)/T^2\leqslant\frac{R_2(\delta)}{(\delta T)^2}\leqslant\frac{R_2(\delta)}{2^2}\frac{G_0}{G_1}\leqslant\frac{R_2(\delta)}{2^2}$$

and

$$\varphi(k)A(2,\delta)R(T)\phi_2(T)\leqslant\varphi(k)\frac{R_2(\delta)}{2^2}R(T)\phi_0(T).$$

Using $\delta \leqslant 2\sqrt{3}/H \leqslant 2\sqrt{3}/1000$, we get $R_2(\delta)/2^2 \leqslant 1.0147$. Under the hypotheses on X in Theorem 5, an upper bound for $\varphi(k)A(2,\delta)R(T)\phi_2(T)$ is therefore

$$0.087 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X).$$

The sum of the two terms can be bounded by

$$0.2 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X). \tag{23}$$

- An upper bound for $(1+\delta)\tilde{E}(T)\frac{\varphi(k)}{\sqrt{x}} + A(2,\delta)\frac{\varphi(k)}{2\sqrt{x}}(\tilde{C}_T + \tilde{D}_T) + \tilde{R}/x$.

  For $f(k) = \sum_{p|k} \frac{1}{p-1}$ observe that (Lemma 10)

$$f(k) \leqslant \frac{\ln k}{\ln 2}.$$

We can explicitly rewrite for $m=2$, $H \geqslant 1000$, and $C_1(k) \leqslant 32\pi$ the following expressions:

$$\begin{aligned}
3\tilde{E}(T) &= 3\left(\frac{1}{2\pi}\ln^2 T + \frac{1}{\pi}\ln\left(\frac{k}{2\pi}\right)\ln T + C_2\right.\\
&\qquad \left. + 2\left(\frac{1}{\pi}\ln\left(\frac{k}{2\pi e}\right) + C_2 \ln k + C_3\right)\right),\\
\tilde{C}_T &= \frac{1}{2\pi T^2}\left(\ln\left(\frac{kT}{2\pi}\right) + 1/2\right),\\
\tilde{D}_T &= (2C_2\ln(kT) + 2C_3 + C_2/3)/T^3,\\
\frac{\tilde{R}}{\varphi(k)\sqrt{x}} &\leqslant \left[(f(k)+0.5)\ln x + 4\ln k + 13.4\right]/\sqrt{x}.
\end{aligned}$$

It is tedious but easy to check that the sum of the above quantities is less than

$$\begin{cases} 1000(\ln T\sqrt{\ln k})^2 & \text{for } k \neq 1,\\ 1000\ln^2 T & \text{for } k = 1.\end{cases}$$

Now we want to find a number $c$ such that

$$A(2,\delta)\varphi(k)\frac{1000(\ln T\sqrt{\ln k})^2}{\sqrt{x}} \leqslant c\left(\frac{k\varphi(k)}{C_1(k)}\right)^{1/2} X^{1/2}\exp(-X)$$

with $X = \sqrt{\frac{\ln x}{R}}$. But $A(2,\delta) \leqslant \frac{R_2(\delta)}{\delta^2}$ and by (16), $T = \left(\frac{G_1}{G_0}\right)^{1/2}\frac{2}{\delta}$, so

$$A(2,\delta) \leqslant \frac{R_2(\delta)}{2^2}T^2\frac{G_0}{G_1}.$$

Moreover, $\frac{1}{\sqrt{x}} = \exp(-RX^2/2)$, hence

$$c \geqslant 1000\frac{R_2(\delta)}{2^2}\frac{G_0}{G_1}T^2\varphi(k)(\ln T\sqrt{\ln k})^2\left(\frac{C_1(k)}{k\varphi(k)}\right)^{1/2} X^{-1/2}\exp(X - RX^2/2).$$

As $\frac{G_0}{G_1} < 1$, $T^2 = \frac{C_1^2(k)}{k^2}\exp(2\nu X) \leqslant \frac{C_1^2(k)}{k^2}\exp(2X)$, hence it suffices to take

$$c \geqslant 1000\frac{R_2(\delta)}{2^2}C_1^2(k)\frac{\ln k}{k^2}(\ln(C_1(k)/k) + X)^2\left(\frac{C_1(k)\varphi(k)}{kX}\right)^{1/2}\exp(3X - RX^2/2).$$

But $\frac{\varphi(k)}{k} \leqslant 1$, $\frac{\ln k}{k^2} \leqslant 1$, and $R_2(\delta) \leqslant (1+3.012\delta/2)^2$ with $\delta \leqslant \frac{2\sqrt{3}}{H} \leqslant \frac{2\sqrt{3}}{1000}$. So, finally, it suffices to take

$$c \geqslant \frac{1000}{4}\left(1+\frac{3.012\sqrt{3}}{1000}\right)^2 C_1^2(k)(\ln C_1(k)+X)^2\sqrt{C_1(k)}X^{-1/2}\exp(3X-RX^2/2).$$

Since $C_1(k) \leqslant 32\pi$ and $X \geqslant 10$, we can take

$$c = 0.643 \cdot 10^{-187}. \tag{24}$$

In the case $k=1$, we can replace the upper bound $\frac{\ln k}{k^2} \leqslant 1$ by 1, and obtain the same result. Combining (22), (23), and (24), we obtain the result in Theorem 5; more precisely, for all $X$ satisfying the conditions of the theorem,

$$\mid \psi(x;k,l) - x/\varphi(k) \mid /x \leqslant 2.9941\sqrt{\frac{k}{\varphi(k)C_1(k)}}X^{1/2}\exp(-X).$$

We also wish to allow $\theta$ instead of $\psi$, which can be done by recalling Theorem 13 of [5]:

$$0 \leqslant \psi(x;k,l) - \theta(x;k,l) \leqslant \psi(x) - \theta(x) \leqslant 1.43\sqrt{x} \quad \text{for } x \geqslant 0.$$

Using $X \geqslant 10$, we find $1.43\sqrt{x}/x \leqslant d \cdot 3(k\varphi(k))/C_1(k)\ X^{1/2}\exp(-X)$, where $d = 1.17 \cdot 10^{-204}$. This difference is absorbed by rounding up the constants. □

## 5. Application for $k=3$

Now we are able to compute $x_0$ and $c$ such that, for $x \geqslant x_0$,

$$\mid \theta(x;3,l) - x/2 \mid < cx/\ln x.$$

This would not have been possible if we had used only the results of [3].

According to Theorem 5,

$$\varepsilon(X) = \frac{3}{2}\sqrt{\frac{6}{20.92}}X^{1/2}\exp(-X)$$

for $k=3$.

To determine for which $x$ this bound is valid, let us solve for the constants $X_0$, $X_1$, $X_2$, $X_3$ in Theorem 5. Noting that $H_3 = 10000$ by the table in Theorem 1, we need $X_0$ to satisfy

$$\exp(X_0 - \frac{1}{2}\ln X_0) \geqslant 10000\sqrt{\frac{6}{2\pi \cdot 20.92}} \approx 2136.51.$$

$X_0 \approx 8.76$ works.

Find $X_1$ such that

$$\exp(X_1 - \ln X_1) \geqslant 20.$$

$X_1 \approx 4.5$ works.

Compute the two other bounds: $X_2 \approx 4.99$, $X_3 \approx 1.22$. Thus we can take $X = \max(10, X_0, X_1, X_2, X_3) = 10$ in Theorem 5.

- For $\sqrt{\frac{\ln x}{R}} \geqslant 10$, write $X = \sqrt{\frac{\ln x}{R}}$, then

  $$\varepsilon(X)\ln x = RX^2\varepsilon(X).$$

  Find the value $c$ such that

  $$\varepsilon(X) < c/\ln(x).$$

For any $x$ such that $\sqrt{\frac{\ln x}{R}} \geqslant 10$, $c \leqslant R \cdot 10^2 \varepsilon(10) \leqslant 0.12$. Hence we have for $x \geqslant \exp(964.59\cdots)$,

$$|\theta(x;3,l) - x/2| \leqslant 0.12\frac{x}{\ln x}.$$

We want to extend the above result for $x \leqslant \exp(964.59\cdots)$. Olivier Ramaré has kindly computed some additional values supplementing Table 1 in [3]. We have

$$|\theta(x;3,l) - x/2| < \tilde{c} \cdot x/2$$

with

$$\begin{aligned}
\tilde{c} &= 0.0008464421 \text{ for } \ln x >= 400 \quad (m = 3, \delta = 0.00042325),\\
\tilde{c} &= 0.0006048271 \text{ for } \ln x >= 500 \quad (m = 3, \delta = 0.00030250),\\
\tilde{c} &= 0.0004190635 \text{ for } \ln x >= 600 \quad (m = 2, \delta = 0.00027950).
\end{aligned}$$

Hence,

- For $e^{600} \leqslant x \leqslant e^{964.59\dots}$

$$c \leqslant 0.0004190635 \cdot 964.6/\varphi(3) \leqslant 0.203.$$

- For $e^{400} \leqslant x \leqslant e^{600}$

$$c \leqslant 0.0008464421 \cdot 600/\varphi(3) \leqslant 0.254.$$

Using the computations of [3],

- For $10^{100} \leqslant x \leqslant e^{400}$

$$c \leqslant 0.001310 \cdot 400/\varphi(3) \leqslant 0.262.$$

- For $10^{30} \leqslant x \leqslant 10^{100}$

$$c \leqslant 0.001813 \cdot 100 \ln 10/\varphi(3) \leqslant 0.42/2 \leqslant 0.21.$$

- For $10^{13} \leqslant x \leqslant 10^{30}$

$$c \leqslant 0.001951 \cdot 30 \ln 10/\varphi(3) \leqslant 0.14/2 \leqslant 0.07.$$

- For $10^{10} \leqslant x \leqslant 10^{13}$

$$c \leqslant 0.002238 \cdot 13 \ln 10/\varphi(3) \leqslant 0.067/2 \leqslant 0.00335.$$

- For $4403 \leqslant x \leqslant 10^{10}$

$$\mid \theta(x;3,l) - x/2 \mid < 2.072\sqrt{x} \quad \text{(Theorem 5.2.1 of Ramaré and Rumely [3])}$$

We choose $c = 0.262$. We check that this bound is also valid for $1531 \leqslant x \leqslant 4403$.

**Theorem 6.** *For $x \geqslant 1531$,*

$$\mid \theta(x;3,l) - x/2 \mid \leqslant 0.262\frac{x}{\ln x}.$$

## 6. Results assuming GRH($k$,$\infty$)

Assuming GRH($k$,$\infty$), we obtain more precise results. Under this hypothesis, one can show that function $\psi$ has the following asymptotic behaviour:

**Proposition 1** ([8, p. 294])**.** *Assume GRH$(k,\infty)$. Then*

$$\psi(x;k,l) = \frac{x}{\varphi(k)} + O(\sqrt{x}\ln^2 x).$$

**Theorem 7.** *Let $x \geqslant 10^{10}$. Let $k$ be a positive integer. Assume GRH$(k,\infty)$.*

1) *If $k \leqslant \frac{4}{5}\ln x$, then*

$$|\psi(x;k,l) - \frac{x}{\varphi(k)}| \leqslant 0.085\sqrt{x}\ln^2 x.$$

2) *If $k \leqslant 432$, then*

$$|\psi(x;k,l) - \frac{x}{\varphi(k)}| \leqslant 0.061\sqrt{x}\ln^2 x.$$

*Proof.* Let $x_0 = 10^{10}$. Applying Theorem 2 in the same way as Theorem 3 (assume that $T \geqslant 1$),

$$\begin{aligned}
&\frac{\varphi(k)}{x}|\psi(x;k,l) - \frac{x}{\varphi(k)}| \\
&\leqslant \quad A(m,\delta)\sum_{\chi}\sum_{|\gamma|>T}\frac{x^{-1/2}}{|\rho(\rho+1)\cdots(\rho+m)|} \\
&\qquad +(1+m\delta/2)\sum_{\chi}\sum_{|\gamma|\leqslant T}\frac{x^{-1/2}}{|\rho|} + m\delta/2 + \tilde{R}/x \\
&\leqslant \quad A(m,\delta)\frac{1}{\sqrt{x}}\sum_{\chi}\sum_{|\gamma|>T}\frac{1}{|\gamma|^{m+1}} + (1+\frac{m\delta}{2})\frac{1}{\sqrt{x}}\sum_{\chi}\sum_{|\gamma|\leqslant T}\frac{1}{|\rho|} + \frac{m\delta}{2} + \frac{\tilde{R}}{x} \\
&\leqslant \quad A(m,\delta)\frac{\varphi(k)}{\sqrt{x}}(\tilde{C}_T + \tilde{D}_T) + (1+\frac{m\delta}{2})\frac{\varphi(k)}{\sqrt{x}}\tilde{E}(T) + \frac{m\delta}{2} + \tilde{R}/x.
\end{aligned}$$

Take $m = 1$ and let

$$\varepsilon_k(x,T,\delta) := \frac{R_1(\delta)}{\delta}\frac{\varphi(k)}{\sqrt{x}}(\tilde{C}_T + \tilde{D}_T) + \left(1+\frac{\delta}{2}\right)\frac{\varphi(k)}{\sqrt{x}}\tilde{E}(T) + \frac{\delta}{2} + \tilde{R}/x, \tag{25}$$

where

$$\begin{aligned}
\tilde{C}_T &= \frac{1}{\pi T}\left(\ln\left(\frac{kT}{2\pi}\right) + 1\right), \\
\tilde{D}_T &= \frac{1}{T^2}\left(2C_2\ln(kT) + 2C_3 + C_2/2\right), \\
\tilde{E}(T) &= \frac{1}{2\pi}\ln^2 T + \frac{1}{\pi}\ln(k/(2\pi))\ln T + C_2 + 2\left(\frac{1}{\pi}\ln\left(\frac{k}{2\pi e}\right) + C_2\ln k + C_3\right).
\end{aligned}$$

Choose

$$T = \frac{2R_1(\delta)}{\delta(2+\delta)} \tag{26}$$

to minimize in (25) the preponderant terms involving $T$. So

$$\begin{aligned}\frac{R_1(\delta)}{\delta}(\tilde{C}_T+\tilde{D}_T) &= \frac{2(2+\delta)}{4\pi}\left[\ln\left(\frac{kR_1(\delta)}{\pi\delta(2+\delta)}\right)+1\right.\\ &\left.+\frac{\pi\delta(2+\delta)}{2R_1(\delta)}\left(2C_2\ln\left(\frac{2kR_1(\delta)}{\delta(2+\delta)}\right)+2C_3+C_2/2\right)\right],\\ (1+\delta/2)\tilde{E}(T) &= \frac{2+\delta}{4\pi}\left[\ln^2\left(\frac{2R_1(\delta)}{\delta(2+\delta)}\right)+2\ln(k/(2\pi))\ln\left(\frac{2R_1(\delta)}{\delta(2+\delta)}\right)\right.\\ &\left.+2\pi C_2+4\pi\left(\frac{1}{\pi}\ln(k/(2\pi e))+C_2\ln k+C_3\right)\right].\end{aligned}$$

With the choice of $T$, the main terms of $\varepsilon_k$ are

$$\frac{\varphi(k)}{\sqrt{x}}\frac{1}{2\pi}\ln^2\left(\frac{2R_1(\delta)}{\delta(\delta+2)}\right)+\frac{\delta}{2}.$$

These terms are minimized by choosing

$$\delta=\frac{\varphi(k)\ln x}{\pi\sqrt{x}}. \tag{27}$$

Now, replacing (26) and (27) in (25), we only have a function of $x$ for fixed $k$:

$$\varepsilon_k(x):=\varepsilon_k(x,T,\delta).$$

We simplify expression (25):

$$\begin{aligned}\frac{\varepsilon_k(x,T,\delta)}{\varphi(k)} &\leqslant \tilde{\varepsilon}_k(x,T,\delta)\\ &:= \frac{R_1(\delta)}{\delta}(\tilde{C}_T+\tilde{D}_T)/\sqrt{x}+(1+\frac{\delta}{2})\tilde{E}(T)/\sqrt{x}+\frac{\delta}{2}+\frac{\tilde{R}}{x\varphi(k)}.\end{aligned}$$

By choosing $T=\frac{2R_1(\delta)}{\delta(2+\delta)}$ and $\delta=\frac{\ln x}{\pi\sqrt{x}}$, $\tilde{\varepsilon}_k(x,T,\delta)$ became $\tilde{\varepsilon}_k(x)$.

Hence,

$$\begin{aligned}\tilde{\varepsilon}_k(x)\sqrt{x} &= \frac{2+\delta}{4\pi}\left[\ln^2\left(\frac{2\pi\sqrt{x}}{\ln x}\cdot\frac{R_1(\delta)}{2+\delta}\right)+2\ln\left(\frac{k}{2\pi}\right)\ln\left(\frac{2\pi\sqrt{x}}{\ln x}\cdot\frac{R_1(\delta)}{2+\delta}\right)\right.\\ &\left.+2\ln\left(\frac{k\sqrt{x}}{\ln x}\cdot\frac{R_1(\delta)}{2+\delta}\right)+\frac{\ln x}{\sqrt{x}}\frac{2+\delta}{R_1(\delta)}(A)\right]+\frac{\ln x}{2\pi\varphi(k)}+\frac{\tilde{R}}{\varphi(k)\sqrt{x}}\\ &+\frac{2+\delta}{4\pi}(2+2\pi C_2+4\pi(\frac{1}{\pi}\ln(k/(2\pi e))+C_2\ln k+C_3)\end{aligned}$$

with

$$A=2C_2\ln\left(\frac{2k\pi\sqrt{x}}{\ln x}\cdot\frac{R_1(\delta)}{2+\delta}\right)+2C_3+C_2/2.$$

Let $\delta_1=\frac{\ln x_0}{\pi\sqrt{x_0}}$. But $\frac{R_1(\delta)}{2+\delta}=\frac{2+2\delta+\delta^2}{2+\delta}=1+\frac{\delta^2+\delta}{2+\delta}\leqslant d_1:=1+\frac{\delta_1^2+\delta_1}{2+\delta_1}$ because $x\geqslant x_0$ and $\frac{2+\delta}{R_1(\delta)}<1$.

By direct computation, for all $k$ between 1 and 432 and $x\geqslant x_0$, of $\frac{\varepsilon_k(x)\sqrt{x}}{\varphi(k)\ln^2 x}$, we find an upper bound 0.06012.

To obtain 1) in Theorem 7, we will study the sum in brackets for $1 \leqslant k \leqslant \frac{4}{5}\ln x$:

$$\begin{aligned}[\cdots] \quad = \quad & \left[\frac{1}{4}\ln^2 x + \ln^2\left(\frac{2\pi d_1}{\ln x}\right) + \ln x \ln\left(\frac{2\pi d_1}{\ln x}\right) + 2\ln\left(\frac{4\ln x}{10\pi}\right)\ln\left(\frac{2\pi d_1}{\ln x}\right)\right.\\ & \left. + \ln\left(\frac{4\ln x}{10\pi}\right)\ln x + \frac{1}{2}\ln x + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}}(A)\right]\\ = \quad & \left[\frac{1}{4}\ln^2 x + \ln x(\ln\left(\frac{2\pi d_1}{\ln x}\right) + 1/2 + \ln(4\ln x/(10\pi)))\right.\\ & \left. + \ln^2\left(\frac{2\pi d_1}{\ln x}\right) + 2\ln\left(\frac{4\ln x}{10\pi}\right)\ln\left(\frac{2\pi d_1}{\ln x}\right) + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}}(A)\right].\end{aligned}$$

We conclude that

$$\lim_{x\to+\infty}\frac{\varepsilon_k(x)\sqrt{x}}{\ln^2 x} = \frac{1}{8\pi},$$

which is the same asymptotic bound as Schoenfeld's [7] for $\psi$.

The bound $\varepsilon_k(x)\sqrt{x}$ is an increasing function of $k$. Choose $k = \frac{4}{5}\ln x$. Now $\varepsilon_k(x)\sqrt{x}/\ln^2 x$ is a decreasing function of $x$ bounded by 0.0849229 for $x \geqslant x_0$. $\square$

*Remark.* If we take $k = 1$ in Theorem 7, our upper bound is twice as bad as the result of Schoenfeld [7, p. 337]: for $x > 73.2$,

$$|\psi(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\ln^2 x.$$

These differences are explained by:

- an exact computation of zeros with $\gamma \leqslant D \approx 158$ (the preponderant ones!) in the sum $\sum \frac{1}{|\rho|}$;
- a better knowledge of $R(T)$ ($k$ fixed, $k = 1$).

**Corollary 3.** *Assume $GRH(k,\infty)$. For all $k$ used in Lemma 2 and $x \geqslant 224$,*

$$\left|\psi(x;k,l) - \frac{x}{\varphi(k)}\right| \leqslant \frac{1}{4\pi}\sqrt{x}\ln^2 x.$$

*Proof.* We use Theorem 5.2.1 of [3]: for all $k$ noted in Lemma 2 and $224 \leqslant x \leqslant 10^{10}$,

$$|\psi(x;k,l) - \frac{x}{\varphi(k)}| \leqslant \sqrt{x}$$

and $\sqrt{x} < \frac{1}{4\pi}\sqrt{x}\ln^2 x$ for $x \geqslant 35$. We conclude by Theorem 7. $\square$

## 7. Estimates for $\pi(x;3,l)$

**Definition 1.** Let

$$\pi(x;k,l) = \sum_{\substack{p\leqslant x\\ p\equiv l \bmod k}} 1$$

be the number of primes smaller than $x$ which are congruent to $l$ modulo $k$.

Our aim is to have bounds for $\pi(x;3,l)$. We show that

**Theorem 8.** *For $l = 1$ or 2,*

(i) $\frac{x}{2\ln x} < \pi(x;3,l)$ *for* $x \geqslant 151$,
(ii) $\pi(x;3,l) < 0.55\frac{x}{\ln x}$ *for* $x \geqslant 229869$.

From this, we can deduce that for all $x \geqslant 151$,

$$\frac{x}{\ln x} < \pi(x)$$

because

$$\pi(x) = \pi(x;3,1) + \pi(x;3,2) + 1.$$

7.1. **The upper bound.** First we give the proof of Theorem 8 (ii).

**Lemma 13.** *Let* $I_n = \int_a^x \frac{dt}{\ln^n t}$*. Then* $I_n = \frac{x}{\ln^n x} - \frac{a}{\ln^n a} + nI_{n+1}$*. Furthermore,*

$$\text{for } a > e, \quad (x-a)/\ln^n(x) \leqslant I_n \leqslant (x-a)/\ln^n(a).$$

**Theorem 9** (Ramaré and Rumely [3])**.** *For* $1 \leqslant x \leqslant 10^{10}$*, for all* $k \leqslant 72$*, for all* $l$ *relatively prime with* $k$*,*

$$\max_{1\leqslant y\leqslant x} \mid \theta(y;k,l) - \frac{y}{\varphi(k)} \mid \leqslant 2.072\sqrt{x}.$$

*Furthermore, for* $x \geqslant 10^{10}$ *and* $k = 3$ *or* $4$*,*

$$\mid \theta(x;k,l) - \frac{x}{\varphi(k)} \mid \leqslant 0.002238 \frac{x}{\varphi(k)}.$$

Write first

$$\pi(x;k,l) - \pi(x_0;k,l) = \frac{\theta(x;k,l)}{\ln(x)} - \frac{\theta(x_0;k,l)}{\ln(x_0)} + \int_{x_0}^{x} \frac{\theta(t;k,l)}{t\ln^2 t}dt.$$

Put $x_0 := 10^5$.

Preliminary computations :

$\theta(10^5,3,1) = 49753.417198\cdots$ $\quad\quad$ $\pi(10^5,3,1) = 4784.$

$\theta(10^5,3,2) = 49930.873458\cdots$ $\quad\quad$ $\pi(10^5,3,2) = 4807.$

Put $c_0 := \frac{1.002238}{2}$ and $K = \max_l(\pi(10^5,3,l) - \theta(10^5,3,l)/\ln(10^5)) \approx 470.$

• For $10^{20} \leqslant x$,

$$\pi(x;k,l) - \pi(10^5;k,l) = \frac{\theta(x;k,l)}{\ln(x)} - \frac{\theta(10^5;k,l)}{\ln(10^5)} + \int_{10^5}^{x} \frac{\theta(t;k,l)}{t\ln^2 t}dt.$$

But

$$\int_{10^5}^{x} \frac{\theta(t;k,l)}{t\ln^2 t}dt = \int_{10^5}^{10^{10}} \frac{\theta(t;k,l)}{t\ln^2 t}dt + \int_{10^{10}}^{\sqrt{x}} \frac{\theta(t;k,l)}{t\ln^2 t}dt + \int_{\sqrt{x}}^{x} \frac{\theta(t;k,l)}{t\ln^2 t}dt$$

and, by Theorem 9

$$\begin{aligned}
\int_{10^5}^{10^{10}} \frac{\theta(t;k,l)}{t\ln^2 t}dt &< M := 1/\varphi(k)\cdot\int_{10^5}^{10^{10}} \frac{dt}{\ln^2 t} + 2.072\cdot\int_{10^5}^{10^{10}} \frac{dt}{\sqrt{t}\ln^2 t}\\
\int_{10^{10}}^{\sqrt{x}} \frac{\theta(t;3,l)}{t\ln^2 t}dt &< c_0\frac{\sqrt{x}-10^{10}}{\ln^2 10^{10}}\\
\int_{\sqrt{x}}^{x} \frac{\theta(t;3,l)}{t\ln^2 t}dt &< c_0\frac{x-\sqrt{x}}{\ln^2\sqrt{x}}.
\end{aligned}$$

We compute $M = 10381055.54\cdots$. Then

$$\begin{aligned}\pi(x;3,l) &< c_0\frac{x}{\ln x} + K + M + c_0\left(\frac{\sqrt{x}-10^{10}}{\ln^2 10^{10}} + \frac{x-\sqrt{x}}{\ln^2\sqrt{x}}\right)\\ &< \frac{x}{\ln x}\left(c_0 + \left(K + M + c_0\frac{10^{20}-10^{10}}{\ln^2 10^{10}}\right)\frac{\ln 10^{20}}{10^{20}}\right)\\ &< 0.545\frac{x}{\ln x}.\end{aligned}$$

- For $10^{10} \leqslant x \leqslant 10^{20}$,

$$\begin{aligned}\pi(x;3,l) &< K + \int_{10^5}^{10^{10}} \frac{\theta(t;3,l)}{t\ln^2 t}dt + \int_{10^{10}}^{x} \frac{\theta(t;3,l)}{t\ln^2 t}dt + c_0\frac{x}{\ln x}\\ &< \frac{x}{\ln x}\left(c_0 + \frac{\ln x}{x}\left(K + M - 10^{10}\frac{c_0}{\ln^2 10^{10}}\right) + \frac{c_0}{\ln^2 10^{10}}\ln x\right)\\ &< 0.5468\frac{x}{\ln x}.\end{aligned}$$

- For $10^5 \leqslant x \leqslant 10^{10}$,

$$\begin{aligned}\int_{10^5}^{x} \frac{\theta(t;k,l)}{t\ln^2 t}dt &< \frac{1}{2}\int_{10^5}^{x}\frac{dt}{\ln^2 t} + 2.072\int_{10^5}^{x}\frac{dt}{\sqrt{t}\ln^2 t}\\ &= \frac{1}{2}\left(\frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2\int_{10^5}^{x}\frac{dt}{\ln^3 t}\right) + 2.072\int_{10^5}^{x}\frac{dt}{\sqrt{t}\ln^2 t}.\end{aligned}$$

Now, $\int_a^b \frac{dt}{\sqrt{t}\ln^2 t} = \left[\frac{2\sqrt{t}}{\ln^2 t}\right]_a^b + 4\int_a^b \frac{dt}{\sqrt{t}\ln^3 t}$.

Therefore

$$\begin{aligned}\pi(x;3,l) &< \frac{1}{2}\frac{x}{\ln x} + 2.072\frac{\sqrt{x}}{\ln x} + K\\ &\quad+\frac{1}{2}\left(\frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2\int_{10^5}^{x}\frac{dt}{\ln^3 t}\right)\\ &\quad+2.072\left(\frac{2\sqrt{x}}{\ln^2 x} - \frac{2\sqrt{10^5}}{\ln^2 10^5} + 4\int_{10^5}^{x}\frac{dt}{\sqrt{t}\ln^3 t}\right)\\ &< 0.55\frac{x}{\ln x} \qquad \text{for } x \geqslant 6\cdot 10^5.\end{aligned}$$

7.2. **The lower bound.** Let $KK = \min_l(\pi(10^5,3,l) - \theta(10^5,3,l)/\ln(10^5)) \approx 462$ and $c = 0.498881 = \frac{1-0.002238}{2}$.

- For $10^{10} \leqslant x$,

$$\begin{aligned}\pi(x;3,l) &> KK + \frac{\theta(x;3,l)}{\ln x} + \int_{10^5}^{x}\frac{\theta(t;k,l)}{t\ln^2 t}dt\\ &> \frac{cx}{\ln x}\end{aligned}$$

because

$$KK > 0 \quad \text{and} \quad \int_{10^5}^{x}\frac{\theta(t;k,l)}{t\ln^2 t}dt > 0.$$

- For $10^5 \leqslant x \leqslant 10^{10}$.

**Lemma 14** (McCurley [2]). *For $x \geqslant 91807$ and $c_2 = 0.49585$, we have $\theta(x;3,l) \geqslant c_2 x$.*

*Remark.* This bound is better than the one given in Theorem 9 for $x \leqslant 2.5 \cdot 10^5$.

$$\pi(x;3,l) > KK + \frac{\theta(x;3,l)}{\ln x} + \int_{10^5}^{x} \frac{\theta(t;k,l)}{t \ln^2 t} dt.$$

Thus for any $x_0,\ x_1$ with $10^5 \leqslant x_0 < x_1$,

$$\begin{aligned} \pi(x;3,l) &> KK + \frac{\theta(x;3,l)}{\ln x} + \int_{10^5}^{x_0} \frac{\theta(t;k,l)}{t \ln^2 t} dt \text{ for } x \geqslant x_0 \\ &> \frac{x}{\ln x} \left( c_2 + \left( KK + \int_{10^5}^{x_0} \frac{\theta(t)}{t \ln^2 t} dt \right) \frac{\ln x_1}{x_1} \right) \text{ for } x_0 \leqslant x \leqslant x_1. \end{aligned}$$

Using the previous remark, we find

$$\begin{aligned} \int_{10^5}^{x} \frac{\theta(t;k,l)}{t \ln^2 t} dt &> c_2 \int_{10^5}^{x} \frac{dt}{\ln^2 t} \text{ if } 10^5 \leqslant x \leqslant 2.5 \cdot 10^5 \\ &\text{and} \\ &> c_2 \int_{10^5}^{2.5 \cdot 10^5} \frac{dt}{\ln^2 t} + \int_{2.5 \cdot 10^5}^{x} \frac{t/2 - 2.072\sqrt{t}}{t \ln^2 t} dt \text{ if } 2.5 \cdot 10^5 \leqslant x. \end{aligned}$$

We use this to make step by step computations with Maple:

| $x_0$ | $x_1$ |
|---|---|
| $10^5$ | $2 \cdot 10^6$ |
| $2 \cdot 10^6$ | $3 \cdot 10^7$ |
| $3 \cdot 10^7$ | $3 \cdot 10^8$ |
| $3 \cdot 10^8$ | $3 \cdot 10^9$ |
| $3 \cdot 10^9$ | $10^{10}$ |

We conclude that $\pi(x;3,l) > 0.499 \frac{x}{\ln x}$ for $10^5 \leqslant x \leqslant 10^{10}$.

7.3. **Small values.** We now check whether $0.49888 \frac{x}{\ln x} < \pi(x;3,l) < 0.55 \frac{x}{\ln x}$ for $x < 6 \cdot 10^5$. It is sufficient to prove that

$$\pi(p;3,l) < 0.55 \frac{p}{\ln p} \text{ for } p \equiv l \bmod 3,$$

and if

$$0.49888 \frac{p}{\ln p} < \pi(p;3,l) - 1 \text{ for } p \equiv l \bmod 3.$$

The highest value not satisfying the first inequality is $p = 229849$, and the highest value not satisfying the second is $p = 151$. Furthermore, $\pi(229869;3,l) \leqslant 10241 < 0.55 \frac{229869}{\ln 229869} \approx 10241.0075$ and $\pi(151;3,l) \geqslant 16 > 0.49888 \frac{151}{\ln 151} \approx 15.01$.

The conclusion is

$$0.49888 \frac{x}{\ln x} \underset{x \geqslant 151}{<} \pi(x;3,l) \underset{x \geqslant 229869}{<} 0.55 \frac{x}{\ln x}.$$

*Remark.* We cannot show that $x/(2 \ln x) < \pi(x;3,l)$ by using the formula $\theta(x) < c \cdot x$. We have obtained other formulas (see Theorem 6) which we will use below.

7.4. **More precise lower bound of $\pi(x;3,l)$.** Now we will give the proof of Theorem 8(i).

Classically,

$$\pi(x;3,l)-\pi(10^5;3,l)=\frac{\theta(x;3,l)}{\ln(x)}-\frac{\theta(10^5;3,l)}{\ln(10^5)}+\int_{10^5}^{x}\frac{\theta(t;3,l)}{t\ln^2 t}dt.$$

Now $\theta(t;3,l)>\frac{x}{\varphi(3)}\left(1-\frac{\alpha}{\ln x}\right)$ with $\alpha=\varphi(3)\cdot 0.262$ by use of Theorem 6. So we write

$$KK=\min_l\left(\pi(10^5;3,l)-\frac{\theta(10^5;3,l)}{\ln(10^5)}\right),$$

$$\pi(x;3,l)>J(x,\alpha)=KK+\frac{x}{\varphi(k)\ln x}\left(1-\frac{\alpha}{\ln x}\right)+\frac{1}{\varphi(k)}\int_{10^5}^{x}\frac{1-\alpha/\ln t}{\ln^2 t}dt.$$

The derivative of $J(x,\alpha)$ with respect to $x$ equals

$$\frac{1}{\varphi(k)}\left(\frac{1-\alpha/\ln x}{\ln x}+\frac{\alpha}{\ln^3 x}\right).$$

Moreover, the derivative of $\frac{x}{\varphi(k)\ln x}$ equals

$$\frac{1}{\varphi(k)}\left(\frac{1}{\ln x}-\frac{1}{\ln^2 x}\right).$$

The inequality

$$\frac{1}{\varphi(k)}\left(\frac{1}{\ln x}-\frac{1}{\ln^2 x}\right)<\frac{1}{\varphi(k)}\left(\frac{1-\alpha/\ln x}{\ln x}+\frac{\alpha}{\ln^3 x}\right)$$

holds if $\alpha-1<\alpha/\ln x$; this holds for all $x>1$. The only thing to do is to find a value $x_1$ such that

$$J(x_1,\alpha)>\frac{x_1}{\varphi(k)\ln x_1}.$$

For $x_1=10^5$, $J(10^5,0.524)\approx 4607.75$ and $\frac{10^5}{2\ln 10^5}\approx 4342.94$. We verify by computer that the inequality holds for $x\leqslant 10^5$ and $l=1$ or 2. We conclude that

$$\frac{x}{2\ln x}<\pi(x;3,l)\text{ for }x\geqslant 151.$$

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