ESTIMATES OF $\theta(x; k, l)$ FOR LARGE VALUES OF $x$

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Abstract. We extend a result of Ramaré and Rumely, 1996, about the Chebyshev function $\theta$ in arithmetic progressions. We find a map $\varepsilon(x)$ such that $\left| \theta(x; k, l) - x/\varphi(k) \right| < x \varepsilon(x)$ and $\varepsilon(x) = O \left( \frac{x}{\ln^a x} \right)$ (for $a > 0$), whereas $\varepsilon(x)$ is a constant. Now we are able to show that, for $x \geq 1531$,

$$| \theta(x; 3, l) - x/2 | < 0.262 \frac{x}{\ln x}$$

and, for $x \geq 151$,

$$\pi(x; 3, l) > \frac{x}{2 \ln x}.$$ 

1. Introduction

Let $R = 9.645908801$ and $X = \sqrt{\frac{\ln x}{R}}$. Rosser [6] and Schoenfeld [7, Th. 11 p. 342] showed that, for $x \geq 101$,

$$| \theta(x) - x |, | \psi(x) - x | < x \varepsilon(x),$$

where

$$\varepsilon(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} \exp(-X).$$

We adapt their work to the case of arithmetic progressions. Let us recall the usual notations for nonnegative real $x$:

$$\theta(x; k, l) = \sum_{\substack{p \equiv l \mod k \leq x \mod k}} \ln p,$$

where $p$ is a prime number,

$$\psi(x; k, l) = \sum_{\substack{n \equiv l \mod k \leq x \mod k}} \Lambda(n),$$

where $\Lambda$ is Von Mangold’s function,

and $\varphi$ is Euler’s function. We show, for $x \geq x_0(k)$ where $x_0(k)$ can be easily computed, that

$$\left| \theta(x; k, l) - x/\varphi(k) \right|, \left| \psi(x; k, l) - x/\varphi(k) \right| < x \varepsilon(x),$$

where

$$\varepsilon(x) = 3 \sqrt{\frac{k}{\varphi(k) C_1(k)}} X^{1/2} \exp(-X)$$

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for an explicit constant $C_1(k)$. We apply the above results for $k = 3$. For small values, we use Ramaré and Rumely’s results \[3\]. We show that for $x \geq 1531$,

\begin{equation}
\left| \theta(x; 3, l) - x/2 \right| < 0.262 \frac{x}{\ln x}.
\end{equation}

If we assume that the Generalized Riemann Hypothesis is true, then we can show that, for $x > 1$ and $k \leq 432$,

\begin{equation}
\left| \psi(x; k, l) - x/\varphi(k) \right| < \frac{1}{4\pi} \sqrt{x} \ln^2 x.
\end{equation}

Let us define, as usual, $\pi(x)$ the number of primes not greater than $x$. In 1962, Rosser and Schoenfeld ([5, p. 69]) found a lower bound for $\pi(x)$:

\begin{equation}
\pi(x) > \frac{x}{\ln x} \quad \text{for } x \geq 17.
\end{equation}

Letting

$$\pi(x; k, l) = \sum_{p \leq x, \ p \equiv l \pmod{k}} 1,$$

we show an analogous result in the case of arithmetic progression with $k = 3$ and $l = 1$ or 2,

$$\pi(x; 3, l) > \frac{x}{2\ln x} \quad \text{for } x \geq 151.$$  

This result, inferred from (1), implies (2) and cannot be proved with Ramaré and Rumely’s results.

The method used for $k = 3$ can also be applied for other fixed integers $k$.

\section*{2. Preliminary lemmas}

\textit{Notations.} We will always denote by $\rho$ a nontrivial zero of Dirichlet’s function $L$, that is to say a zero such that $0 < \Re \rho < 1$. We write $\rho = \beta + i\gamma$. Let $\varphi(\chi)$ be the set of the zeros $\rho$ of the function $L(s, \chi)$, with $0 < \beta < 1$.

For a positive real $H$, following Ramaré and Rumely, we say that GRH($k, H$) holds if, for all $\chi$ modulo $k$, all the nontrivial zeros of $L(s, \chi)$ with $|\gamma| \leq H$ are such that $\beta = 1/2$.

As in Rosser and Schoenfeld (in [6, 7] where the case $k = 1$ is studied), we must know the distribution of $L(s, \chi)$’s zeros; namely, find a real $H$ such that GRH($k, H$) is satisfied and is a zero-free region.

\subsection*{2.1. Zero-free region.}

\textbf{Theorem 1} (Ramaré and Rumely \[3\]). \textit{If $\chi$ is a character with conductor $k$, $H \geq 1000$, and $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ with $|\gamma| \geq H$, then there exists a computable constant $C_1(\chi, H)$ such that}

$$1 - \beta \geq \frac{1}{R \ln(k|\gamma|/C_1(\chi, H))}.$$  

\footnote{Note that our GRH is an acronym for the usual Generalized Riemann Hypothesis.}

<table>
<thead>
<tr>
<th>$k$</th>
<th>$H_k$</th>
<th>$C_1(\chi, H_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5450000000</td>
<td>38.31</td>
</tr>
<tr>
<td>3</td>
<td>10000</td>
<td>20.92</td>
</tr>
<tr>
<td>420</td>
<td>2500</td>
<td>56.59</td>
</tr>
</tbody>
</table>

Proof. See Theorem 3.6.3 of Ramaré and Rumely [3] p. 409. \qed

Remark. For $k \geq 1$ and $H_k \geq 1000$, $C_1(\chi, H) \geq C_1(\chi_0, 1000) \geq 9.14$.

As $C_1(\chi, H)$ could be large, we limit $C_1(k;H_k)$ up to $32\pi$ to make some computations. So we have in our hypothesis

\[ 9.14 \leq C_1(\chi, H) \leq 32\pi. \]

From now on,

\[ C_1(k) = \min(\min_{\chi \mod k} C_1(\chi, H_\chi), 32\pi). \]

2.2. GRH($k, H$) and $N(T, \chi)$.

Lemma 1 (McCurley [1]). Let $C_2 = 0.9185$ and $C_3 = 5.512$. Write $F(y, \chi) = \frac{y}{\pi} \ln \left( \frac{ky}{2\pi e} \right)$ and $R(y, \chi) = C_2 \ln(ky) + C_3$. If $\chi$ is a character of Dirichlet with conductor $k$, if $T \geq 1$ is a real number, and if $N(T, \chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s, \chi)$ in the rectangle $0 < \beta < 1$, $|\gamma| \leq T$, then

\[ |N(T, \chi) - F(T, \chi)| \leq R(T, \chi). \]

Lemma 2 (deduced from [3] Theorem 2.1.1, p. 399 and [3]).

- GRH($1, H$) is true for $H = 5.45 \times 10^8$.
- GRH($k, H$) is true for $H = 10000$ and $k \leq 13$.
- GRH($k, 2500$) is true for sets
  
  $E_1 = \{k \leq 72\}$,
  $E_2 = \{k \leq 112, k \text{ not prime}\}$,
  $E_3 = \{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}$.

2.3. Estimates of $|\psi(x; k, l) - x/\varphi(k)|$ using properties of zeros of $L(s, \chi)$. As in Ramaré and Rumely, we remove the zeros with $\beta = 0$ and we consider only primitive $L$-series by adding small terms. Here we take the version stated in [3] Theorem 4.3.1 which is deduced from [1].

Theorem 2 (McCurley [1]). Let $x > 2$ be a real number, $m$ and $k$ two positive integers, $\delta$ a real number such that $0 < \delta < \frac{x-2}{m}$, and $T$ a positive real. Let

\[ A(m, \delta) = \frac{1}{\delta^m} \sum_{j=0}^{m} \binom{m}{j} (1 + j\delta)^{m+1}. \]
Assume GRH($k, 1$). Then
\[
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} \left| \psi(y; k, l) - \frac{y}{\varphi(k)} \right| < A(m, \delta) \sum_{\chi} \sum_{\rho \in \nu(\chi)} \frac{\chi^{\beta-1}}{|\rho(\rho + 1) \cdots (\rho + m)}} + \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \sum_{\rho \in \nu(\chi)} \frac{\chi^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x,
\]
where \(\sum_{\chi}\) denotes the summation over all characters modulo \(k\), \(\tilde{R} = \varphi(k)(f(k) + 0.5) \ln x + 4 \ln k + 13.4\) and \(f(k) = \sum_{p|k} \frac{1}{p^2}\).

2.4. One more explicit form of estimates. The next lemma can be found in [3] with the difference that the authors assumed GRH($k, H$) but in fact they used only GRH($k, 1$). Since we must apply it with \(T > H\), we repeat the proof.

**Lemma 3.** Let \(\chi\) be a character modulo \(k\). Assume GRH($k, 1$). Then, for any \(T \geq 1\), we have
\[
\sum_{\substack{|\gamma| \leq T \\ \rho \in \nu(\chi)}} \frac{1}{|\rho|} \leq \tilde{E}(T)
\]
with \(\tilde{E}(T) = \frac{1}{2\pi} \ln^2(T) + \frac{\ln(k)}{\pi} \ln(T) + C_2 + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right)\).

**Proof.** For \(|\gamma| \leq 1\), we have GRH($k, 1$) and so
\[
\sum_{\substack{|\gamma| \leq 1 \\ \rho \in \nu(\chi)}} \frac{1}{|\rho|} \leq \sum_{\substack{|\gamma| \leq 1 \\ \rho \in \nu(\chi)}} \frac{1}{|1/2 + i\gamma|} \leq 2N(1, \chi).
\]

For \(|\gamma| > 1\),
\[
\sum_{\substack{1 < |\gamma| \leq T \\ \rho \in \nu(\chi)}} \frac{1}{|\rho|} \leq \int_{1}^{T} \frac{dN(t, \chi)}{t} = \int_{1}^{T} \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} - \frac{N(1, \chi)}{1}.
\]
Thus,
\[
\sum_{\substack{1 < |\gamma| \leq T \\ \rho \in \nu(\chi)}} \frac{1}{|\rho|} \leq \int_{1}^{T} \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} + N(1, \chi).
\]

We conclude by Lemma [4] that
\[
\int_{1}^{T} \frac{N(t, \chi)}{t^2} dt \leq \int_{1}^{T} F(t, \chi) + R(t, \chi) dt = \left( \frac{1}{\pi} \int_{1}^{T} \frac{\ln(kt/(2\pi e))}{t} dt \right) + C_2 \int_{1}^{T} \frac{\ln(kt)}{t^2} dt + C_3 \int_{1}^{T} \frac{1}{t^2} dt
\]
\[
= \frac{1}{\pi} \left[ \frac{1}{2} \ln^2 \left( \frac{kt}{2\pi e} \right) \right]_{1}^{T} + C_2 \left[ \frac{\ln(kt)}{t} \right]_{1}^{T} + \int_{1}^{T} \frac{1}{t^2} dt + C_3 \left[ -1/t \right]_{1}^{T}
\]
\[
= \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) \ln T + C_2 \left( -\frac{\ln(kt)}{T} + \ln k - \frac{1}{T} + 1 \right) + C_3 (1 - 1/T).
\]
In the same way, we have an upper bound of
\[
\frac{N(T, \chi)}{T} \quad \text{with} \quad \frac{F(T, \chi) + R(T, \chi)}{T}
\]
and
\[
N(1, \chi) \quad \text{with} \quad F(1, \chi) + R(1, \chi).
\]
Finally, we obtain
\[
\sum_{|\gamma| \leq T, \rho \in \nu(\chi)} \frac{1}{|\rho|} \leq \frac{1}{2 \pi} \ln^2(T) + \frac{\ln \left( \frac{k}{2\pi} \right)}{\pi} \ln(T)
\]
\[
+ C_2 + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi} \right) + C_2 \ln k + C_3 \right) - \frac{C_2}{T}.
\]
\[\square\]

Using the facts that
- if \( \rho \) is a zero of \( L(s, \chi) \) then \( \overline{\rho} \) is zero of \( L(s, \overline{\chi}) \),
- these zeros are symmetrical with to the line \( \Re(z) = 1/2 \),
we obtain Lemma 4 by examining the proof of [3, Lemma 4.1.3].

**Lemma 4 ([3]).** Let
\[
(5) \quad \phi_m(t) = \frac{1}{|t|^{m+1}} \exp \left( -\frac{-\ln x}{R \ln(k|t|/C_k(k))} \right)
\]
with \( R = 9.645908801 \). Let \( T \geq H \). We have
\[
\sum_{|\gamma| \leq T, \rho \in \nu(\chi)} \frac{x^\beta}{|\gamma|^{m+1}} + \sum_{|\gamma| > T, \rho \in \nu(\chi)} \frac{x^\beta}{|\gamma|^{m+1}} \leq x \sum_{|\gamma| > T, \rho \in \nu(\chi)} \phi_m(\gamma) + \sqrt{x} \sum_{|\gamma| \leq T, \rho \in \nu(\chi)} \frac{1}{|\gamma|^{m+1}}.
\]

Let us rewrite Lemma 7 of [3] to adapt it to the new functions \( F(y, \chi) \) and \( R(y, \chi) \) which we use.

**Lemma 5.** Write \( N(y) = N(y, \chi) \), \( F(y) = F(y, \chi) \), and \( R(y) = R(y, \chi) \). Let \( 1 < U \leq V \) and \( \phi(y) \) be a positive and differentiable function for \( U \leq y \leq V \). Let \( (W - y)\phi'(y) \geq 0 \) for \( U < y < V \), where \( W \) does not necessarily belong to \([U, V]\). Let \( Y \) be that one of the numbers \( U, V, W \) which is not numerically the least or greatest (or is the repeated one, if two among \( U, V, W \) are equal). Take \( j = 0 \) or \( 1 \), accordingly as \( W < V \) or \( W \geq V \). Then
\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \frac{1}{\pi} \int_U^V \phi(y) \ln \left( \frac{ky}{2\pi} \right) dy + (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy + B_j(Y, U, V),
\]
where
\[
B_0(Y, U, V) = 2R(Y)\phi(Y) + \{N(V) - F(V) - R(V)\}\phi(V) - \{N(U) - F(U) + R(U)\}\phi(U),
\]
\[
B_1(Y, U, V) = \{N(V) - F(V) + R(V)\}\phi(V) - \{N(U) - F(U) + R(U)\}\phi(U).
\]
Proof. We have
\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) = \int_U^V \phi(y) dN(y)
\]
\[
= - \int_U^V N(y) \phi'(y) dy + N(V) \phi(V) - N(U) \phi(U).
\]

- **j = 1.** We have \(W > V\) and so \(Y = \min(V, W) = V\). According to Theorem 1, \(N(y) \geq F(y) - R(y)\).

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \left[(N(y) - F(y) + R(y))\phi(y)\right]_U^V + \frac{1}{\pi} \int_U^V \ln \left(\frac{ky}{2\pi}\right) \phi(y) dy
\]
\[
- \int_U^V R'(y) \phi(y) dy
\]

because \(F'(y) = \frac{1}{\pi} \left(\ln \left(\frac{ky}{2\pi}\right) + 1\right) = \frac{1}{\pi} \ln \left(\frac{ky}{2\pi}\right)\). Moreover,
\[
- \int_U^V R'(y) \phi(y) dy = -C_2 \int_U^V \frac{\phi(y)}{y} dy.
\]

- **j = 0.** We have \(V > W\). Take \(Y = \max(U, W)\). Split the integral at \(Y\). Then \(-\phi'(y) \leq 0\) for \(y \in [U, Y]\) and \(-\phi'(y) \geq 0\) for \(y \in [Y, V]\). Replacing \(N(y)\) by \(F(y) - R(y)\) in the first part and by \(F(y) + R(y)\) in the second part, we obtain

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \frac{1}{\pi} \int_U^V \ln \left(\frac{ky}{2\pi}\right) \phi(y) dy + \int_U^V R'(y) \phi(y) dy - \int_U^V R'(y) \phi(y) dy
\]
\[
+ B_0(Y, U, V).
\]

Moreover,
\[
\int_U^V R'(y) \phi(y) dy \leq (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy
\]
and
\[
- \int_U^Y R'(y) \phi(y) dy \leq 0.
\]

We want to apply Lemma 5 with \(\phi = \phi_m\) defined by (5) and with \(W = W_m\) being the root of \(\phi'_m\). Let
\[
X = \sqrt{\frac{\ln x}{R}}
\]
and, for \(m \geq 0\),
\[
W_m = \frac{C_1(k)}{k} \exp(X/\sqrt{m + 1}).
\]

**Corollary 1** (Corollary from Lemma 5). Under the hypothesis of Lemma 5 if moreover \(\frac{k}{X} \leq U\), then
\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \left\{1/\pi + (-1)^j q(Y)\right\} \int_U^V \phi(y) \ln(ky/2\pi) dy + B_j(Y, U, V),
\]
where \(q(y) = \frac{C_2}{y \ln(2\pi)}\).
Proof. The map \( y \mapsto 1/(y \ln(ky/2\pi)) \) is decreasing if \( y \geq 2\pi/(ke) \).

- Case \((j = 0)\), then \( Y = \max(U, W) \).

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) < B_0(Y, U, V) + \frac{1}{\pi} \int_{U}^{V} \phi(y) \ln \left( \frac{ky}{2\pi} \right) dy + \int_{Y}^{V} R'(y)\phi(y) dy.
\]

\[
\int_{Y}^{V} R'(y)\phi(y) dy = C_2 \int_{Y}^{V} \frac{\phi(y)}{y} dy = C_2 \int_{Y}^{V} \frac{\phi(y) \ln(ky/2\pi)}{y \ln(ky/2\pi)} dy 
\leq \frac{C_2}{\ln(kY/2\pi)} \int_{Y}^{V} \phi(y) \ln(ky/2\pi) dy.
\]

- Case \((j = 1)\), then \( Y = V \).

\[
- \int_{U}^{V} R'(y)\phi(y) dy \leq -\frac{C_2}{V \ln(kV/2\pi)} \int_{U}^{V} \phi(y) \ln(ky/2\pi) dy.
\]

\[\blacksquare\]

**Theorem 3.** Let \( k \geq 1 \) an integer, \( H \geq 1000 \) a real number. Assume GRH\((k, H)\).

Let \( x_0 > 2 \) be a real number, \( m \) a positive integer, and \( \delta \) a real number such that \( 0 < \delta < (x_0 - 2)/(mx_0) \) and let \( Y \) be defined as in Lemma \[\] We write

\[
\hat{A}_H = \frac{1}{\pi} \int_{H}^{\infty} \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy + C_2 \int_{H}^{\infty} \frac{\phi_m(y)}{y} dy,
\]

\[
\hat{B}_H = B_0(Y, H, \infty),
\]

\[
\hat{C}_H = \frac{1}{m\pi H^m} \left( \ln \left( \frac{kH}{2\pi} \right) + 1/m \right),
\]

\[
\hat{D}_H = \left( 2C_2 \ln(kH) + 2C_3 + \frac{C_2}{m + 1} \right) / H^{m+1}.
\]

Then for all \( x \geq x_0 \), we have

\[
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l) - \frac{y}{\varphi(k)}| \leq A(m, \delta) \frac{\varphi(k)}{2} \left( \hat{A}_H + \hat{B}_H + (\hat{C}_H + \hat{D}_H)/\sqrt{x} \right) + \left( 1 + \frac{m\delta}{2} \right) \varphi(k) E(H)/\sqrt{x} + \frac{m\delta}{2} + \hat{R}/x.
\]

**Remark.** We find a version of Theorem 4.3.2 of \[\] where \( x_0 \) is replaced by \( x \) in \( \hat{A} \) and \( \hat{B} \).

**Proof.** According to Theorem \[\]

\[
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l) - \frac{y}{\varphi(k)}| < A(m, \delta) \sum_{\chi \mod \rho \in \chi | \gamma| \leq H} \frac{x^{\beta-1}}{|\rho| (\rho + 1) \cdots (\rho + m)}
\]

\[
+ \left( 1 + \frac{m\delta}{2} \right) \sum_{\chi \mod \rho \in \chi | \gamma| \leq H} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \hat{R}/x.
\]
We separately examine the different parts:

- We have
  \[
  \sum_{\chi} \sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} |\rho|^{\beta-1} \ll \sum_{\chi} \sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} |\gamma|^{m+1}.
  \]

By Lemma 4

\[
\sum_{\chi} \sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} |\rho|^{\beta-1} = \sum_{\chi} \frac{1}{2} \left( \sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} |\gamma|^{m+1} + \sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} x^{\beta-1} \right)
\]

\[
\leq \frac{1}{2} \sum_{\chi} \left( \sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} \phi_m(\gamma) + \frac{1}{\sqrt{x}} \sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} \frac{1}{|\gamma|^{m+1}} \right).
\]

Using Lemma 4 with \(U = H\), \(V = \infty\), \(\phi = \phi_m\), and \(W = W_m\),

\[
\sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} \phi_m(\gamma) \leq \tilde{A}_H + \tilde{B}_H.
\]

Integration by parts gives

\[
\sum_{\rho \in \mathcal{P}(x) \mid \gamma \geq H} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C}_H + \tilde{D}_H.
\]

- By GRH(\(k, H\)) we have \(\beta = 1/2\) for all \(|\gamma| \leq H\), and by Lemma 3

\[
\sum_{\rho \in \mathcal{P}(x) \mid \gamma \leq H} x^{\beta-1} \leq \tilde{E}(H)/\sqrt{x}.
\]

2.5. The leading term \(\tilde{A}_H\). To obtain an upper bound for the leading term, we proceed like Rosser and Schoenfeld with upper bounds on the integrals. The next three lemmas are issued directly from [6, p. 251-255].

**Lemma 6** (Functions of incomplete Bessel type). Let

\[K_\nu(z, u) = \frac{1}{2} \int_u^\infty t^{\nu-1} H^z(t)dt,\]

where \(z > 0\), \(u \geq 0\), and

\[H^z(t) = \{H(t)\}^z = \exp\left\{-\frac{z}{2}(t + 1/t)\right\}.
\]

Further, write \(K_\nu(z, 0) = K_\nu(z)\). Then

(12) \[K_1(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left(1 + \frac{3}{8z}\right),\]

(13) \[K_2(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left(1 + \frac{15}{8z} + \frac{105}{128z^2}\right).
\]

**Lemma 7.**

\[K_\nu(z, x) + K_{-\nu}(z, x) = K_\nu(z).
\]

Hence, \(K_\nu(z, x) \leq K_\nu(z) (\nu \geq 0)\).
Lemma 8. Let
\[ Q_\nu(z, x) = \frac{x^{\nu+1}}{z(x^2 - 1)} \exp\{-z(x + 1/x)/2\}. \]
If \( z > 0 \) and \( x > 1 \), then
\[ K_1(z, x) < Q_1(z, x) \]
and
\[ K_2(z, x) < (x + 2/z)Q_1(z, x). \]

The term \( \tilde{A}_H \) can be expressed using incomplete Bessel functions.

Lemma 9. Let \( X \) be defined by (6). Let \( z_m = 2X \sqrt{m} = 2\sqrt{\frac{m \ln x}{R}} \) and \( U_m = \frac{2m}{z_m} \ln \left( \frac{kH}{C_1(k)} \right) = \sqrt{\frac{Rm}{\ln x}} \ln \left( \frac{kH}{C_1(k)} \right)^m \).

\[ \tilde{A}_H = \frac{2 \ln x}{\pi Rm} \left( \frac{k}{C_1(k)} \right)^m K_2(z_m, U_m) \]
\[ + \frac{2 \ln \left( \frac{C_1(k)}{2\pi} \right)}{\ln x} K_1(z_m, U_m) \]
\[ + 2C_2 \sqrt{\frac{\ln x}{R(m + 1)}} \left( \frac{k}{C_1(k)} \right)^{m+1} K_1(z_{m+1}, U_{m+1}). \]

Proof. This is by straightforward algebraic manipulation; for example, we write
\[ I = \int_H^{\infty} \frac{C_2}{y^{m+1}} \exp \left( \frac{-\ln x}{R \ln(ky/C_1(k))} \right) dy. \]

Changing variables:
\[ t = \sqrt{\frac{R(m + 1)}{\ln x}} \ln \left( \frac{ky}{C_1(k)} \right), \]
\[ dt = \sqrt{\frac{R(m + 1)}{\ln x}} \frac{dy}{y}. \]

Now
\[ \exp \left( \frac{-\ln x}{R \ln(ky/C_1(k))} \right) = \exp \left( \frac{-\ln x}{R t/\sqrt{\frac{R(m + 1)}{\ln x}}} \right) = \exp \left( \frac{-\ln x}{R t/\sqrt{(m + 1) \ln x}} \right) = \exp \left( \frac{-z_{m+1}^{m+1}}{2t} \right) \]
and
\[ \frac{1}{y^{m+1}} = \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( -\frac{(m + 1)t}{\sqrt{R(m + 1)/\ln x}} \right) = \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( -t \frac{z_{m+1}}{2} \right). \]

Consequently,
\[ I = \int_{U_{m+1}}^{\infty} C_2 \sqrt{\frac{\ln x}{R(m + 1)} \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( \frac{-z_{m+1}^{m+1}}{2}(t + 1/t) \right)}. \]
2.6. **Study of** \( f(k) \) **which appears in the expression of** \( \tilde{R} \). **Remember that**
\[
 f(k) = \sum_{p|k} \frac{1}{p}.
\]

**Lemma 10.** **For an integer** \( k \geq 1 \),
\[
f(k) \leq \frac{\ln k}{\ln 2}.
\]

**Proof.** **We prove by recursion that**
\[
f(k) \leq \frac{\ln k}{\ln 2}.
\]

For \( k = 1 \), it is obvious. For \( k = 2 \), \( f(k) = 1 \leq \frac{\ln 2}{\ln 2} \). Assume \( f(k) \leq \frac{\ln k}{\ln 2} \) holds for \( k \leq n \). Find an upper bound for \( f(n + 1) \).

If \( n + 1 \) is prime, then \( f(n + 1) = 1 \leq \frac{\ln n}{\ln 2} \). If \( n + 1 \) is not prime, then there exists \( p \leq n \), which divides \( n \). If \( p = 2 \) and \( 2^\alpha \parallel n + 1 \),
\[
f(n + 1) = f \left( \frac{n + 1}{2^\alpha} \cdot 2^\alpha \right) = f \left( \frac{n + 1}{2^\alpha} \right) + f(2)
\]
\[
= 1 + f \left( \frac{n + 1}{2^\alpha} \right) \leq \frac{\ln(n + 1)}{\ln 2} + 1 - \frac{\ln 2}{\ln 2}
\]
\[
\leq \frac{\ln(n + 1)}{\ln 2}.
\]

If \( p > 2 \) and \( p^\alpha \parallel n + 1 \),
\[
f(n + 1) = f \left( \frac{n + 1}{p^\alpha} \cdot p^\alpha \right) = f \left( \frac{n + 1}{p^\alpha} \right) + f(p)
\]
\[
= \frac{1}{p - 1} + f \left( \frac{n + 1}{p^\alpha} \right) \leq \frac{\ln(n + 1)}{\ln 2} + \frac{1}{p - 1} - \frac{\ln p}{\ln 2}
\]
\[
\leq \frac{\ln(n + 1)}{\ln 2} \quad \text{because} \quad \frac{1}{p - 1} - \frac{\ln p}{\ln 2} < 0 \quad \text{for} \quad p > 2.
\]

3. **The method with** \( m = 1 \)

**Theorem 4.** **Let** \( k \) **be an integer,** \( H \geq 1250 \), **and** \( H \geq k \). **Assume** GRH\((k, H)\). **Let**
\( C_1(k) \) **defined by** \( \textbf{[3]} \). **Let** \( x > 1 \). **Write** \( X = \sqrt{\frac{\ln x}{H}} \) **and**
\[
\varepsilon(x) = 2 \sqrt{\frac{k \varphi(k)}{C_1(k) \sqrt{\pi}}} \left( 1 + \frac{1}{2x} (15/16 + \ln(C_1(k)/(2\pi))) \right) X^{3/4} \exp(-X).
\]

If \( \varepsilon(x) \leq 0.2 \) **and** \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \), **then**
\[
\max_{1 \leq y < x} | \psi(y; k, l) - \psi(x; k, l) | \leq x \varepsilon(x) / \varphi(k).
\]

**Proof.** **Take** \( m = 1 \) **in Theorem** \( \textbf{[3]} \). **Assuming** \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \), **then** \( W_1 \geq H \).

In this situation, \( Y = W_1 \) **and** \( \tilde{B}_H < 2R(W_1)\phi_1(W_1) \). **For** \( y > 1 \), \( R(y) / \ln y \) **is
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Decreasing; hence,

\[
\hat{B}_H < 2R(W_1)\phi_1(W_1) < 2\frac{R(H)}{\ln H} \phi_1(W_1) \ln W_1
\]

\[
= 2\frac{R(H)}{\ln H} \left( \frac{X}{\sqrt{2}} + \ln \left( \frac{C_1(k)}{k} \right) \right) \phi_1(W_1)
\]

\[
= 2\frac{R(H)}{\ln H} \left( \frac{X}{\sqrt{2}} + \ln \left( \frac{C_1(k)}{k} \right) \right) (k/C_1(k))^2 \exp(-2\sqrt{2}X).
\]

Inserting the upper bounds (12) and (13) into the bound for $\hat{A}_H$ in Lemma 9,

\[
\hat{A}_H < 2 \left( \frac{k}{C_1(k)} \right) \left[ \frac{\pi}{\sqrt{4X}} \exp(-2X) \left( 1 + \frac{15}{16X} + \frac{105}{512X^2} \right) X^2/\pi \right.
\]

\[
+ \frac{1}{\pi} \ln \frac{C_1(k)}{2\pi} X^{3/2} \left( \frac{\pi}{\sqrt{4X}} \exp(-2X) \left( 1 + \frac{3}{16X} \right) \right.
\]

\[
\left. + C_2 \frac{kX}{\sqrt{C_1(k)}} \left( \frac{\pi}{\sqrt{4X}} \exp(-2\sqrt{2}X) \left( 1 + \frac{3}{16X} \right) \right) \right].
\]

Put

\[
F_1 := \frac{1}{\pi} \frac{k}{C_1(k)} X^{3/2} \exp(-2X) \left[ 1 + \left( \frac{15}{16} + \ln \frac{C_1(k)}{2\pi} \right) \frac{1}{2X} \right]^2.
\]

In Lemma 11 below it is shown that

\[
\hat{A}_H + \hat{B}_H + (\hat{C}_H + \hat{D}_H + 3\hat{E}(H))/\sqrt{x} + \hat{R} \frac{2}{x \phi(k)} < F_1.
\]

We must choose $\delta$ to minimize

\[
\frac{A(1, \delta)}{2} \frac{\phi(k) F_1}{6} + \delta/2.
\]

Write $f = \phi(k) F_1$. As $A_1(\delta) = (\delta^2 + 2\delta + 2)/\delta$, we must minimize $g(\delta) = (\delta/2 + 1 + 1/\delta)f + \delta/2$. The minimum value here is at $\delta = \sqrt{2f(1+f)}$, and the value there is $g(\sqrt{2f(1+f)}) = f + \sqrt{2f(1+f)}$.

It is a simple matter to prove that for $0 < f < 0.202$,

\[
f + \sqrt{2f(1+f)} < 2\sqrt{f}.
\]

As $X \geq X_0 := \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right)$, then $x_0 \geq \exp(122.5)$, and it is obvious that $\delta$ meets the hypothesis $0 < \delta < (x_0 - 2)/x_0$ in Theorem 3 since

\[
0 < \delta < \sqrt{2} \sqrt{f} < 0.6357 < \frac{x_0}{x_0 - 2}.
\]

\[\Box\]

Lemma 11.

\[
\hat{A}_H + \hat{B}_H + (\hat{C}_H + \hat{D}_H + 3\hat{E}(H))/\sqrt{x} + \hat{R} \frac{2}{x \phi(k)} < F_1.
\]
Proof. First we prove that $\hat{A}_H + \hat{B}_H < F_1$:

$$F_1 = \frac{k}{C_1(k)\sqrt{\pi}} \cdot X^{3/2} e^{-2X} \left(1 + \frac{15}{16} + \ln(C_1(k)/2\pi)/X + \frac{15}{32} \ln(C_1(k)/2\pi) + \frac{1}{4} \ln^2(C_1(k)/2\pi)/X^2\right),$$

$$\hat{A}_H < \frac{k}{C_1(k)\sqrt{\pi}} \cdot X^{3/2} e^{-2X} \left(1 + \frac{15}{16X} + \frac{105}{512X^2} + \ln \left(\frac{C_1(k)}{2\pi}\right) \left(\frac{1}{X} + \frac{3}{16X^2}\right)\right),$$

$$\hat{B}_H < \frac{k}{\sqrt{\pi} C_1(k)} \cdot X^{3/2} \exp(-2X) \exp(-2(\sqrt{2} - 1)X) \times \left[\frac{2k}{C_1(k)} \ln H \left(C_2\ln(kH) + C_3\left(\frac{1}{\sqrt{2X}} + \frac{1}{X} \ln(C_1(k)/k)\right)\right)\right].$$

This yields $F_1 - \hat{A}_H - \hat{B}_H > 0$ if

$$F_2 := \frac{1}{X^2} \left(\frac{15}{1024} + \frac{9}{32} \ln \left(\frac{C_1(k)}{2\pi}\right) + \frac{1}{4} \ln^2 \left(\frac{C_1(k)}{2\pi}\right)\right)$$

$$> \frac{C_2\sqrt{\pi k}}{C_1(k)} \exp(-2(\sqrt{2} - 1)X) \frac{1}{\sqrt{2X}}$$

$$\times \left[\sqrt{X} \ln \left(\frac{2}{16X^{3/2}}\right)\right]$$

$$= 2 \left(1 + \frac{\ln k + C_3/C_2}{\ln H}\right) \left(1 + \frac{\sqrt{2}}{X} \ln \frac{C_1(k)}{k}\right).$$

This holds if we can show that

$$F_2 > \frac{C_2\sqrt{\pi k}}{C_1(k)} \exp(-2(\sqrt{2} - 1)X) \frac{1}{\sqrt{2X}} \cdot 16.9,$$

since $C_1(k) \leq 32\pi$, $H \geq 1250$, $X \geq \sqrt{2} \ln(1250/32\pi)$, and $k \leq H$.

It remains to be proved that

$$\frac{\sqrt{2}C_1(k)}{kC_2\sqrt{\pi} \cdot 16.9} (15/1024 + \ldots) > X^{3/2} \exp(-2(\sqrt{2} - 1)X).$$

But for $X \geq X_0 := \sqrt{2} \ln \left(\frac{k H}{C_1(k)}\right)$,

$$X^{3/2} \exp(-2(\sqrt{2} - 1)X) < X_0^{3/2} \left(\frac{k H}{C_1(k)}\right)^{(1+a)}$$

$$= \frac{1}{k} \cdot 2^{3/4} \left(C_1(k)/H\right)^{1+a} \left(\frac{\ln^{3/2}(kH/C_1(k))}{k^a}\right),$$

where $a = 2(\sqrt{2} - 1) - 1 \approx 0.17157$. The map $k \mapsto \ln^{3/2}(kH/C_1(k))$ reaches its maximum for $k = e^{2\frac{C_1(k)}{H}}$. Hence

$$X^{3/2} \exp(-2(\sqrt{2} - 1)X) < \frac{C_1(k)}{kH} 2^{3/4} \left(\frac{3}{2a}\right)^{3/2} / e^{3/2}.$$
We must compare
\[
\frac{\sqrt{2}}{C_2 \sqrt{\pi} \cdot 16.9} \left(\frac{15}{1024} + \cdots\right) \text{ with } \frac{2^{3/4} (\frac{x}{2})^{3/2}}{H e^{3/2}}.
\]
Since \( C_1(k) \geq 9.14 \) (see the remark above (3)) and \( C_2 = 0.9185 \), it remains to be proved that
\[
0.007976 > \frac{2^{3/4} (\frac{x}{2})^{3/2}}{H e^{3/2}} \approx 0.00776,
\]
which is true since \( H \geq 1250 \).

We show below that the remaining terms \((\tilde{C}_H + \tilde{D}_H + 3 \tilde{E}(H)) / \sqrt{x} + \tilde{R} / \varphi(k) \) are negligible.

- We will find an upper bound for \( A(1, \delta) \frac{\varphi(k)}{2} (\tilde{C}_H + \tilde{D}_H) + \frac{3}{2} \varphi(k) \frac{\tilde{E}(H)}{\sqrt{x}} + \tilde{R} / x \).

We assume that \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \); hence, \( X \geq X_0 := \sqrt{2} \ln \left( \frac{1250}{32} \right) \approx 3.5644 \). It is straightforward but tedious to check that
\[
\text{Rest} := \tilde{C}_H + \tilde{D}_H + 3 \tilde{E}(H) + \frac{2 \tilde{R}}{\varphi(k) \sqrt{x}} \leq \begin{cases} 
1250 (\ln H \ln k)^2 & \text{if } k \neq 1, \\
1250 (\ln H)^2 & \text{if } k = 1.
\end{cases}
\]

Let us consider the case \( k \neq 1 \). As \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \),
\[
\exp \left( \frac{X}{\sqrt{2}} \right) \geq \frac{kH}{C_1(k)}.
\]

This yields
\[
\text{Rest} \leq 1250 (\ln H \ln k)^2 \leq 1250 \left( \frac{\ln H \ln k}{C_1(k)} \right)^2 \exp(2X)
\]
\[
\leq 1250 C_1^2(k) \frac{1}{e^2} \left( \frac{\ln 1250}{1250} \right)^2 \exp(2X)
\]
\[
\leq K \exp(2X) \text{ because } C_1(k) \leq 32 \pi,
\]
where \( K := 55.65 \). Now compare
\[
\frac{K \exp(2X)}{\sqrt{x}} = K \exp(2X - RX^2/2)
\]
with the term involving \( 1/X^2 \) in \( F_1 \)
\[
\frac{1}{X^2} \times \frac{k}{C_1(k) \sqrt{\pi}} X^{3/2} \exp(-2X).
\]

We may compute \( c \) such that
\[
K \exp(2X - RX^2/2) \leq c \times \frac{1}{X^2} \times \frac{k}{C_1(k) \sqrt{\pi}} X^{3/2} \exp(-2X)
\]
\[
\Leftrightarrow c \geq K \sqrt{32 \pi} \exp(X \sqrt{2} - RX^2/2 + 2X) \left( \frac{X^2}{X^{3/2}} \right)
\]
\[
\Leftrightarrow c \geq 0.7 \cdot 10^{-18} \text{ for } X \geq X_0.
\]

Thus, the rest is negligible and absorbed by rounding up the constants. \( \square \)
4. The method with $m = 2$

Lemma 12. Let $A(m, \delta)$ be defined as in formula [4]. Write

$$R_m(\delta) = (1 + (1 + \delta)^{m+1})^m.$$ 

Then

$$A(m, \delta) \lesssim \frac{R_m(\delta)}{\delta^m}.$$ 

Proof. The proof appears in [4, p. 222].

Theorem 5. Let an integer $k \geq 1$. Remember that $R = 9.645908801$. Let $H \geq 1000$. Assume GRH$(k, H)$. Let $C_1(k)$ be defined by [5]. Let $X_0$, $X_1$, $X_2$, and $X_3$ be such that

$$e^{X_0} = H \frac{k\varphi(k)}{2\pi C_1(k)}, \quad e^{X_1} = 10\varphi(k),$$

$$X_2 = kC_1(k)/(2\pi\varphi(k)), \quad X_3 = \frac{2k\pi e}{C_1(k)\varphi(k)}.$$ 

Let $X_4 := \max(10, X_0, X_1, X_2, X_3)$. Write

$$\varepsilon(X) = 3\sqrt{\frac{k}{\varphi(k)C_1(k)}}X^{1/2}\exp(-X).$$

Then for all real $x$ such that $X = \sqrt{\frac{\ln x}{R}} \geq X_4$, we have

$$\max_{1 \leq y \leq x} |\psi(y; k, l) - y/\varphi(k)| < x\varepsilon\left(\sqrt{\frac{\ln x}{R}}\right),$$

$$\max_{1 \leq y \leq x} |\theta(y; k, l) - y/\varphi(k)| < x\varepsilon\left(\sqrt{\frac{\ln x}{R}}\right).$$

Corollary 2. With the notations and the hypothesis of Theorem 5 let $X_5 \geq X_4$ and $c := \varepsilon(X_5)$. For $x \geq \exp(RX_5^2)$, we have

$$|\psi(x; k, l) - x/\varphi(k)|, \quad |\theta(x; k, l) - x/\varphi(k)| < cx.$$ 

Proof. The idea is to judiciously split the integral into two parts, and bound each part optimally, using an $m = 0$ estimate in the first part and an $m = 2$ estimate in the second part.

We want to split the integral at $T$, where $T$ will optimally be chosen later. We take $T$ in the same form as $W_m$ (formula [11]):

$$T := \frac{C_1(k)}{k} \exp(\nu X),$$

where $\nu$ is a parameter.

Assume that $T \geq H$ and $1/\sqrt{m+1} \leq \nu \leq 1$. Hence $W_m \leq T \leq W_0$. This last hypothesis is needed to apply Corollary 1.

We use Theorem 2 and split the sums at $T$:

$$A(m, \delta) \sum_{\chi} \sum_{\substack{\rho \in \rho(\chi) \cap [\gamma, \kappa T] \neq \nu \chi}} \frac{x^{\beta-1}}{|\rho(p+1)\cdots(p+m)|} \left(1 + m\delta \sum_{\gamma \in \gamma(\chi)} \frac{x^{\beta-1}}{|\rho|} + m\delta \frac{2}{x} + \frac{\tilde{R}}{x}\right).$$
Define
\[ \hat{A}_1 := \sum_{\chi} \sum_{\rho \in \rho(\chi) \setminus \gamma \leq T} \frac{x^{\beta-1}}{|\rho|}, \]
\[ \hat{A}_2 := \sum_{\chi} \sum_{\rho \in \rho(\chi) \setminus \gamma > T} \frac{x^{\beta-1}}{|\rho(\rho + 1) \cdots (\rho + m)|}. \]

Bounding the term \( \hat{A}_1 \), we get
\[ \hat{A}_1 = \sum_{\chi} \left( \sum_{\rho \in \rho(\chi) \setminus \gamma \leq T} \frac{\sqrt{x}}{|\rho|} + \sum_{\rho \in \rho(\chi) \setminus H < \gamma \leq T} \frac{x^\beta}{|\rho|} \right) \]
by GRH(\( k, H \))
\[ = \frac{1}{x} \sum_{\chi} \left( \sum_{\rho \in \rho(\chi) \setminus \gamma \leq H} \frac{\sqrt{x}}{|\rho|} + \sum_{\rho \in \rho(\chi) \setminus H < \gamma \leq T} \frac{x^\beta}{|\rho|} \right) \]
\[ \leq \frac{1}{\sqrt{x}} \varphi(k) \hat{E}(H) + \frac{1}{2x} \sum_{\chi} \left( \sum_{\rho \in \rho(\chi) \setminus H < \gamma \leq T} x\phi_0(\gamma) + \sqrt{x} \sum_{\rho \in \rho(\chi) \setminus H < \gamma \leq T} \frac{1}{|\gamma|} \right) \]
by Lemmas 3 and 4
\[ \leq \varphi(k) \hat{E}(T)/\sqrt{x} + \frac{1}{2} \sum_{\chi} \sum_{\rho \in \rho(\chi) \setminus H \leq \gamma \leq T} \phi_0(\gamma). \]

Apply Corollary 1 (\( j = 1, m = 0 \)) for the interval \([H, T]\) with \( \phi = \phi_0 \) and \( W = W_0 \)
\[ \sum_{\rho \in \rho(\chi) \setminus H \leq \gamma \leq T} \phi_0(\gamma) = \{1/\pi - q(T)\} \int_H^T \phi_0(y) \ln(ky/2\pi)dy + B_1(T, H, T). \]
Moreover, \( B_1(T, H, T) < 2R(T)\phi_0(T) \).

We want to find an upper bound for
\[ I_1 := \frac{1}{\pi} \int_H^T \phi_0(y) \ln \left( \frac{ky}{2\pi} \right) dy. \]
Write \( V'' = X^2/\ln \left( \frac{kT}{C_1(k)} \right) = X/\nu = Y'' + 2X - \nu X \), where \( Y'' := X(1 - \nu)^2/\nu \).
Write \( U'' = X^2/\ln \left( \frac{kH}{C_1(k)} \right) \) and \( \Gamma(\alpha, x) = \int_x^\infty e^{-u}u^{\alpha-1}du \). Now
\[ \int_H^T \ln \left( \frac{ky}{2\pi} \right) \phi_0(y)dy = \int_H^T \ln \left( \frac{ky}{2\pi} \right) \exp \left( -X^2/\ln \left( \frac{ky}{C_1(k)} \right) \right) \frac{dy}{y} \]
\[ = X^4 \{ \Gamma(-2, V'') - \Gamma(-2, U'') \} \]
\[ + X^2 \ln \left( \frac{C_1(k)}{2\pi} \right) \{ \Gamma(-1, V'') - \Gamma(-1, U'') \} \]
by making the change of variables \( y = \frac{C_1(k)}{k} \exp(X^2/u) \). Now if \( \alpha \leq 1 \) and \( x > 0 \), then \( \Gamma(\alpha, x) \leq x^{\alpha-1} \int_0^\infty e^{-t} \, dt = x^{\alpha-1}e^{-x} \). Hence,

\[
\int_T^H \ln \left( \frac{k y}{2 \pi} \right) \phi_0(y) \, dy \leq X^4 V'' \exp(-V'') + X^2 \ln \left( \frac{C_1(k)}{2 \pi} \right) V'' e^{-V''}.
\]

This yields

\[
I_1 \leq \frac{1}{\pi} X^2 \left( X^2 V'' + \ln \left( \frac{C_1(k)}{2 \pi} \right) V'' \right) e^{-V''} = \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) \left( \frac{X^4}{(X/\nu)^3} + \frac{dX^2}{(X/\nu)^2} \right) = \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0,
\]

where \( d := \ln \left( \frac{C_1(k)}{2 \pi} \right) \) and \( G_0 := \nu^2(\nu + d/X) \). With the help of Corollary 4, we write

\[
\tilde{A}_1 \leq \varphi(k)\tilde{E}(T) / \sqrt{X} + \varphi(k) \left\{ \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0 + 2R(T)\phi_0(T) \right\}.
\]

Bounding the term \( \tilde{A}_2 \), we get

\[
\tilde{A}_2 = \frac{1}{x} \sum_X \sum_{\rho \in \rho(x)} \frac{x^\beta}{|\rho(\rho + 1) \cdots (\rho + m)|} \leq \frac{1}{2x} \sum_X \left( \sum_{\rho \in \rho(x)} \frac{x^\beta}{|\rho(\rho + 1) \cdots (\rho + m)|} + \sum_{\rho \in \rho(x)} \frac{x^\beta}{|\rho(\rho + 1) \cdots (\rho + m)|} \right) \leq \frac{1}{2x} \sum_X \left( \sum_{\rho \in \rho(x)} \frac{x^\beta}{|\rho|^{m+1}} + \sum_{\rho \in \rho(x)} \frac{x^\beta}{|\rho|^{m+1}} \right) = \frac{1}{2x} \sum_X \left( x \sum_{\rho \in \rho(x)} \phi_m(\gamma) + \sqrt{x} \sum_{\rho \in \rho(x)} \frac{1}{|\gamma|^{m+1}} \right)
\]

by Lemma 4.

By using Corollary 3 \((j = 0)\) on \([U, V] = [T, \infty)\),

\[
\sum_{\rho \in \rho(x)} \phi_m(\gamma) \leq \{1/\pi + q(T)\} \int_T^\infty \phi_m(y) \ln \left( \frac{ky}{2 \pi} \right) \, dy + B_0(T, T, \infty).
\]

We have

\[
B_0(T, T, \infty) < 2R(T)\phi_m(T).
\]

Moreover,

\[
\sum_{\rho \in \rho(x)} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C}_T + \tilde{D}_T.
\]
Let us study more precisely
\[
I_2 := \int_0^\infty \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy
\]
\[
= \frac{z_m}{2m^2} \left( \frac{k}{C_1(k)} \right)^m \left( K_2(z_m, U_m) + \frac{2md}{z_m} K_1(z_m, U_m) \right),
\]
where \( d = \ln \left( \frac{G_1(k)}{2k} \right) \) and \( U' := U_m = \frac{z_m}{2m} \ln \left( \frac{kT}{C_1(k)} \right) = \nu \sqrt{m}. \) Now, by writing \( z = z_m \) and using Lemma 8,
\[
K_2(z, U') + \frac{2dm}{z} K_1(z, U') < (U' + 2/z + 2dm/z) Q_1(z, U')
\]
\[
\leq \sqrt{m} \left( \nu + \frac{1 + dm}{mX} \right) \frac{U'^2}{z(U'^2 - 1)} e^{-\frac{2}{3}(U' + 1/U')}.
\]
But \( \frac{2}{3}(U' + 1/U') = X \sqrt{m} \nu + 1/(\nu \sqrt{m}) = mX + X/\nu = mX + (Y'' + 2X - X \nu), \) where \( Y'' = X(1 - \nu)^2/\nu. \) Hence
\[
K_2(z, U') + \frac{2dm}{z} K_1(z, U') < G_1 e^{-Y''} \frac{m}{2(m - 1)} X^{-1} e^{-2X} \left( \frac{kT}{C_1(k)} \right)^{-(m-1)},
\]
where \( G_1 := \frac{m-1}{m} \frac{U'^2}{U'^2 - 1} (\nu + \frac{1 + dm}{mX}) \) because
\[
e^{\nu X(m-1)} = \left( \frac{kT}{C_1(k)} \right)^{m-1}
\]
and \( \sqrt{m} = \frac{1}{2} X^{-1}. \) This yields
\[
I_2 = \int_0^\infty \phi_m(y) \ln(ky/2\pi) dy < \frac{G_1 e^{-Y''}}{m-1} \frac{k}{C_1(k)} X e^{-2XT^{-(m-1)}}
\]
Let \( G_2 := \frac{B_m(\delta)}{2\delta}(1 + \pi q(T)). \) So, by using Lemma 12,
\[
A(m, \delta) \frac{\varphi(k)}{2} (1/\pi + q(T)) \int_0^\infty \phi_m(y) \ln(ky/2\pi) dy
\]
\[
< \left( \frac{2}{\delta} \right)^m \frac{\varphi(k)}{2} \left\{ \frac{G_2}{\pi} \frac{kG_1 e^{-Y''}}{(m - 1)C_1(k)} X e^{-2XT^{-(m-1)}} \right\}.
\]
The results above yield
\[
(1 + m\delta/2) \tilde{A}_1 + A(m, \delta) \tilde{A}_2
\]
\[
< \frac{XG_2 e^{-2X} e^{-Y''} \varphi(k)}{2\pi} \left( \frac{k}{C_1(k)} \right) \left\{ \frac{G_1}{m-1} T^{-(m-1)} \left( \frac{2}{\delta} \right)^m + G_0 T \right\} + r
\]
because \( 1 + m\delta/2 < R_m(\delta)/2m < G_2, \) with
\[
r = \varphi(k)(1 + m\delta/2) R(T) \phi_0(T) + A(m, \delta) \varphi(k) R(T) \phi_m(T)
\]
\[
+ \frac{\varphi(k)}{\sqrt{\pi}} ((1 + m\delta/2) \tilde{E}(T) + A(m, \delta)(\tilde{C}T + \tilde{D}T)/2).
\]
Suppose \( G_0/G_1 \) were independent of \( \nu; \) then the expression between braces in (15) would be minimized for
\[
T = (G_1/G_0)^{1/m} \cdot \frac{2}{\delta}.
\]
With this choice,
\[
\frac{G_1}{m - 1} T^{-(m-1)} \left( \frac{2}{\delta} \right)^m + G_0 T = \frac{m}{m - 1} G_1^{1/m} G_0^{1-1/m} 2^{\frac{m}{\delta}}
\]
and we obtain \((G_2 > 1)\)
\[
\varepsilon_1 := (1 + m \delta / 2) \tilde{A}_1 + A(m, \delta) \tilde{A}_2 + \frac{1}{2} m \delta + \frac{\bar{R}}{x}
\]
\[
< \frac{1}{2} m G_2 \left\{ X e^{-2X} e^{-Y''} \frac{2k \varphi(k)}{(m-1)\pi C_1(k)} G_1^{1/m} G_0^{1-1/m} + \delta \right\} + r + \frac{\bar{R}}{x}.
\]
The expression between braces can be minimized by choosing
\[
\delta = \left\{ G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k \varphi(k)}{(m-1)\pi C_1(k)} \right\}^{1/2} X^{1/2} e^{-X}.
\]
Hence, we write (by replacing the above value of \(\delta\) in \((16)\))
\[
T = \left( \frac{G_1}{G_0} \right)^{1/2m} \left( \frac{2C_1(k)}{k \varphi(k)} (m - 1) \pi e^{Y''} / G_0 \right)^{1/2} X^{-1/2} e^X
\]
and
\[
\varepsilon_1 < G_2 \left( G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k \varphi(k)}{\pi C_1(k)} \right)^{1/2} \frac{m}{\sqrt{m-1}} X^{1/2} e^{-X} + r + \frac{\bar{R}}{x}.
\]
The value \(m = 2\) minimizes the expression \(\frac{m}{\sqrt{m-1}}\). For the remainder of the argument, we fix \(m = 2\).

We now have two definitions for \(T\). On the one hand (equation \((18)\)),
\[
T = \left( \frac{G_1}{G_0} \right)^{1/4} e^{Y''/2} \sqrt{\frac{2 \pi C_1(k)}{k \varphi(k)}} X^{-1/2} e^X
\]
with \(Y'' = X(1 - \nu)^2 / \nu\), and on the other hand (equation \((14)\))
\[
T = \frac{C_1(k)}{k} \exp(\nu X).
\]
These two equations are compatible if and only if there exists \(\nu\) such that \(f(\nu) = 1\), where
\[
f(\nu) = \frac{C_1(k) \varphi(k)}{2\pi k} \left( \frac{G_0^3}{G_1} \right)^{1/2} X e^{-X(1-\nu)^2 / \nu} e^{-2X(1-\nu)}.
\]
Here we have \(m = 2\) and our assumption \(1/\sqrt{m + 1} \leq \nu \leq 1\) gives \(1/\sqrt{3} \leq \nu \leq 1\). Note that
\[
G_0 = \nu^2 (\nu + d / X),
\]
\[
G_1 = \frac{m - 1}{m} \frac{U''}{U'^2 - 1} \left( \nu + \frac{1 + dm}{mX} \right) = \frac{\nu^2}{2\nu^2 - 1} \left( \nu + \frac{1 + 2d}{2X} \right).
\]
It is easy to check that on the interval \(1/\sqrt{3} \leq \nu \leq 1\), \(G_0 / G_1\) is increasing, and hence, \(f(\nu)\) is strictly increasing. Moreover, \(\lim_{\nu \to (1/\sqrt{3})^+} f(\nu) = 0\) and \(f(1) > 1\)
(for all $X \geq \frac{2\pi k}{C_1(k)\varphi(k)}$). So there exists a unique $\nu \in [1/\sqrt{2}, 1]$ such that $f(\nu) = 1$.

For $1/\sqrt{2} < \nu < 1$, we have $(m = 2)$

$$H(\nu) := \frac{G_0^3}{G_1} = \frac{[\nu^2(\nu + d/X)]^2}{\nu^2 + 1} < (\nu + d/X)^2.$$ 

Write, for $X \geq X_3 := \frac{2\pi k}{C_1(k)\varphi(k)}$,

$$\nu_0 = 1 - \frac{1}{2X} \ln \left( \frac{C_1(k)\varphi(k)X}{2\pi} \right).$$  \hspace{1cm} (20)

Let us study $H(\nu_0)$:

$$H(\nu_0) < 1 \text{ if } \nu_0 + d/X \leq 1,$$

equivalently

$$1 - \frac{1}{2X} \ln \left( \frac{C_1(k)\varphi(k)X}{2\pi k} \right) + \frac{\ln(C_1(k)/2\pi)}{X} \leq 1,$$

which holds if $X \geq X_2 := \frac{kC_1(k)}{2\pi\varphi(k)}$.

As

$$f(\nu) = \frac{C_1(k)\varphi(k)}{2\pi} \left( \frac{G_0^3}{G_1} \right)^{1/2} X \exp(-X(1 - \nu)^2/\nu) \exp(-2X(1 - \nu)),$$

replacing $\nu_0$ by (20), we obtain

$$f(\nu_0) = \left( \frac{G_0^3}{G_1} \right)^{1/2} \exp \left( -\ln^2 \left( \frac{C_1(k)\varphi(k)X}{2\pi k} \right) / (4\nu_0 X) \right).$$

Assume that $\nu_0 > 0$, then, for $X \geq X_2$, $f(\nu_0) < 1 = f(\nu)$ and hence $\nu_0 < \nu$. We will require $X \geq X_2$.

The assumption $T \geq H$ holds if $T \geq \frac{C_1(k)}{k} \text{exp}(\nu_0 X) \geq H$. Using (20), rewrite

$$\frac{C_1(k)}{k} \text{exp}(\nu_0 X) = \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} \text{e}^{-\frac{1}{2} \ln X} \text{.}$$

Let $X_0$ satisfy

$$e^{X_0 - \frac{1}{2} \ln X_0} = H \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}.$$

We have $T \geq H$ provided that $X \geq X_0$. We will require $X \geq X_0$.

For $X \geq X_3 := \frac{2\pi k}{C_1(k)\varphi(k)}$, $\nu_0$ is an increasing function of $X$. We will require that $X \geq \max(X_3, 10)$. Then since $C_1(k) \leq 32\pi$ and $X \geq 10$, we have

$$\nu_0 > 0.7462413 \text{ and } 0 < \nu_0 < 1.$$ 

The assumption $\nu > 1/\sqrt{2}$ is satisfied.

We want to evaluate

$$K := G_2(\sqrt{G_0G_1}e^{-Y''})^{1/2},$$

which appears in (19). Again using $C_1(k) \leq 32\pi$ and $X \geq 10$, we find

$$G_0G_1 < \frac{(1 + d/X)}{2\nu_0^2} - 1 \left( \nu_0 + \frac{1 + 2d}{2X} \right) < 8.995.$$ 

The following results will be needed in later computations.
1. Since $X \geq X_0$ and $\exp(X)/\sqrt{X}$ is increasing for $X \geq 1/2$, 
\[ \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} \exp(-X) \leq \frac{1}{H}. \]

2. Since $G_0 G_1 < 9$,
\[ \delta = 2 \sqrt{G_0 G_1} \exp(-Y''/2) \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} e^{-X} \leq 2 \sqrt{3}/H. \]
In particular, for $H \geq 1000$, we have $\delta \leq 0.00347$.

3. \[ G_2 = \frac{R_2(\delta)}{22} (1 + \pi q(T)) < (1 + 3.012 \cdot \delta/2)^2 (1 + \pi q(T)), \]
because
\[ \frac{R_2(\delta)}{22} = \left( \frac{(1 + \delta)^3 + 1}{2} \right)^2 \]
\[ = \left( 1 + \frac{1}{2} \delta (3 + 3\delta + \delta^2) \right)^2 < \left( 1 + \frac{3.012}{2} \delta \right)^2 \]
since $1 + \delta + \delta^2/3 < 1.0035$.

4. Since $T \geq H$,
\[ q(T) = \frac{C_2}{T \ln(kT/2\pi)} \leq \frac{C_2}{H \ln(kH/2\pi)}. \]
But $\exp(-Y''/2) \leq 1$ and $H \geq 1000$, so this yields
\[ K < (8.995)^{1/4} G_2 \]
\[ < (8.995)^{1/4} \left( 1 + \frac{\pi C_2}{1000 \ln(1000/(2\pi))} \right) \times \left( 1 + \frac{3.012}{2} \frac{2\sqrt{3}}{1000} \right)^2 \]
\[ < 1.751. \]
Inserting this upper bound of $K$ (see formula (21) in [14]), we obtain
\[ \varepsilon_1 < 2 \sqrt{\frac{2}{\pi}} K \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r + \frac{\tilde{R}}{x} \]
\[ < 2.7941 \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r + \frac{\tilde{R}}{x}. \]

(22)

Now we want to bound $r$ and $\tilde{R}$.
- An upper bound for $\varphi(k)(1 + \delta) R(T) \phi_0(T)$ and $\varphi(k) A(2, \delta) R(T) \phi_2(T)$. Recall that
\[ R(T) = C_2 \ln(kT) + C_3, \]
\[ \phi_0(T) = \frac{1}{T} \exp \left( -X^2 / \ln(kT/C_1(k)) \right), \]
\[ \phi_m(T) = \phi_0(T) T^{-m}. \]
ESTIMATES OF \theta(x;k,l) FOR LARGE VALUES OF x

Now
\[
\phi_0(T) = \frac{1}{T} \exp(-X^2/(\nu X)) = \frac{1}{T} \exp(-1/\nu X) \leq \frac{1}{T} \exp(-X)
\]
and
\[
\frac{1}{T} = X^{1/2} \exp(-X) \sqrt{\frac{k \phi(x)}{C_1(\nu)}} \left( \frac{G_0}{2\pi e^{Y''}} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4},
\]
hence
\[
R(T)\phi_0(T) \leq \frac{C_2 \ln(kT) + C_3}{T} \exp(-X)
\]
\[
\leq \sqrt{X}e^{-X} \sqrt{\frac{k \phi(x)}{C_1(\nu)}} \left[ C_2 \ln(kT) + C_3 \left( \frac{G_0}{2\pi e^{Y''}} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4} e^{-X} \right].
\]

But
\[
G_0 \leq 1 + \frac{\ln(C_1(k)/2\pi)}{X},
\]
\[
\frac{G_0}{G_1} \leq 2\nu^2 - 1 < 1 \quad (m = 2),
\]
\[
\exp(Y'') \geq 1,
\]
\[
\ln(kT) = \nu X + \ln(C_1(k)) \leq X + \ln(C_1(k)) \leq X + \ln(32\pi).
\]
So, since \(X \geq 10\) and \(C_1(k) \leq 32\pi\),
\[
(1+\theta)\phi(x) \left[ (C_2 \ln(kT) + C_3) \left( \frac{G_0}{2\pi e^{Y''}} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4} \exp(-X) \right]
\]
\[
\leq \phi(x) \left( 1 + \frac{2\sqrt{3}}{1000} \right) \left[ C_2 (X + \ln 32\pi) + C_3 \right] \sqrt{\frac{1 + \ln 16/10}{2\pi}} \exp(-X)
\]
\[
\leq 0.857 \phi(x)X \exp(-X).
\]
Furthermore, if \(X_1\) is defined by \(\exp(X_1)/X_1 = 10\phi(x)\), and if we require that \(X \geq X_1\), then this term is bounded by 0.0857. Hence, under the hypotheses on \(X\) in Theorem 5 an upper bound for \(\phi(x)(1 + \delta)R(T)\phi_0(T)\) is
\[
0.09 \sqrt{\frac{k \phi(x)}{C_1(\nu)}} X^{1/2} \exp(-X).
\]

Next, by (10)
\[
\delta T = 2 \sqrt{\frac{G_1}{G_0}}.
\]
Hence, by Lemma 12
\[
A(2, \delta)/T^2 \leq \frac{R_2(\delta)}{(\delta T)^2} \leq \frac{R_2(\delta) G_0}{2^2 G_1} \leq \frac{R_2(\delta)}{2^2}
\]
and
\[
\phi(x)A(2, \delta)R(T)\phi_0(T) \leq \phi(x)\frac{R_2(\delta)}{2^2} R(T)\phi_0(T).
\]
Using $\delta \leq 2\sqrt{3}/H \leq 2\sqrt{3}/1000$, we get $R_2(\delta)/2^2 \leq 1.0147$. Under the hypotheses on $X$ in Theorem 5, an upper bound for $\varphi(k)A(2, \delta)R(T)\phi_2(T)$ is therefore

$$0.087 \cdot \frac{k\varphi(k)}{C_1(k)} X^{1/2} \exp(-X).$$

The sum of the two terms can be bounded by

$$0.2 \cdot \frac{k\varphi(k)}{C_1(k)} X^{1/2} \exp(-X).$$

- An upper bound for $(1 + \delta)\tilde{E}(T)\frac{\varphi(k)}{\sqrt{T}} + A(2, \delta)\frac{\varphi(k)}{2\sqrt{T}}(\tilde{C}_T + \tilde{D}_T) + \tilde{R}/x$.

For $f(k) = \sum_{p|k} \frac{1}{p-1}$ observe that (Lemma 10)

$$f(k) \leq \frac{\ln k}{\ln 2}.$$

We can explicitly rewrite for $m = 2$, $H \geq 1000$, and $C_1(k) \leq 32\pi$ the following expressions:

$$3\tilde{E}(T) = 3 \left( \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi} \right) \ln T + C_2 \right) + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right),$$

$$\tilde{C}_T = \frac{1}{2\pi T^2} \left( \ln \left( \frac{kT}{2\pi} \right) + 1/2 \right),$$

$$\tilde{D}_T = (2C_2 \ln(kT) + 2C_3 + C_2/3)/T^3,$$

$$\frac{\tilde{R}}{\varphi(k)\sqrt{x}} \leq \left[ (f(k) + 0.5) \ln x + 4 \ln k + 13.4 \right]/\sqrt{x}.$$  

It is tedious but easy to check that the sum of the above quantities is less than

$$\begin{cases} 
1000(\ln T \sqrt{\ln k})^2 & \text{for } k \neq 1, \\
1000 \ln^2 T & \text{for } k = 1.
\end{cases}$$

Now we want to find a number $c$ such that

$$A(2, \delta)\varphi(k)\frac{1000(\ln T \sqrt{\ln k})^2}{\sqrt{x}} \leq c \left( \frac{k\varphi(k)}{C_1(k)} \right)^{1/2} X^{1/2} \exp(-X)$$

with $X = \sqrt{\frac{\ln x}{\ln 2}}$. But $A(2, \delta) \leq \frac{R_2(\delta)}{2^2}$ and by (16), $T = \left( \frac{G_0}{C_0} \right)^{1/2} \frac{2}{3}$, so

$$A(2, \delta) \leq \frac{R_2(\delta)}{2^2} T^2 G_0 \frac{G_0}{G_1}.$$

Moreover, $\frac{1}{\sqrt{x}} = \exp(-RX^2/2)$, hence

$$c \geq 1000 \frac{R_2(\delta)}{2^2} \frac{G_0}{G_1} T^2 \varphi(k)(\ln T \sqrt{\ln k})^2 \left( \frac{C_1(k)}{k\varphi(k)} \right)^{1/2} X^{-1/2} \exp(X - RX^2/2).$$

As $\frac{C_0}{G_1} < 1$, $T^2 = \frac{C_1^2(k)}{k^2} \exp(2\nu X) \leq \frac{C_1^2(k)}{k^2} \exp(2X)$, hence it suffices to take

$$c \geq 1000 \frac{R_2(\delta)}{2^2} \frac{C_1^2(k)}{k^2} (\ln(C_1(k)/k) + X)^2 \left( \frac{C_1(k)}{kX} \right)^{1/2} \exp(3X - RX^2/2).$$
But \( \frac{\varphi(k)}{k} \leq 1, \frac{\ln k}{k} \leq 1, \) and \( R_2(\delta) \leq (1 + 3.012\delta/2)^2 \) with \( \delta \leq \frac{2\sqrt{3}}{1000} \). So, finally, it suffices to take
\[
c \geq \frac{1000}{4} \left( 1 + \frac{3.012\sqrt{3}}{1000} \right)^2 C_1^2(k)(\ln C_1(k) + X)^2 \sqrt{C_1(k)} X^{-1/2} \exp(3X - RX^2/2).
\]

Since \( C_1(k) \leq 32\pi \) and \( X \geq 10 \), we can take
\[
(24) \quad c = 0.643 \cdot 10^{-187}.
\]

In the case \( k = 1 \), we can replace the upper bound \( \frac{\ln k}{k} \leq 1 \) by 1, and obtain the same result. Combining (22), (23), and (24), we obtain the result in Theorem 5:

\[
| \psi(x; k, l) - x/\varphi(k) | / x \leq 2.9941 \sqrt{\frac{k}{\varphi(k) C_1(k)}} X^{1/2} \exp(-X).
\]

We also wish to allow \( \theta \) instead of \( \psi \), which can be done by recalling Theorem 13 of [3]:
\[
0 \leq \psi(x; k, l) - \theta(x) \leq \psi(x) - \theta(x) \leq 1.43\sqrt{x} \quad \text{for} \quad x \geq 0.
\]

Using \( X \geq 10 \), we find \( 1.43\sqrt{x}/x \leq d \cdot 3(k\varphi(k))/C_1(k) \cdot X^{1/2} \exp(-X) \), where \( d = 1.17 \cdot 10^{-204} \). This difference is absorbed by rounding up the constants.

5. Application for \( k = 3 \)

Now we are able to compute \( x_0 \) and \( c \) such that, for \( x \geq x_0 \),
\[
| \theta(x; 3, l) - x/2 | < cx/\ln x.
\]

This would not have been possible if we had used only the results of [3]. According to Theorem [5]
\[
\varepsilon(X) = \frac{3}{2} \sqrt{\frac{6}{20.92}} X^{1/2} \exp(-X)
\]
for \( k = 3 \).

To determine for which \( x \) this bound is valid, let us solve for the constants \( X_0, X_1, X_2, X_3 \) in Theorem [5] Noting that \( H_3 = 10000 \) by the table in Theorem [1] we need \( X_0 \) to satisfy
\[
\exp(X_0 - \frac{1}{2} \ln X_0) \geq 10000 \sqrt{\frac{6}{2\pi \cdot 20.92}} \approx 2136.51.
\]

\( X_0 \approx 8.76 \) works.

Find \( X_1 \) such that
\[
\exp(X_1 - \ln X_1) \geq 20.
\]

\( X_1 \approx 4.5 \) works.

Compute the two other bounds: \( X_2 \approx 4.99, X_3 \approx 1.22 \). Thus we can take \( X = \max(10, X_0, X_1, X_2, X_3) = 10 \) in Theorem [5]

- For \( \sqrt{\ln x} / \ln x \geq 10 \), write \( X = \sqrt{\ln x} / \ln x \), then
\[
\varepsilon(X) \ln x = RX^2 \varepsilon(X).
\]

Find the value \( c \) such that
\[
\varepsilon(X) < c/\ln(x).
\]
For any $x$ such that $\sqrt{\frac{\ln x}{x}} \geq 10$, $c \leq R \cdot 10^2 \varepsilon(10) \leq 0.12$. Hence we have for $x \geq \exp(964.59 \cdots)$,

$$|\theta(x; 3, l) - x/2| \leq 0.12 \frac{x}{\ln x}.$$  

We want to extend the above result for $x \leq \exp(964.59 \cdots)$. Olivier Ramaré has kindly computed some additional values supplementing Table 1 in [3]. We have

$$|\theta(x; 3, l) - x/2| < \tilde{c} \cdot x/2$$

with

\[
\begin{align*}
\tilde{c} &= 0.0008464421 \text{ for } \ln x \geq 400 \quad (m = 3, \delta = 0.00042325), \\
\tilde{c} &= 0.0006048271 \text{ for } \ln x \geq 500 \quad (m = 3, \delta = 0.00030250), \\
\tilde{c} &= 0.0004190635 \text{ for } \ln x \geq 600 \quad (m = 2, \delta = 0.00027950).
\end{align*}
\]

Hence,

- For $e^{600} \leq x \leq e^{964.59\cdots}$
  
  $$c \leq 0.0004190635 \cdot 964.6/\varphi(3) \leq 0.203.$$

- For $e^{400} \leq x \leq e^{600}$

  $$c \leq 0.0008464421 \cdot 600/\varphi(3) \leq 0.254.$$

Using the computations of [3],

- For $10^{100} \leq x \leq e^{400}$

  $$c \leq 0.001310 \cdot 400/\varphi(3) \leq 0.262.$$

- For $10^{30} \leq x \leq 10^{100}$

  $$c \leq 0.001813 \cdot 100 \ln 10/\varphi(3) \leq 0.42/2 \leq 0.21.$$

- For $10^{13} \leq x \leq 10^{30}$

  $$c \leq 0.001951 \cdot 30 \ln 10/\varphi(3) \leq 0.14/2 \leq 0.07.$$

- For $10^{10} \leq x \leq 10^{13}$

  $$c \leq 0.002238 \cdot 13 \ln 10/\varphi(3) \leq 0.067/2 \leq 0.00335.$$

- For $4403 \leq x \leq 10^{10}$

  $$|\theta(x; 3, l) - x/2| < 2.072\sqrt{x} \quad (\text{Theorem 5.2.1 of Ramaré and Rumely [3]})$$

We choose $c = 0.262$. We check that this bound is also valid for $1531 \leq x \leq 4403$.

**Theorem 6.** For $x \geq 1531$,

$$|\theta(x; 3, l) - x/2| \leq 0.262 \frac{x}{\ln x}.$$
6. Results assuming GRH\((k, \infty)\)

Assuming GRH\((k, \infty)\), we obtain more precise results. Under this hypothesis, one can show that function \(\psi\) has the following asymptotic behaviour:

**Proposition 1** ([8] p. 294]. Assume GRH\((k, \infty)\). Then

\[
\psi(x; k, l) = \frac{x}{\varphi(k)} + O(\sqrt{x} \ln^2 x).
\]

**Theorem 7.** Let \(x \geq 10^{10}\). Let \(k\) be a positive integer. Assume GRH\((k, \infty)\).

1) If \(k \leq \frac{1}{3} \ln x\), then

\[
|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.085 \sqrt{x} \ln^2 x.
\]

2) If \(k \leq 432\), then

\[
|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.061 \sqrt{x} \ln^2 x.
\]

**Proof.** Let \(x_0 = 10^{10}\). Applying Theorem 2 in the same way as Theorem 3 (assume that \(T \geq 1\),

\[
\frac{\varphi(k)}{x} |\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq A(m, \delta) \sum_{|\gamma| \geq T} \frac{x^{-1/2}}{|\rho(\rho + 1) \cdots (\rho + m)|}
\]

\[
+ (1 + m\delta/2) \sum_{|\gamma| \leq T} \frac{x^{-1/2}}{|\rho|} + m\delta/2 + \tilde{R}/x
\]

\[
\leq A(m, \delta) \frac{1}{\sqrt{x}} \sum_{|\gamma| \geq T} \frac{1}{|\gamma|^{m+1}} + \frac{1 + m\delta}{2} \frac{1}{\sqrt{x}} \sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x.
\]

\[
\leq A(m, \delta) \frac{\varphi(k)}{\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + (1 + \frac{m\delta}{2}) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{m\delta}{2} + \tilde{R}/x.
\]

Take \(m = 1\) and let

\[
\varepsilon_k(x, T, \delta) := \frac{R_1(\delta)}{\delta} \frac{\varphi(k)}{\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + \left(1 + \frac{\delta}{2}\right) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{\delta}{2} + \tilde{R}/x,
\]

where

\[
\tilde{C}_T = \frac{1}{\pi T} \left( \ln \left( \frac{kT}{2\pi} \right) + 1 \right),
\]

\[
\tilde{D}_T = \frac{1}{T^2} \left( 2C_2 \ln(kT) + 2C_3 + C_2/2 \right),
\]

\[
\tilde{E}(T) = \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln(k/(2\pi)) \ln T + C_2 + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right).
\]

Choose

\[
T = \frac{2R_1(\delta)}{\delta(2 + \delta)}
\]
to minimize in (25) the preponderant terms involving T. So
\[
\frac{R_1(\delta)}{\delta} (\tilde{C}_T + \tilde{D}_T) = \frac{2(2 + \delta)}{4\pi} \left[ \ln \left( \frac{kR_1(\delta)}{\pi\delta(2 + \delta)} \right) + 1 + \frac{\pi\delta(2 + \delta)}{2R_1(\delta)} \left( 2C_2 \ln \left( \frac{2kR_1(\delta)}{\delta(2 + \delta)} \right) + 2C_3 + C_2/2 \right) \right],
\]
\[
(1 + \delta/2) \tilde{E}(T) = \frac{2 + \delta}{4\pi} \left[ \ln^2 \left( \frac{2R_1(\delta)}{\delta(2 + \delta)} \right) + 2 \ln(k/(2\pi)) \ln \left( \frac{2R_1(\delta)}{\delta(2 + \delta)} \right) + 2\pi C_2 + 4\pi \left( \frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right) \right].
\]

With the choice of T, the main terms of \(\varepsilon_k\) are
\[
\frac{\varphi(k)}{\sqrt{x}} \frac{1}{2\pi} \ln^2 \left( \frac{2R_1(\delta)}{\delta(\delta + 2)} \right) + \frac{\delta}{2}.
\]

These terms are minimized by choosing
\[
(27) \quad \delta = \frac{\varphi(k) \ln x}{\pi \sqrt{x}}.
\]

Now, replacing (26) and (27) in (25), we only have a function of x for fixed k:
\[
\varepsilon_k(x) := \varepsilon_k(x, T, \delta).
\]

We simplify expression (25):
\[
\frac{\varepsilon_k(x, T, \delta)}{\varphi(k)} \leq \tilde{\varepsilon}_k(x, T, \delta)
\]
\[
:= \frac{R_1(\delta)}{\delta} (\tilde{C}_T + \tilde{D}_T)/\sqrt{x} + (1 + \delta/2) \tilde{E}(T)/\sqrt{x} + \frac{\delta}{2} + \frac{\tilde{R}}{x \varphi(k)}.
\]

By choosing
\[
T = \frac{2R_1(\delta)}{\delta(2 + \delta)}
\]
and
\[
\delta = \frac{\ln x}{\pi \sqrt{x}},
\]
\(\tilde{\varepsilon}_k(x, T, \delta)\) became \(\varepsilon_k(x)\).

Hence,
\[
\tilde{\varepsilon}_k(x) \sqrt{x} = \frac{2 + \delta}{4\pi} \left[ \ln^2 \left( \frac{2\pi \sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + 2 \ln \left( \frac{k}{2\pi} \right) \ln \left( \frac{2\pi \sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + \ln x \ln \left( \frac{k \sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) \cdot \frac{\ln x}{\sqrt{x}} R_1(\delta) (A) \right] + \frac{\ln x}{2\pi \varphi(k)} + \frac{\tilde{R}}{x \varphi(k) \sqrt{x}}
\]
\[
+ \frac{2 + \delta}{4\pi} (2 + 2\pi C_2 + 4\pi \left( \frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right))
\]
with
\[
A = 2C_2 \ln \left( \frac{2\pi \sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + 2C_3 + C_2/2.
\]

Let
\[
\delta_1 = \frac{\ln x}{\pi \sqrt{x}}.
\]
But
\[
\frac{R_1(\delta)}{2 + \delta} = \frac{2 + 2\delta + \delta^2}{2 + \delta} = 1 + \frac{\delta^2 + \delta}{2 + \delta} \leq d_1 := 1 + \frac{\delta^2 + \delta_1}{2 + \delta_1} \quad \text{because} \quad x \geq x_0 \quad \text{and} \quad \frac{\delta_1}{R_1(\delta)} \leq 1.
\]

By direct computation, for all \(k\) between 1 and 432 and \(x \geq x_0\), of \(\frac{\varepsilon_k(x)}{\sqrt{x}}\), we find an upper bound 0.06012.
To obtain 1) in Theorem 7, we will study the sum in brackets for $1 \leq k \leq \frac{4}{3} \ln x$:

\[
\left[ \frac{1}{4} \ln^2 x + \ln x \left( \frac{2\pi d_1}{\ln x} \right) + \ln x \ln \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) + \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln x + \frac{1}{2} \ln x + \ln(4d_1/5) + \ln x \sqrt{x} (A) \right]
\]

\[
= \left[ \frac{1}{4} \ln^2 x + \ln x \ln \left( \frac{2\pi d_1}{\ln x} \right) + 1/2 + \ln(4 \ln x/(10 \pi)) \right]
\]

\[
+ \ln^2 \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) + \ln(4d_1/5) + \ln x \sqrt{x} (A).
\]

We conclude that

\[
\lim_{x \to +\infty} \frac{\varepsilon_k(x) \sqrt{x}}{\ln^2 x} = \frac{1}{8\pi},
\]

which is the same asymptotic bound as Schoenfeld’s [7] for $\psi$.

The bound $\varepsilon_k(x) \sqrt{x}$ is an increasing function of $k$. Choose $k = \frac{4}{3} \ln x$. Now $\varepsilon_k(x) \sqrt{x}/\ln^2 x$ is a decreasing function of $x$ bounded by 0.0849229 for $x \geq x_0$. □

**Remark.** If we take $k = 1$ in Theorem 7, our upper bound is twice as bad as the result of Schoenfeld [7, p. 337]: for $x > 793$,

\[
|\psi(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \ln^2 x.
\]

These differences are explained by:

- an exact computation of zeros with $\gamma \leq D \approx 158$ (the preponderant ones!) in the sum $\sum \frac{1}{|\gamma|}$,
- a better knowledge of $R(T)$ ($k$ fixed, $k = 1$).

**Corollary 3.** Assume GRH $(k, \infty)$. For all $k$ used in Lemma 4 and $x \geq 224$,

\[
\left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| \leq \frac{1}{4\pi} \sqrt{x} \ln^2 x.
\]

**Proof.** We use Theorem 5.2.1 of [8]: for all $k$ noted in Lemma 4 and $224 \leq x \leq 10^{10}$,

\[
|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq \sqrt{x}
\]

and $\sqrt{x} < \frac{1}{4\pi} \sqrt{x} \ln^2 x$ for $x \geq 35$. We conclude by Theorem 7. □

### 7. Estimates for $\pi(x; 3, l)$

**Definition 1.** Let

\[
\pi(x; k, l) = \sum_{\substack{p \leq x \mod k \neq 0}} 1
\]

be the number of primes smaller than $x$ which are congruent to $l$ modulo $k$.

Our aim is to have bounds for $\pi(x; 3, l)$. We show that

**Theorem 8.** For $l = 1$ or $2$,

(i) $\frac{\pi(x)}{1.2x} < \pi(x; 3, l)$ for $x \geq 151$,

(ii) $\pi(x; 3, l) < 0.55 \frac{x}{\ln x}$ for $x \geq 229869$. 

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From this, we can deduce that for all \( x \geq 151 \),
\[
\frac{x}{\ln x} < \pi(x)
\]
because
\[
\pi(x) = \pi(x; 3, 1) + \pi(x; 3, 2) + 1.
\]

7.1. The upper bound. First we give the proof of Theorem [3](ii).

**Lemma 13.** Let \( I_n = \int_a^x \frac{dt}{\ln^n t} \). Then \( I_n = \frac{x}{\ln^n x} - \frac{a}{\ln^n a} + n I_{n+1} \). Furthermore, for \( a > e \), \((x-a)/\ln^n(x) \leq I_n \leq (x-a)/\ln^n(a)\).

**Theorem 9** (Ramaré and Rumely [3]). For \( 1 \leq x \leq 10^{10} \), for all \( k \leq 72 \), for all \( l \) relatively prime with \( k \),
\[
\max_{1 \leq y \leq x} |\theta(y; k, l) - \frac{y}{\varphi(k)}| \leq 2.072 \sqrt{x}.
\]
Furthermore, for \( x \geq 10^{10} \) and \( k = 3 \) or 4,
\[
|\theta(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.002238 \frac{x}{\varphi(k)}.
\]
Write first
\[
\pi(x; k, l) - \pi(x_0; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(x_0; k, l)}{\ln(x_0)} + \int_{x_0}^x \frac{\theta(t; k, l)}{t\ln^2 t} dt.
\]
Put \( x_0 := 10^5 \).

Preliminary computations :
\[
\theta(10^5, 3, 1) = 49753.417198 \cdots \quad \pi(10^5, 3, 1) = 4784.
\]
\[
\theta(10^5, 3, 2) = 49930.873458 \cdots \quad \pi(10^5, 3, 2) = 4807.
\]
Put \( c_0 := \frac{10^{0.2238}}{2} \) and \( K = \max_i (\pi(10^5, 3, l) - \theta(10^5, 3, l) / \ln(10^5)) \approx 470. \)

* For \( 10^{20} \leq x \),
\[
\pi(x; k, l) - \pi(10^5; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(10^5; k, l)}{\ln(10^5)} + \int_{10^5}^x \frac{\theta(t; k, l)}{t\ln^2 t} dt.
\]

But
\[
\int_{10^5}^x \frac{\theta(t; k, l)}{t\ln^2 t} dt = \int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t\ln^2 t} dt + \int_{10^5}^{\sqrt{x}} \frac{\theta(t; k, l)}{t\ln^2 t} dt + \int_{\sqrt{x}}^x \frac{\theta(t; k, l)}{t\ln^2 t} dt
\]
and, by Theorem [3]
\[
\int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t\ln^2 t} dt < M := 1/\varphi(k) \cdot \int_{10^5}^{10^{10}} \frac{dt}{\ln^2 t} + 2.072 \cdot \int_{10^5}^{10^{10}} \frac{dt}{\sqrt{t}\ln^2 t}
\]
\[
\int_{10^5}^{\sqrt{x}} \frac{\theta(t; 3, l)}{t\ln^2 t} dt < c_0 \frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}}
\]
\[
\int_{\sqrt{x}}^{x} \frac{\theta(t; 3, l)}{t\ln^2 t} dt < c_0 \frac{x - \sqrt{x}}{\ln^2 \sqrt{x}}.
\]
We compute \( M = 10381055.54 \cdots \). Then
\[
\pi(x; 3, l) < c_0 \frac{x}{\ln x} + K + M + c_0 \left( \frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}} + \frac{x - \sqrt{x}}{\ln^2 x} \right) \\
< \frac{x}{\ln x} \left( c_0 + \frac{1}{\ln x} \left( K + M + c_0 \left( \frac{10^{20} - 10^{10}}{\ln^2 10^{10}} \right) \right) \right) \\
< 0.545 \frac{x}{\ln x}.
\]

- For \( 10^{10} \leq x \leq 10^{20} \),
\[
\pi(x; 3, l) < K + \int_{10^{10}}^{x} \frac{\theta(t; 3, l)}{t \ln^2 t} \, dt + \int_{10^{10}}^{x} \frac{\theta(t; 3, l)}{t \ln^2 t} \, dt + c_0 \frac{x}{\ln x} \\
< \frac{x}{\ln x} \left( c_0 + \frac{\ln x}{x} \left( K + M - 10^{10} \frac{c_0}{\ln^2 10^{10}} \right) + \frac{c_0}{\ln^2 10^{10}} \ln x \right) \\
< 0.5468 \frac{x}{\ln x}.
\]

- For \( 10^{5} \leq x \leq 10^{10} \),
\[
\int_{10^{5}}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt < \frac{1}{2} \int_{10^{5}}^{x} \frac{dt}{\ln^2 t} + 2.072 \int_{10^{5}}^{x} \frac{dt}{\sqrt{t} \ln^2 t} \\
= \frac{1}{2} \left( \frac{x}{\ln^2 x} - \frac{10^{5}}{\ln^2 10^{5}} + \frac{x}{\ln^2 x} \right) + 2.072 \int_{10^{5}}^{x} \frac{dt}{\sqrt{t} \ln^2 t}.
\]

Now, \( \int_{a}^{b} \frac{dt}{\sqrt{t} \ln^2 t} \) and \( \frac{2\sqrt{x}}{\ln x} \) are.

Therefore
\[
\pi(x; 3, l) < \frac{1}{2} \frac{x}{\ln x} + 2.072 \frac{\sqrt{x}}{\ln x} + K \\
+ 1 \left( \frac{x}{\ln^2 x} - \frac{10^{5}}{\ln^2 10^{5}} + 2 \int_{10^{5}}^{x} \frac{dt}{\ln^4 t} \right) \\
+ 2.072 \left( \frac{2\sqrt{x}}{\ln^2 x} - \frac{2\sqrt{10^{5}}}{\ln^2 10^{5}} + 4 \int_{10^{5}}^{x} \frac{dt}{\sqrt{t} \ln^3 t} \right) \\
< 0.55 \frac{x}{\ln x} \quad \text{for } x \geq 6 \cdot 10^{5}.
\]

7.2. The lower bound. Let \( KK = \min(\pi(10^{5}, 3, l) - \theta(10^{5}, 3, l)/\ln(10^{5})) \approx 462 \) and \( c = 0.498881 = \frac{1-0.002238}{2} \).

- For \( 10^{10} \leq x \),
\[
\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \frac{\int_{10^{5}}^{x} \theta(t; k, l)}{t \ln^2 t} \, dt \\
> c x \frac{e}{\ln x}
\]
because
\[
KK > 0 \quad \text{and} \quad \int_{10^{5}}^{x} \theta(t; k, l) \, dt > 0.
\]

Lemma 14 (McCurley [2]). For \( x \geq 91807 \) and \( c_2 = 0.49585 \), we have \( \theta(x; 3, l) \geq c_2 x \).
Remark. This bound is better than the one given in Theorem 9 for $x \leq 2.5 \cdot 10^5$. 

$$
\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt.
$$

Thus for any $x_0, x_1$ with $10^5 \leq x_0 < x_1$,

$$
\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x_0} \frac{\theta(t; k, l)}{t \ln^2 t} dt \quad \text{for } x \geq x_0
$$

$$
> \frac{x}{\ln x} \left( c_2 + \left( KK + \int_{10^5}^{x_0} \frac{\theta(t)}{t \ln^2 t} dt \right) \frac{\ln x_1}{x_1} \right) \quad \text{for } x_0 \leq x \leq x_1.
$$

Using the previous remark, we find

$$
\int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt > c_2 \int_{10^5}^{x} \frac{dt}{\ln^2 t} \quad \text{if } 10^5 \leq x \leq 2.5 \cdot 10^5
$$

and

$$
> c_2 \int_{10^5}^{2.5 \cdot 10^5} \frac{dt}{\ln^2 t} + \int_{2.5 \cdot 10^5}^{x} \frac{t/2 - 2.072 \sqrt{t}}{t \ln^2 t} dt \quad \text{if } 2.5 \cdot 10^5 \leq x.
$$

We use this to make step by step computations with Maple:

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$x_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^5$</td>
<td>$2 \cdot 10^6$</td>
</tr>
<tr>
<td>$2 \cdot 10^6$</td>
<td>$3 \cdot 10^7$</td>
</tr>
<tr>
<td>$3 \cdot 10^7$</td>
<td>$3 \cdot 10^8$</td>
</tr>
<tr>
<td>$3 \cdot 10^8$</td>
<td>$3 \cdot 10^9$</td>
</tr>
<tr>
<td>$3 \cdot 10^9$</td>
<td>$10^{10}$</td>
</tr>
</tbody>
</table>

We conclude that $\pi(x; 3, l) > 0.499 \frac{x}{\ln x}$ for $10^5 \leq x \leq 10^{10}$.

7.3. Small values. We now check whether $0.49888 \frac{x}{\ln x} < \pi(x; 3, l) < 0.55 \frac{x}{\ln x}$ for $x < 6 \cdot 10^5$. It is sufficient to prove that

$$
\pi(p; 3, l) < 0.55 \frac{p}{\ln p} \quad \text{for } p \equiv l \mod 3,
$$

and if

$$
0.49888 \frac{p}{\ln p} < \pi(p; 3, l) - 1 \quad \text{for } p \equiv l \mod 3.
$$

The highest value not satisfying the first inequality is $p = 229849$, and the highest value not satisfying the second is $p = 151$. Furthermore, $\pi(229869; 3, l) \leq 10241 < 0.55 \frac{229869}{\ln 229869} \approx 10241.0075$ and $\pi(151; 3, l) \geq 16 > 0.49888 \frac{151}{\ln 151} \approx 15.01$.

The conclusion is

$$
0.49888 \frac{x}{\ln x} \lesssim _{>151} \pi(x; 3, l) \lesssim _{>229869} 0.55 \frac{x}{\ln x}.
$$

Remark. We cannot show that $x/(2 \ln x) < \pi(x; 3, l)$ by using the formula $\theta(x) < c \cdot x$. We have obtained other formulas (see Theorem 6) which we will use below.
7.4. More precise lower bound of $\pi(x;3,l)$. Now we will give the proof of Theorem 8(i).

Classically,

$$\pi(x;3,l) - \pi(10^5;3,l) = \frac{\theta(x;3,l)}{\ln(x)} - \frac{\theta(10^5;3,l)}{\ln(10^5)} + \int_{10^5}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} dt.$$ 

Now $\theta(t;3,l) > \frac{x}{\varphi(3)} (1 - \frac{\alpha}{\ln x})$ with $\alpha = \varphi(3) \cdot 0.262$ by use of Theorem 5. So we write

$$KK = \min_{l} \left( \pi(10^5;3,l) - \frac{\theta(10^5;3,l)}{\ln(10^5)} \right),$$

$$\pi(x;3,l) > J(x,\alpha) = KK + \frac{x}{\varphi(k) \ln x} \left(1 - \frac{\alpha}{\ln x} \right) + \frac{1}{\varphi(k)} \int_{10^5}^{x} \frac{1 - \alpha/\ln t}{\ln^2 t} dt.$$ 

The derivative of $J(x,\alpha)$ with respect to $x$ equals

$$\frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right).$$

Moreover, the derivative of $\frac{x}{\varphi(k) \ln x}$ equals

$$\frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right).$$

The inequality

$$\frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right) < \frac{1}{\varphi(k)} \left( \frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right)$$

holds if $\alpha - 1 < \alpha/\ln x$; this holds for all $x > 1$. The only thing to do is to find a value $x_1$ such that

$$J(x_1,\alpha) > \frac{x_1}{\varphi(k) \ln x_1}.$$ 

For $x_1 = 10^5$, $J(10^5, 0.524) \approx 4607.75$ and $\frac{10^5}{2 \ln 10^5} \approx 4342.94$. We verify by computer that the inequality holds for $x \leq 10^5$ and $l = 1$ or 2. We conclude that

$$\frac{x}{2 \ln x} < \pi(x;3,l)$$

for $x \geq 151$.

References


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