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NEW AMICABLE PAIRS OF TYPE $(2,2)$ AND TYPE $(3,2)$

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Abstract. A UBASIC computer program was developed to implement a method of te Riele for finding amicable pairs of type $(2,2)$. Hundreds of new pairs were found, including a new largest $(2,2)$ pair and several "daughter", "granddaughter", and "great granddaughter" pairs.

1. Introduction

A pair of positive integers (m,n) is called *amicable* if n is equal to the sum of the proper divisors of m and vice versa. If we let $\sigma(x)$ denote the sum of all divisors of x, then (m,n) is an amicable pair when $\sigma(m)-m=n$ and $\sigma(n)-n=m$. The smallest pair of amicable numbers $(220,284)$ was known to the Pythagoreans. The pair $(17296,18416)$ has often been attributed to Fermat, but it was actually first found in the early 14th century by Ibn al-Banna in Marakesh and also by Kamaladdin Farisi in Bagdad. The pair $(9363584,9437056)$ has often been attributed to Descartes, but it was actually first discovered by Muhammad Baqir Yazdi in Iran [3]. Euler found 59 more pairs by noticing that each member of a known pair had a common factor and then a product of different primes on the end. Euler came up with several methods for finding new pairs. In fact, the majority of known pairs have a common factor and then additional primes on the end of the two numbers. Special terminology has been developed for pairs of Euler's form. Suppose you have a pair of the form (Ex,Ey) where E,x, and y are all relatively prime and x and y are products of distinct primes. This pair is considered to be of *type* (i,j) where i is the number of primes in x and j is the number of primes in y. In a previous work by the author, new pairs of type $(i,1)$ were found [4]. In this article, we report on the discovery of 2235 new pairs of type $(2,2)$ and 22 pairs of type $(3,2)$.

2. The method

In 1984, Herman te Riele [12] published a paper that gave methods for generating new amicable pairs from known amicable pairs. By looking at previously known pairs, te Riele was able to observe that he could construct new amicable pairs of the form (a_1x,a_2y), where a_1 and a_2 were factors of a previously known pair. He cleverly named these "daughter" pairs.

In this paper, we concentrate on applying his Method 2. Let $\sigma(n)'=\sigma(n)-n$ and $D=a_1a_2-\sigma(a_1)'\sigma(a_2)'$.

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Method 2. Choose a prime p_2 such that $\gcd(a_1, p_2) = 1$ and $Dp_2 - \sigma(a_2)'\sigma(a_1) > 0$. Find a solution (q_1, q_2) of the bilinear Diophantine equation

$$(*)\qquad \begin{aligned}&(Dp_2 - \sigma(a_2)'\sigma(a_1))q_1q_2 - (\sigma(a_1)'p_2 + \sigma(a_1))\sigma(a_2)(q_1 + q_2)\\ &\quad = a_1\sigma(a_1)p_2^2 + (a_1\sigma(a_1) + \sigma(a_1)'\sigma(a_2))p_2 + \sigma(a_1)\sigma(a_1)\end{aligned}$$

for which both q_1 and q_2 are distinct primes and $\gcd(a_2, q_1q_2) = 1$. For such a solution, compute p_1 from the equation

$$(**)\qquad (\sigma(a_1)'p_2 + \sigma(a_1))p_1 = a_2q_1q_2 - \sigma(a_1)(p_2 + 1).$$

If p_1 is prime and different from p_2 and $\gcd(a_1, p_1) = 1$, then $(a_1p_1p_2, a_2q_1q_2)$ is an amicable pair.

3. Computer program

A computer program was written in UBASIC to implement Method 2. UBASIC was chosen because of its ability to work with large integers quickly. UBASIC also contains a number of useful number-theoretic functions that are built-in and very quick. The program written looks for amicable pairs of the form $(a_1p_1p_2, a_2q_1q_2)$ where p_1, p_2, q_1, q_2 are all distinct primes not dividing a_1 or a_2. The initial version of the program accepted two values e_1 and e_2 and then multiplied e_1 and e_2 by the prime *lastpr* (which ran through 2500 consecutive primes that follow an input value and did not divide e_1 or e_2) to create the values a_1 and a_2. The values of sigma of e_1 and e_2 were computed in a user-defined subroutine, Sigma, and then multiplied by $(\mathit{lastpr} + 1)$, which is sigma of *lastpr*. These products are then $\sigma(a_1)$ and $\sigma(a_2)$, respectively. The current version of the program multiplies e_1 and e_2 by the odd integer *lastpr*, which is relatively prime to e_1 and e_2, to create the values a_1 and a_2. The values of sigma of e_1, e_2, and *lastpr* are computed in the subroutine. Then $\sigma(a_i) = \sigma(e_i)\sigma(\mathit{lastpr})$ for each i. The prime p_2 is chosen to start at the smallest prime larger than $\sigma(a_2)'\sigma(a_1)/D$. Consecutive values for p_2 can easily be found using the built-in NXTPRM function (which finds the smallest prime greater than the input). The number of p_2 values to try is predetermined by the user at the beginning of the program. As te Riele indicated in his article, we can abbreviate the equation $(*)$ in the form

$$c_1q_1q_2 - c_2(q_1 + q_2) = c_3,$$

which is equivalent to

$$(c_1q_1 - c_2)(c_1q_2 - c_2) = c_1c_3 + c_2^2.$$

The right-hand side of this last equation we denote by the variable C. Divisors of C were used to produce q_1 and q_2. When q_1 and q_2 were found to be integers and primes, p_1 was computed using equation $(**)$. If p_1 was determined to be a prime integer, the resulting amicable pair was printed.

A few of the specific details of the program will now be mentioned. When you obtain the UBASIC package, it comes with a UBASIC program which is an extremely fast implementation of the Elliptic Curve Method [10] for factoring integers. This program, ECMX, was appended to our main program and slightly modified so that it became a subroutine in our program. The ECMX subroutine was used at several points. It was called by the subroutine Sigma after all small prime divisors were handled by the built-in PRMDIV function (which finds the least prime divisor of the input). Similarly, the subroutine to factor the above-mentioned C value called on ECMX after all the small prime divisors were found by PRMDIV. Once C was factored, backtracking was used to run through each of the divisors of C up to its square root to form the value q_1. Whenever prime testing was required and could not be handled by the test PRMDIV(N)=N, the subroutine ECMX was called on to determine primality. However, inside ECMX, the first test done on the input is the Adleman, Pomerance, Rumely primality test algorithm [1]. When a determination of primality is all that is required, an exit is made after the Adleman test is completed. This is how the primality of q_1, q_2, and p_1 are determined.

4. Inputs to the program

Most of the time e_1 and e_2 were chosen to be the same value. Consequently, most of the pairs found are of type $(2, 2)$. Sometimes e_1 contained one more prime than e_2. In this case, an amicable pair produced is of type $(3, 2)$. At first, the choices for e_1 and e_2 were often determined by looking at known pairs of other types. As te Riele did in his discovery of "daughter" pairs, either the whole common factor or part of it was chosen for the e_1 and e_2 values. For example, the $(3, 2)$ pair

$$10360830371746725 = 3^2 * 5^2 * 13 * 31 * 149 * 449 * 1707947$$
$$10453453678813275 = 3^2 * 5^2 * 13 * 31 * 73547 * 1567499$$

was discovered by Escott in 1946 [5]. Using the common factor of Escott's pair, we let $e_1 = e_2 = 3^2 * 5^2 * 13 * 31$ and found the two pairs of 19-digit numbers

$$5239885167665187975 = 3^2 * 5^2 * 13 * 31 * 439 * 229 * 574823087,$$
$$5262737823841858425 = 3^2 * 5^2 * 13 * 31 * 439 * 363491 * 363719$$

and

$$8711938551986013825 = 3^2 * 5^2 * 13 * 31 * 1531 * 131 * 479049899,$$
$$8778331314497800575 = 3^2 * 5^2 * 13 * 31 * 1531 * 89399 * 707321,$$

which are both previously unknown pairs of type $(2, 2)$. According to te Riele's terminology, these two pairs are "daughter" pairs of Escott's pair.

Besides using common factors from known pairs, the e_1 and e_2 values were often chosen to be a value that allowed for a substitution. (A list of current substitutions can be obtained via E-mail from Jan Munch Pedersen at jmp@vejlehs.dk.) For example, a very productive choice for producing odd amicable pairs is

$$e_1 = e_2 = 3^3 * 5^2 * 19 * 31.$$

In addition to lots of different *lastpr* values that this number can be multiplied by to get an amicable pair, there are six substitutions that can be used to produce pairs with the same p_1p_2 and q_1q_2 values on the end.

$$\begin{aligned}
&3^3 * 5^2 * 19 * 31 \text{ can be replaced by each of the following :}\\
&3^4 * 7 * 11^2 * 19^2 * 127,\\
&3^5 * 7^2 * 13 * 19^2 * 127,\\
&3^4 * 7 * 11^2 * 19^4 * 151 * 911,\\
&3^5 * 7^2 * 13 * 19^4 * 151 * 911, \quad \text{and}\\
&3^3 * 5^2 * 31 \text{ can be replaced by each of the following :}\\
&3^{10} * 5 * 19 * 23 * 107 * 3851,\\
&3^6 * 5 * 19 * 23 * 137 * 547 * 1093.
\end{aligned}$$

Consequently, if one pair is discovered using $e_1 = e_2 = 3^3 * 5^2 * 19 * 31$, it can possibly generate six additional pairs with the substitutions listed above. However, if p_1, p_2, q_1, q_2, or *lastpr* is a prime in a particular substitution, then that substitution cannot be used. For example, a pair was discovered with $e_1 = e_2 = 3^3 * 5^2 * 19 * 31$ and $lastpr = 547$. This pair generated only five additional pairs from the substitutions, because the last substitution above could not be used as it contains 547.

One feature of the design of the program is that it can be run on several machines with different choices for e_1 and e_2. Running lots of machines in parallel like this did produce lots of new amicable pairs. In order to find previously unknown pairs, the primes p_1, p_2, q_1, and q_2 all need to be large values. This requires then that the $\sigma(e_i)/e_i$ values must be fairly close to 2. Hence a separate program to find all possible 5-digit odd values for e where $\sigma(e)/e$ is between 1.9 and 2.0 was used to obtain the values to use for $e_1 = e_2 = e$. By running our amicable pair program on several machines in parallel in our computer lab, it was possible to systematically test all 5-digit odd values for e where $\sigma(e)/e$ is between 1.9 and 2.0. These runs produced many of the odd pairs found. Lists of n-digit odd values for e where $\sigma(e)/e$ is between 1.9 and 2.0 were also produced. Many of these values were also used as input to the program.

At first, an intensive effort was made to find just odd amicable pairs by inputting only odd values for $e_1 = e_2 = e$. This was a result of successful work by the author on pairs of type $(i, 1)$ [4]. Recently, an effort was also made to discover even amicable pairs with $e_1 = e_2 = 2^k$.

Lastly, inputs included the common factors of newly discovered pairs. For example, two new pairs were discovered having common factor $2^9 * 1031$ using inputs $e_1 = e_2 = 2^9$. Using $e_1 = e_2 = 2^9 * 1031$ as input, the program generated a pair with 48-digit numbers with common factor $2^9 * 1031 * 134923$, which is displayed later.

5. Results

Many previously known pairs of type $(2, 2)$ were produced by the program when small values of e_1 and e_2 were used. For example, one run of the program choosing $e_1 = e_2 = 81$ produced a set of four pairs that were discovered separately between 1921 and 1982. One pair was discovered by Mason in 1921 [11]. One pair was discovered by Garcia in 1957 [7]. One pair was discovered by Lee in 1969 and published in 1972 [9]. One pair was discovered by Woods in 1982 [14].

Over 2250 new amicable pairs were discovered with the computer program and more are being discovered weekly. The search for odd amicable pairs yielded a tremendous number of the new amicable pairs. The smallest new pair found is the pair of odd numbers

$$12735506841255 = 3^3 * 5 * 11^2 * 47 * 263 * 63073,$$
$$12777310556505 = 3^3 * 5 * 11^2 * 47 * 2683 * 6203.$$

The largest new odd pair found is the pair of 98-digit numbers

$$3^6 * 5 * 29 * 37^3 * 73 * 79 * 137^2 * 157 * 521 * 547 * 1093 * 27238423 * 16128936730187$$
$$* 4654107998788087305671340069010810560441694949999,$$
$$3^6 * 5 * 29 * 37^3 * 73 * 79 * 137^2 * 157 * 521 * 547 * 1093 * 27238423$$
$$* 9155570536947514276063 * 81989225188081007428926971636751152437 49.$$

This is currently the largest known odd pair of type $(2, 2)$.

While the search for odd amicable pairs was quite successful, the search for even pairs has been even more fruitful. In fact, a slight modification of the basic program to search for pairs with a common factor being a power of 2 found the current largest pair of type $(2, 2)$, which is the even pair of 233-digit numbers

$$2^{140} * 139379657490816394634598239204052259412 4799$$
$$* 1663700371916779583665012396448580274219478362251155014096175318 07$$
$$59794094132466314565505647369117465109268647783170295107679535799 6$$
$$3188935636549319,$$
$$2^{140} * 18971375900641885458197870186767997487790827204074383 8$$
$$6759562171348008138159685631$$
$$* 12222939929056956897648019981175927831073328558286802033405$$
$$3579639403327790146021945708506337899820068372479 99,$$

which was discovered by letting $e_1 = e_2 = 2^{140}$. It was also the case that while running the program in the search for odd pairs, the program ran long periods of time without giving any possibilities. On the other hand, as soon as the program was run with $e_1 = e_2 = 2^k$, possibilities usually came quickly and often.

In some cases, pairs were found that contained the same common factor as some known pairs. It appears that some previous searches were restricted to the case where a p_2 value caused c_1 to be 1. For example, the following pairs were found by te Riele in 1982 [13]:

$$574284829770135 = 3^3 * 5 * 11^2 * 43 * 2837 * 288191,$$
$$574423766883945 = 3^3 * 5 * 11^2 * 43 * 9803 * 83423$$

and

$$3970936953946215 = 3^3 * 5 * 11^2 * 43 * 2837 * 1992719,$$
$$3971874070386585 = 3^3 * 5 * 11^2 * 43 * 8663 * 652739.$$

The following pair has the same common factor

$$150506795002165242879 15 = 3^3 * 5 * 11^2 * 43 * 2129 * 10064515008767,$$
$$150577438839493194752 85 = 3^3 * 5 * 11^2 * 43 * 3023759 * 7089655583,$$

and was found by Borho and Battiato in 1988 [2]. All three of these pairs have $c_1 = 1$. By using $e_1 = e_2 = 3^3 * 5 * 11^2$, we found both of te Riele's pairs, Borho and Battiato's pair, and the following four new pairs with the indicated c_1 values:

$$695112182897827365 = 3^3 * 5 * 11^2 * 43 * 2267 * 436531699, \qquad (c_1 = 31)$$
$$695398712394460635 = 3^3 * 5 * 11^2 * 43 * 34649 * 28572983;$$

and

$$804274558747421355 = 3^3 * 5 * 11^2 * 43 * 2153 * 531829847, \qquad (c_1 = 17)$$
$$804643646368190805 = 3^3 * 5 * 11^2 * 43 * 185243 * 6184067;$$

and

$$1345949335183059585 = 3^3 * 5 * 11^2 * 43 * 2887 * 663734411, \qquad (c_1 = 1519)$$
$$1346249194534055295 = 3^3 * 5 * 11^2 * 43 * 8093 * 236825423;$$

and

$$3827041334257185 = 3^3 * 5 * 11^2 * 43 * 2287 * 2382371, \qquad (c_1 = 7)$$
$$3828592540912095 = 3^3 * 5 * 11^2 * 43 * 40013 * 136223.$$

To find the last pair requires that the number of p_2 values tried be greater than 12365. For much of the time in our computer runs, the number of p_2 values to try was limited to 10000 or less.

Once a pair was found and believed to be new, it was sent to Jan Munch Pedersen for verification that it was indeed new. He is maintaining the list of all discovered amicable pairs at the web site:

`http://www.vejlehs.dk/staff/jmp/aliquot/knwnap.htm`

Pairs of just type $(2, 2)$, including ones found by our program, can be found at

`http://www.vejlehs.dk/staff/jmp/aliquot/apreg22.txt`

With an exhaustive search using many different e_1 and e_2 values, the program has found 2235 pairs of type $(2, 2)$, 22 pairs of type $(3, 2)$, and 3 pairs which are called exotic pairs because the powers on the common primes in e_1 and e_2 are not the same (e_1 contains 3^3 and e_2 contains 3^2 in the exotic pairs that were discovered).

6. Daughters, granddaughters, and great granddaughters

If we restrict "daughter" pairs to be of the same type as the "mother" pair and the "daughter" pairs must contain at least one more prime divisor in the common factor, we have some interesting genealogy.

Suppose we start with the following pair discovered by Lee in 1966 [8] as the "mother" pair:

$$7074650624 = 2^9 * 947 * 14591,$$
$$7076729344 = 2^9 * 1367 * 10111.$$

When $e_1 = e_2 = 2^9$ was used as input to the program, eleven new "daughter" pairs were obtained. Three of these were

$$3127099868507907584 = 2^9 * 1031 * 75679 * 78277643,$$
$$3127135197141761536 = 2^9 * 1031 * 544367 * 10882439;$$

and

$$231974355636586737 6128 = 2^9 * 1031 * 66751 * 65834504999,$$
$$231977790414096398 3872 = 2^9 * 1031 * 5779199 * 760414049;$$

and

$$6263273477289739140029 44 = 2^9 * 1031 * 67043 * 17697801068639,$$
$$6263365398253373373148 16 = 2^9 * 1031 * 4173487 * 284302093319.$$

When $e_1 = e_2 = 2^9 * 1031$, which is the common factor in the above pairs, was used as input to the program, the following new "granddaughter" was obtained.

$$\begin{aligned}&181355921259459017640118201505140867001444628992\\&\quad = 2^9 * 1031 * 134923 * 3011759 * 8454674801160802956512741042 23,\\&181355981475383819587609861085138326465532282368\\&\quad = 2^9 * 1031 * 134923 * 8056783096529 * 3160498560537268269934 07.\end{aligned}$$

Let us start a new genealogical chain. Suppose we start with the following pair, discovered by Euler in 1747 [6], as the "mother" pair:

$$67095 = 3^3 * 5 * 7 * 71,$$
$$71145 = 3^3 * 5 * 17 * 31.$$

Then the following pair, discovered by Garcia in 1957 [7], is a "daughter" of Euler's pair:

$$11123243055 = 3^3 * 5 * 17 * 19 * 79 * 3229,$$
$$11202516945 = 3^3 * 5 * 17 * 19 * 199 * 1291.$$

The following pair, discovered by Borho and Battiato in 1987 [2], is then a "granddaughter" of Euler's pair.

$$1629533973784736214045 = 3^3 * 5 * 17 * 19 * 107 * 17333 * 20149747759,$$
$$1629627696266737193955 = 3^3 * 5 * 17 * 19 * 107 * 6220979 * 56144807.$$

Finally, the following three pairs were discovered in April, 2000 by our computer program and are "great granddaughters" of Euler's pair:

$$\begin{aligned}&317819549181139958723115216635830 35\\&\quad = 3^3 * 5 * 17 * 19 * 107 * 34679 * 5036047 * 39003555504842837,\\&317819612288044801655193440680009 65\\&\quad = 3^3 * 5 * 17 * 19 * 107 * 34679 * 183683149571 * 1069361986391;\end{aligned}$$

and

$$3717466418721520968504020605231223520781 35$$

$$= 3^3 * 5 * 17 * 19 * 107 * 34613 * 11285503 * 2039702473875676023376 19,$$

$$3717466748122474198227749458273069711858 65$$

$$= 3^3 * 5 * 17 * 19 * 107 * 34613 * 3823415900159 * 602055100172872877;$$

and

$$1301376175418399515840226263818928 95$$

$$= 3^3 * 5 * 17 * 19 * 107 * 34667 * 5599109 * 143696966668797119,$$

$$1301376407827813133823487250472271 05$$

$$= 3^3 * 5 * 17 * 19 * 107 * 34667 * 81140923099 * 9915774831071.$$

7. Future work

As machines get faster, some of the most productive choices of e_1 and e_2 should probably be retested with more than just the 2500 values we used for *lastpr*. In many cases, the program was stopped before it could run through all 2500 *lastpr* values in order to allow the machine to work on another set of choices for e_1 and e_2. To allow the *lastpr* value to go through a much larger range of integers simply requires increasing the limit of a counter. The program allows an input of where to start *lastpr* values and prints out the ones tried, so that further runs can pick up where previous runs left off. In addition, those same productive choices of e_1 and e_2 should be run with a larger number of p_2 values. This too can be handled by simply increasing the limit of a counter.

As you can see by the discussion, the genealogical chains cause the pairs to get fairly large fairly quickly. Maybe a "great-great granddaughter" of the Euler pair can be found in the near future.

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