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CORRIGENDA AND ADDITION TO "COMPUTER VERIFICATION OF THE ANKENY-ARTIN-CHOWLA CONJECTURE FOR ALL PRIMES LESS THAN 100 000 000 000"

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Abstract. An error in the program for verifying the Ankeny-Artin-Chowla (AAC) conjecture is reported. As a result, in the case of primes p which are $\equiv 5 \bmod 8$, the AAC conjecture has been verified using a *different* multiple of the regulator of the quadratic field $\mathbb{Q}(\sqrt{p})$ than was meant. However, since *any* multiple of this regulator is suitable for this purpose, provided that it is smaller than $8p$, the main result that the AAC conjecture is true for all the primes $\equiv 1 \bmod 4$ which are $< 10^{11}$, remains valid.

As an addition, we have verified the AAC conjecture for all the primes $\equiv 1 \bmod 4$ between 10^{11} and 2×10^{11}, with the corrected program.

1. Corrigenda

Let p be a prime $\equiv 1 \bmod 4$, and consider the Dirichlet L-function

$$L(1, \chi_p) = \prod_q (1 - \chi_p(q)/q)^{-1},$$

where the product is taken over all the primes q, and the character $\chi_p(q)$ is the same as the Kronecker symbol (p/q). This coincides with the Legendre symbol when q is an odd prime. When $q = 2$, the Kronecker symbol $(p/2)$ is only defined when p is 0 or 1 mod 4. We have

$$(p/2) = 1 \text{ when } p \equiv 1 \bmod 8, \ (p/2) = -1 \text{ when } p \equiv 5 \bmod 8$$

and

$$(p/2) = 0 \text{ when } p \equiv 0 \bmod 4$$

(but the latter case does not apply here). In [1, p. 1314], an approximation $S(T, p)$ of the function $\log L(1, \chi_p)$ is used, given by

$$S(T, p) = \sum_{q<2T-1} w(q) \log\left(\frac{q}{q - \chi_p(q)}\right), \tag{1}$$

where T is some parameter which varies between 500 and 5000, $w(q) = 1$ when $q < T$, and $w(q)$ is a more complicated function of q and T when $T \le q < 2T - 1$, explicitly given in [1, p. 1314].

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In the computer program written by one of the authors (HJJtR), a mistake was made with the implementation of the Kronecker symbol (p/q). Namely, this was assumed to coincide with the Legendre symbol not only for $q > 2$ (which is correct), but also for $q = 2$, which is correct for primes $p \equiv 1 \bmod 8$ *but not* for primes $p \equiv 5 \bmod 8$: in that case, we have $(p/2) = 1$ for the Legendre symbol and $(p/2) = -1$ for the Kronecker symbol. Hence, for $q = 2$, and $p \equiv 5 \bmod 8$, for the value of

$$w(q) \log\left(\frac{q}{q - \chi_p(q)}\right) = w(q) \log\left(\frac{q}{q - (p/q)}\right)$$

we took $\log(2/(2-1)) = \log 2$, where we should have taken $\log(2/(2-(-1))) = \log(2/3)$. As a consequence, in the approximation $\exp(S(T,p))$ of $L(1,\chi_p)$ we used the factor $\exp(\log(2)) = 2$, where we should have taken $\exp(\log(2/3)) = 2/3$. This means that in our computation of an estimate E for a multiple of R_2 according to the formula

$$E = \frac{\sqrt{p}}{\log 4} \exp(S(T,p))$$

[1, p. 1314, line -6], we computed a value of E which was too big by a factor of 3 in those cases where $p \equiv 5 \bmod 8$. As a consequence, the integral multiple of R_2, computed from E with Algorithm 5.4 of [1, p. 1322], was also too big by a factor of 3. *However*, in the verification of the AAC conjecture with Algorithm 6.1 of [1, p. 1324], *any* multiple of R_2 is suitable, as long as it is smaller than $8p$. So the programming mistake was fortunate enough not to influence the main result of [1]!

Corrections in [1] as a result of the incorrect implementation of the Kronecker symbol: In Table 2.1, the values of E and E/R_2 should be divided by 3 for the second and third prime (which are 5 mod 8). The detailed example given in Section 8 of [1] provides a correct proof of the AAC conjecture for $p = 97\,843\,343\,893$, but a correct implementation of the Kronecker symbol would have given a value of E which is 1/3 of the value of E given there. The other computational results in Section 8 would have changed correspondingly.

Additional correction: The values of $\xi(kR_2)$ and $\eta(kR_2)$ given on page 1327 of [1] should be interchanged.

2. Addition

With our corrected program, we have extended the verification of the AAC conjecture to all the primes $\equiv 1 \bmod 4$ between 10^{11} and 2×10^{11}. No exceptions were found. The computations were carried out on one CPU of CWI's SGI Origin 2000 computer, and took 950 CPU hours.

Reference

[1] A. J. van der Poorten, H. J. J. te Riele and H. C. Williams, Computer verification of the Ankeny-Artin-Chowla conjecture for all primes less than 100 000 000 000, *Math. Comp*, **70** (2001), 1311–1328. MR **2001j:**11125

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