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PSEUDOZEROS OF MULTIVARIATE POLYNOMIALS

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Abstract. The pseudozero set of a system f of polynomials in n complex variables is the subset of $\mathbf{C}^n$ which is the union of the zero-sets of all polynomial systems g that are near to f in a suitable sense. This concept is made precise, and general properties of pseudozero sets are established. In particular it is shown that in many cases of natural interest, the pseudozero set is a semialgebraic set. Also, estimates are given for the size of the projections of pseudozero sets in coordinate directions. Several examples are presented illustrating some of the general theory developed here. Finally, algorithmic ideas are proposed for solving multivariate polynomials.

1. Introduction

1.1. **Summary.** The study of systems of polynomial equations in several variables is important in computer science and engineering because of applications in diverse areas such as robot motion and computer vision. Developing fast and reliable methods for solving polynomial systems is a high priority. Broadly speaking, there are two general approaches: symbolic and numeric. Among the former belong methods from the theory of Gröbner bases (see [5]) and methods based on the theory of resultants; see [6, 13]. Among the latter are methods based on iteration, such as Newton's method, and homotopy continuation methods; see [7, 11, 18, 34]. Lately, there have been serious efforts to implement systems that combine both methods; see, e.g., [19, 23].

In practically all situations arising in science or engineering, the data are known only to limited accuracy. For a system modeled by polynomial equations this means that the coefficients of those polynomials are known only to within certain tolerances. Thus it is important to understand the variation of the roots of a polynomial system in the presence of uncertainty of the coefficients. For systems of linear equations this is a classical subject in numerical analysis. For nonlinear systems, new phenomena occur, whose study is rather recent. See for instance [4, 15, 16], and [25, chapter 5]. One key concept for this study is that of a *pseudozero* of a system of polynomials, which has been studied in [3, 22, 27, 28, 29, 33]. The present work is an outgrowth of a previous investigation of the third author of this paper [33], which dealt with pseudozeros only of univariate polynomials. In this paper we begin an investigation of pseudozeros of multivariate polynomial systems. Some of this work has an overlap with previous investigations. Especially relevant are

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the papers of Stetter just cited, but as will become clear, our viewpoint is quite different.

Recall that the pseudozero set of a system of polynomials $f(z)$ in several complex variables $z = (z_1, ..., z_n)$ is the subset of complex n-space $\mathbf{C}^n$ which consists of the union of the zero sets of all polynomial systems $g(z)$ whose coefficients differ from those of $f(z)$ by no more than some specified amount. The motivation for studying this set is that the roots of many polynomials are extremely sensitive to small changes in the coefficients. Often in practice, the coefficients are known only approximately or, even if they are known exactly, roundoff errors occurring in the course of numerically solving the system may have an effect equivalent to introducing small changes. In either case, we can get a damaging fuzziness in the roots from seemingly harmless deformations of the coefficients, and in numerical applications it is important to understand the range over which roots may thus vary. In fact, as we will show, this even leads to useful algorithmic ideas for the robust computation of roots. As is pointed out in the introduction to [28], the idea of a pseudozero belongs to the philosophy of backward error analysis, in which an *exact* solution to a nearby problem is regarded as meaningful.

The mathematical domains that study polynomial systems are algebraic geometry and commutative algebra. The particular study of variation of polynomial systems belongs to deformation theory. We bring in several nontrivial tools from these disciplines in order to understand pseudozeros. This has the disadvantage that several sections of this paper are rather abstract. We have tried to counterbalance the abstraction by the consideration of examples. At any rate, we believe that a true understanding of the phenomena that can appear in solving systems of polynomial equations requires some of the powerful machinery of these subjects.

We now outline the contents of this paper. The latter sections of the introduction present notation and background information, including a brief reminder of the standard notations for projective space. Let f be a system of polynomials in any number of variables. In section 2, we define $Z(f, B, \varepsilon)$, the ε-pseudozero set of f, relative to a linear deformation space. It is difficult to compute $Z(f, B, \varepsilon)$ directly from the definition, since it involves an existential quantifier. In Proposition 2.1, we show that $Z(f, B, \varepsilon)$ can be described by a system of inequalities that is easily computed from the data defining the pseudozero set. This result is the key to our further investigations. Actually, our Proposition 2.1 is a special case of a more general result of Stetter's (see [27], and Remark 2.1). In section 2.2, we consider generalizations of the definition of pseudozero set that retain the useful property of being semialgebraic. In section 2.3, we examine pseudozero sets in projective space, where it is possible to generalize some results obtained by Mosier in the one-variable case. We finish this section with an example that shows how the projective completion of a pseudozero set repairs pathology that arises in affine space.

Section 3 treats problems related to estimating the size of pseudozero sets. We begin with some general observations that are based on properties of semialgebraic sets. In particular, we examine how the diameter d of a projection of $Z(f, B, \varepsilon)$ onto a lower dimensional subspace varies with the parameter ε. In the case of an isolated zero, we show that there are positive rational constants c and ν such that $d \leq c\varepsilon^{\nu}$. This is based on a powerful and general inequality of Hörmander and Łojasiewicz, but it is rather difficult to calculate the exponent ν from this. Therefore we give a different, more effective proof of this result in the case of an isolated zero in section 3.2. The optimal possible value of ν depends on the direction of projection. Roughly speaking, what is important are various tangency conditions of the defining equations with the projection direction, but the exact formulation is rather subtle. In section 3.2 we give a theoretical analysis of some local properties of the pseudozero set that are used to derive estimates for these projections. In section 3.3 we illustrate the abstract theory with some of the examples that are later studied using the computer visualization tools. Our experiments provide vivid illustrations of the theory we have developed.

A motivation for the special study of projections of pseudozero sets is that one of the methods for solving multivariate polynomial systems is via successive projections onto spaces of lower dimension, for instance by means of Gröbner basis techniques or by resultants. In fact the examples worked in section 3.3 were done with the aid of a multiresultant package written in Mathematica by the second author.

In section 4, we discuss two algorithmic ideas for solving systems of polynomials based on the concept of the pseudozero set. The first studies the conditioning of the polynomial zeros by projections in various directions. The second discusses the change from one Taylor basis to another. This is applied in particular to systems of linear equations, where the Iterative Refinement Algorithm is employed. We give a new interpretation of the well-known theorem on the order of error reduction of the refinement (see Theorem 4.1).

1.2. **Notation.**

- $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ denote respectively the natural numbers, the integers, the field of rational numbers, the field of real numbers and the field of complex numbers.
- The n coordinate functions on real affine n-space $\mathbf{R}^n$ are denoted $x_1, \ldots, x_n$. In complex affine n-space $\mathbf{C}^n$, we use $z_1, \ldots, z_n$ to denote the coordinate functions. Each z_j can be written as $z_j := x_j + iy_j$, where x_j and y_j are real-valued functions.
- We often use x to stand for the n-tuple $(x_1, \ldots, x_n)$. A monomial in the functions x_i is written in the form $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha \in \mathbf{N}^n$. The degree of x^α is $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Analogous notation is used with z (a tuple of complex numbers), a (a tuple of constants), *etc.*
- Elements of $\mathbf{C}[x]$ or $\mathbf{R}[x]$ are denoted f, g, etc. We also use f, g, etc. to denote systems of polynomials. We use capital letters to denote homogeneous polynomials, or systems of homogeneous polynomials.
- A point in $\mathbf{R}^n$ or $\mathbf{C}^n$ is typically referred to by the values of the coordinate functions at that point, which are written as an n-tuple: $(a_1, \ldots, a_n)$. Projective coordinates on complex projective n-space $\mathbf{P}^n(\mathbf{C})$ are denoted $z_0, \ldots, z_n$. A point in $\mathbf{P}^n(\mathbf{C})$ is denoted $[a_0, \ldots, a_n]$ (with at least one of the a_i non-zero). As usual, if $\lambda \neq 0$, then $[a_0, \ldots, a_n] = [\lambda a_0, \ldots, \lambda a_n]$.
- If $f \in \mathbf{C}[x]$, then the *complex zero set of* f is $Z^{\mathbf{C}^n}(f) := \{ a \in \mathbf{C}^n \mid f(a) = 0 \}$. This is also often denoted simply $Z(f)$. If $f \in \mathbf{R}[x]$, then the *real zero set of* f is $Z^{\mathbf{R}^n}(f) := \{ a \in \mathbf{R}^n \mid f(a) = 0 \}$ — or simply $Z^{\mathbf{R}}(f)$. The *positivity set of* f is $P(f) := \{ a \in \mathbf{R}^n \mid f(a) > 0 \}$.
- Let $f = \{f_1, \ldots, f_k\} \in \mathbf{C}[z_1, \ldots, z_n]$ be a collection of polynomials with complex coefficients. We can write each $f_j(x + iy) = g_j(x, y) + ih_j(x, y)$, to

get a system of real polynomials

$$(g,h) = \{g_1, \dots, g_k, h_1, \dots, h_k\} \subseteq \mathbf{R}[x_1, \dots, x_n, y_1, \dots, y_n].$$

If we identify $\mathbf{C}^n = \mathbf{R}^{2n}$, then

$$Z(f) = Z^{\mathbf{R}}(g,h).$$

Thus, any discussion of complex zeros of systems of polynomials with complex coefficients can be reduced to the real zeros of systems of polynomials with real coefficients. (In 2.2, when we consider pseudozero sets from the point of view of elimination theory, the real language will be more natural. In solving polynomial systems, and in discussing algebro-geometric properties, on the other hand, the complex language is more suitable.)

- A *semialgebraic set* in $\mathbf{R}^n$ is a subset of $\mathbf{R}^n$ that belongs to the boolean algebra of subsets generated by $\{ P(f) \mid f \in \mathbf{R}[x] \}$. In other words, a semialgebraic set is a set that can be constructed from positivity sets using the operations of union, intersection and complement finitely many times. By distributivity, any semialgebraic set is a finite union of finite intersections of positivity sets and their complements.

1.3. **Affine and projective varieties.** For a general introduction in the spirit of this paper, see [6]. Let $F = (F_1, \dots, F_k) \in \mathbf{C}[z_0, \dots, z_n]$ be a collection of complex polynomials, each homogeneous of some degree $d_j = \deg F_j$. The zero set is

$$Z(F) = \{a \in \mathbf{P}^n(\mathbf{C}) : F_j(a) = 0, \ \forall j = 1, \dots, k\}.$$

Recall that the compact complex manifold $\mathbf{P}^n(\mathbf{C})$ is covered by the $n+1$ open subsets

$$U_i = \{[z_0, \dots, z_n] \in \mathbf{P}^n(\mathbf{C}) : z_i \neq 0\}$$

and each of these is isomorphic with $\mathbf{C}^n$ via

$$\varphi_i : U_i \xrightarrow{\sim} \mathbf{C}^n : [z_0, \dots, z_n] \mapsto \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i} \right)$$

with inverse

$$\psi_i : \mathbf{C}^n \xrightarrow{\sim} U_i : (z_1, \dots, z_n) \mapsto [z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n].$$

The intersection $Z(F) \cap U_i \cong Z(f)$, where $f = (f_1, \dots, f_k)$ is the set of complex polynomials obtained by dehomogenizing the polynomials F with respect to the variable z_i, that is,

$$f(z_1, \dots, z_n) := F(z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n).$$

Note that the polynomials f are no longer necessarily homogeneous. Conversely, starting with a system of polynomials f, we have their common zero locus $Z(f) \subset \mathbf{C}^n$; viewing $\mathbf{C}^n$ as $U_0 \subset \mathbf{P}^n$, say, we define corresponding homogeneous polynomials F via

$$F(z_0, \dots, z_n) = z_0^{\deg f} f\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0} \right).$$

Thus we get a subset $Z(F) \subset \mathbf{P}^n(\mathbf{C})$. Informally, we think of the locus $Z(F) \cap \{z_0 = 0\}$ as the part of $Z(f)$ "lying at infinity." The sets $Z(f)$ (resp. $Z(F)$) depend only on the ideal generated by the polynomials f in the polynomial ring $\mathbf{C}[z]$ (resp., on the homogeneous ideal generated by the F in $\mathbf{C}[z]$). In general, subsets $Z(f) \subset \mathbf{C}^n$ (resp. $Z(F) \subset \mathbf{P}^n(\mathbf{C})$) will be called affine (resp. projective) varieties.

2. Pseudozero sets of multivariate polynomials

2.1. **Definition and examples.** In this section, we define and discuss *pseudozero sets* of systems of multivariate polynomial equations. Let $f = (f_1, \dots, f_k) \in (\mathbf{C}[z])^k$, where $\mathbf{C}[z] := \mathbf{C}[z_1, \dots, z_n]$. To state the main idea as directly as possible, the pseudozero set of f is the union of the zero sets of all systems $\widehat{f}$ that are "acceptable approximations" of f in the sense that they come from f by small changes in the coefficients. What counts as an "acceptable approximation" depends on context. As we flesh out this idea in the following paragraphs, it is useful to think of two steps in specifying the approximations. First, one fixes a "domain of variation" from which the (coefficients of the) $\widehat{f}$ may be chosen. After this, one fixes a way of determining, in that space, which $\widehat{f}$ are "sufficiently close" to f. We now describe what seems to be the most useful way of making these specifications.

Let V_i, $i = 1, 2, \dots, k$, be a finite-dimensional linear subspace of the $\mathbf{C}$-vector space $\mathbf{C}[z]$. For example, V_i could be all polynomials of degree less than or equal to some fixed bound d_i. The V_i do not need to be the same, but in many typical applications they will be. (See the end of this section for a list of some typical cases.) Let $V := V_1 \oplus \cdots \oplus V_k$. Our "domain of variation" will be $\{ f + g : g \in V \}$. We view V as the space of permissible deformations of f.

In order to measure "closeness" we use a norm on V. To specify the norm, pick a basis $B_i = \{ b_{i\gamma}(z) : \gamma \in \Gamma_i \}$ for each V_i. Sometimes we will use the notation $B_i(z)$, and view this collection of polynomials as a vector of polynomials, with some order chosen on the set Γ_i. Then an expression such as $||B_i(z)||_p$ (p-norm of the vector) becomes meaningful as a function of z.

Let B denote the basis for V built up from the B_i. If $g = (g_1, \dots, g_k) \in V$, then $g_{i\gamma}$ denotes the component of g_i at $b_{i\gamma}(z)$ (i.e., the scalar coefficient of $b_{i\gamma}(z)$ in the expression for g_i), so $g_i(z) = \sum_\gamma g_{i\gamma} b_{i\gamma}(z)$. Finally, we let $\|\_\|_B$ be the sup-norm on V induced by B:

$$\|g\|_B := \sup\{ |g_{i\gamma}| : i = 1, \dots, k \,; \gamma \in \Gamma_i \}.$$

Let $\varepsilon > 0$. We define the *ε-neighborhood of f relative to B* to be

$$N(f, B, \varepsilon) := \{ f + g : g \in V \;\&\; \|g\|_B \le \varepsilon \},$$

and we define the *ε-pseudozero set of f relative to B* to be

(2.1)
$$Z(f, B, \varepsilon) := \bigcup \{ Z(\widehat{f}) : \widehat{f} \in N(f, B, \varepsilon) \} = \bigcup \{ Z(f + g) : g \in V \;\&\; \|g\|_B \le \varepsilon \}.$$

This set is exactly the collection of zeroes of all systems whose B-coefficients differ from those of f by no more than ε, and so it very well suits the purposes described in the introduction. However, $Z(f, B, \varepsilon)$ cannot be computed directly because it is an *infinite union*. This is remedied by the following proposition, which on the one hand generalizes [22, Theorem 1] and [31, Proposition 2.1] and on the other is implied by results in [26], as explained and generalized in [27]. See our Remark 2.1, below. For the convenience of the reader, and to illustrate the methods, we have included a short self-contained proof.

Proposition 2.1.

$$Z(f, B, \varepsilon) = \{ z \in \mathbf{C}^n : \forall i \;\; |f_i(z)| \le \varepsilon \sum_{\gamma \in \Gamma_i} |b_{i\gamma}(z)| = \varepsilon ||B_i(z)||_1 \}.$$

Proof. We will show inclusions both ways.

($\subseteq$) Let $a \in Z(f, B, \varepsilon)$. Then there exists $g \in V$ such that $\|g\|_B \leq \varepsilon$ and $f(a) + g(a) = 0$. Then

$$|f_i(a)| = |g_i(a)| = \left| \sum_{\gamma \in \Gamma_i} g_{i\gamma} b_{i\gamma}(a) \right| \leq \sum_{\gamma \in \Gamma_i} |g_{i\gamma}|\, |b_{i\gamma}(a)| \leq \varepsilon \sum_{\gamma \in \Gamma_i} |b_{i\gamma}(a)|.$$

($\supseteq$) Suppose that a is a point such that $|f_i(a)| \leq \varepsilon \sum_{\gamma \in \Gamma_i} |b_{i\gamma}(a)|$ for $i = 1, \ldots, k$. Define

$$r_i(z) := \sum_{\gamma \in \Gamma_i} r_{i\gamma} b_{i\gamma}(z), \quad \text{where} \quad r_{i\gamma} := e^{-i\arg(b_{i\gamma}(a))}.$$

Then,

$$r_i(a) = \sum_{\gamma \in \Gamma_i} |b_{i\gamma}(a)|.$$

If $r_i(a) = 0$, then $f_i(a) = 0$, so let $g_i(z) \equiv 0$. If $r_i(a) \neq 0$, define

$$g_i(z) := -\frac{f_i(a)}{r_i(a)} r_i(z).$$

By construction, $f(a) + g(a) = 0$. Moreover,

$$\|g\|_B = \sup_i \{ \sup_{\gamma \in \Gamma_i} \left| \frac{f_i(a)}{r_i(a)} r_{i\gamma} \right| \} = \sup_i \frac{|f_i(a)|}{|r_i(a)|} \leq \varepsilon,$$

showing that $a \in Z(f, B, \varepsilon)$. □

Observe that $Z(f) \subseteq Z(f, B, \epsilon)$, but $Z(f)$ is *not necessarily* in the interior of $Z(f, B, \epsilon)$. The simplest example would be when $V_1 = \{0\}$. In this case, no variation in f_1 is allowed, and $Z(f, B, \epsilon) \subseteq Z(f_1)$. For a second example, let $f = (z_1, z_2)$, so $Z(f)$ is the origin in $\mathbf{C}^2$. Let $B_1 = \{az_2 : a \in \mathbf{C}\}$, and let $B_2 = \{bz_1 : b \in \mathbf{C}\}$. Then for small ε, $Z(f, B, \varepsilon) = Z(f)$.

Sometimes we want to incorporate weights into the pseudozero set $Z(f, B, \varepsilon)$ to make it compatible with numerical application (as, for example, in the discussion of componentwise perturbation [8]). This can be accomplished by modifying the basis. Because we use this later, we provide details. A weight vector will be a set $w = \{ w_{i\gamma} : i = 1, \ldots, k \,;\, \gamma \in \Gamma_i \}$, with $w_{i\gamma} \in \mathbf{R}_{\geq 0}$. Let $\Gamma_i' = \{ \gamma \in \Gamma_i : w_{i\gamma} > 0 \}$, and let V' be the subspace of $V = V_1 \oplus \cdots \oplus V_k$ spanned by the k-tuples $(0, \ldots 0, b_{i\gamma}(z), 0, \ldots, 0)$ with $\gamma \in \Gamma_i'$. Finally, let $B_w := \{ w_{i\gamma}^{-1} b_{i\gamma}(z) : \gamma \in \Gamma_i' \}$. Then we have a norm on V' defined by

$$\|g\|_{B_w} := \sup_{i\gamma} |w_{i\gamma} g_{i\gamma}|.$$

The defining equation (2.1) then becomes

$$Z(f, B_w, \varepsilon) := \bigcup \{ Z(f + g) : g \in V' \;\&\; \|g\|_{B_w} \leq \varepsilon \}.$$

By Proposition 2.1,

$$Z(f, B_w, \varepsilon) = \{ z \in \mathbf{C}^n : \forall i \;\; |f_i(z)| \leq \varepsilon \sum_{\gamma \in \Gamma_i'} {w_{i\gamma}}^{-1} |b_{i\gamma}(z)| = \varepsilon ||w^{-1} B_i(z)||_1 \}.$$

It is important to realize that, when working with weights w, the set V' of allowed deformations consists only of those elements of V that correspond to nonzero weights.

Sometimes, we will want to use the same basis and the same weight vectors at each of the k components, i.e., there is $B = \{ b_\gamma(z) : \gamma \in \Gamma \}$ such that $B_i = B$ for $i = 1, \ldots, k$, and there is $w = \{ w_\gamma : \gamma \in \Gamma \}$ such that $w_i = w$ for $i = 1, \ldots, k$. If V_0 is the space spanned by B, then the permissible deformations are systems $g \in V_0^k$. In this case, we write $Z(f, B_w, \varepsilon)$ for the pseudozero set, understanding that B_w is not itself a basis for the deformation space. An actual basis consists of k "copies" of B_w. In this notation,

$$Z(f, B_w, \varepsilon) = \{ z \in \mathbf{C}^n : ||f(z)||_\infty \leq \varepsilon \sum_{\gamma \in \Gamma'} {w_\gamma}^{-1} |b_\gamma(z)| = \varepsilon ||w^{-1} B(z)||_1 \}, \tag{2.2}$$

where $||f(z)||_\infty = \sup_i |f_i(z)|$.

While there are numerous interesting bases (e.g., Taylor, Bernstein, orthogonal), the simplest and most useful bases are subsets of the standard monomial base. Often the base used in defining a pseudozero set for f will itself depend on f, as we describe in the table below. Let $\operatorname{supp} f_i$ denote the set of $\alpha \in \mathbf{N}^n$ such that f_i has a non-zero term of multi-degree α, and let $\operatorname{conv} X$ denotes the convex hull of $X \subseteq \mathbf{R}^n$. In the examples below, B_i is a set of monomials z^α.

Name of base	$B_i = \{ z^\alpha : \alpha \in \mathbf{N}$ & *...see below...* $\}$
degree-bounded	$\lvert\alpha\rvert \leq \max \{\deg f_1, \ldots, \deg f_k\}$
convex	$\alpha \in \operatorname{conv} (\bigcup_i \operatorname{supp} f_i)$
sparse	$\alpha \in \bigcup_i \operatorname{supp} f_i$
equation-wise degree-bounded	$\lvert\alpha\rvert \leq \deg f_i$
equation-wise convex	$\alpha \in \operatorname{conv} \operatorname{supp} f_i$
equation-wise sparse	$\alpha \in \operatorname{supp} f_i$
When all f_i homogeneous:	
equation-wise degree-exact	$\lvert\alpha\rvert = \deg f_i$

Remark 2.1. As we mentioned in the introduction, Proposition 2.1 is proved in greater generality in [27]. As is pointed out in that work, this result has origins in an older theorem of Oettli and Prager [26]. The exposition in [27] is elegant and lucid, but we have retained our own proof, which closely follows Mosier's, because we need the specific form of it for later applications, and our proof is in any case short.

Remark 2.2. There are interesting variations on the concept of a pseudozero. For instance, consider a univariate polynomial $p(z)$, and suppose that a is an m-fold zero of it, for some $m \geq 2$. Then this m-fold zero will become a cluster of m generally distinct zeros in a pseudozero neighborhood of a, that is, a typical nearby polynomial $\tilde{p}(z)$ will have m zeros lying close to a. One can instead ask only for those $\tilde{p}(z)$ near to $p(z)$ which also have an m-fold zero, and one defines the m-fold pseudozero set $Z^{(m)}(p, B, \varepsilon)$ to be the union of all the b such that b is an m-fold zero of a polynomial $\tilde{p}(z) \in B$ whose distance from $p(z)$ is less than or equal to ε. Actually, this concept can be subsumed under the more general definition of pseudozero set given in the next section, 2.2. It turns out that there is no simple criterion such as Proposition 2.1 for multiple pseudozeros. In general, describing multiple pseudozero domains requires solving a linear programming problem. See [28, §6] and [29, §5].

2.2. **General pseudozero sets.** We will now discuss pseudozero sets from a much more general perspective. If U is any subset of the k-fold product $(\mathbf{C}[z])^k$, then we put

$$Z_U := \bigcup \{ Z(g) \mid g \in U \}.$$

If $U \subseteq (\mathbf{R}[x])^k$, we define $Z_U^{\mathbf{R}}$ similarly:

$$Z_U^{\mathbf{R}} := \bigcup \{ Z^{\mathbf{R}}(g) \mid g \in U \}.$$

Any rule for specifying $U = U(f, \varepsilon)$, given f and a parameter ε, will determine a set $Z_{U(f,\varepsilon)}$, which we can view as a generalized pseudozero set. (Usually, we would expect $U(f, \varepsilon)$ to contain f.) A general goal is to find canonical choices for U for which the set Z_U has some meaning or significance in relation to numerical computing, or has other interesting theoretical properties. For this reason, it is natural to look to semialgebraic sets. Not only does one frequently encounter such sets in applications, but they are well-understood algorithmically. Thus, it is reasonable to ask for conditions under which $Z_{U(f,\varepsilon)}$ will be semialgebraic. In dealing with semialgebraic sets, the elimination of quantifiers (often referred to as the Tarski-Seidenberg Theorem) is the most powerful tool.

Theorem 2.2. *Let N and n be positive integers and let $\pi : \mathbf{R}^N \times \mathbf{R}^n \to \mathbf{R}^n$ be the projection function:*

$$(\xi_1, \dots, \xi_N, x_1, \dots, x_n) \mapsto (x_1, \dots, x_n).$$

If $S \subseteq \mathbf{R}^N \times \mathbf{R}^n$ is semialgebraic, then so is

$$\pi(S) := \{ x \in \mathbf{R}^n \mid \exists \xi \in \mathbf{R}^N \text{ such that } (\xi, x) \in S \}.$$

□

For references and a proof, see [2, Theorem 2.2.1, p. 26]. A function $\phi : \mathbf{R}^m \to \mathbf{R}^n$ is called semialgebraic if $Graph(\phi)$ (the graph of ϕ) is a semialgebraic subset of $\mathbf{R}^{m+n}$. Tarski's theorem shows immediately that the image of a semialgebraic set under a semialgebraic function is semialgebraic. To see this, note that for any semialgebraic $S \subseteq \mathbf{R}^m$, $\phi(S)$ is the projection onto $\mathbf{R}^n$ of the semialgebraic set

$$Graph(\phi) \cap (S \times \mathbf{R}^n).$$

A similar argument shows that the inverse image of a semialgebraic set under a semialgebraic function is semialgebraic. Another interpretation of Tarski's theorem is that it says that any subset of $\mathbf{R}^n$ that can be defined explicitly from systems of polynomial inequalities using existential and universal quantifiers is semialgebraic.

In applying this theorem, it will be useful to view the coefficients of a system f as variables. In order to do this, we will use the notation from 2.1. As above, let $\Gamma = \bigcup_{i=1}^k \Gamma_i$ be a finite index set, and let $b_{i\gamma}(x)$, $\gamma \in \Gamma_i$, be a basis for a domain of variation $\mathbf{R}^\Gamma$. For each $i = 1, \dots, k$ and each $\gamma \in \Gamma_i$, we have a coordinate function $\xi_{i\gamma} : \mathbf{R}^\Gamma \to \mathbf{R}$. Let ξ_i denote the vector of coordinate functions $(\xi_{i\gamma}, \gamma \in \Gamma_i)$. Let

$$\xi_i \cdot b_i(x) := \sum_{\gamma \in \Gamma_i} \xi_{i\gamma} b_{i\gamma}(x).$$

This is a function from $\mathbf{R}^{\Gamma_i} \times \mathbf{R}^n$ to $\mathbf{R}$. Similarly,

$$\xi \cdot b(x) := (\xi_1 \cdot b_1(x), \ldots, \xi_k \cdot b_k(x))$$

is a function from $\mathbf{R}^\Gamma \times \mathbf{R}^n$ to $\mathbf{R}^k$. Choosing a point in $\mathbf{R}^\Gamma$ is tantamount to fixing a value for each $\xi_{i\gamma}$, and this is the same thing as choosing a specific polynomial system. We can use the notation $f(\xi, x)$ to denote the system of polynomials corresponding to this choice of coefficients. Or, given a system f in the span of the $b_{i\gamma}$, we let $f_{i\gamma}$ be the coordinates of f in that basis. A subset $U \subset \mathbf{R}^\Gamma$ may equally well be considered as a family of polynomial systems in $(\mathbf{R}[x_1, \ldots, x_n])^k$.

Proposition 2.3. *If U is a semialgebraic subset of $\mathbf{R}^\Gamma$, then so is $Z_U^{\mathbf{R}}$.*

Proof. Let $\pi_1 : \mathbf{R}^\Gamma \times \mathbf{R}^n \to \mathbf{R}^\Gamma$ and $\pi_2 : \mathbf{R}^\Gamma \times \mathbf{R}^n \to \mathbf{R}^n$ be the canonical projections. Let $Y := Z^{\mathbf{R}}(\xi \cdot b(x))$, the zero set of $\xi \cdot b(x)$ in $\mathbf{R}^\Gamma \times \mathbf{R}^n$. If U is a subset of $\mathbf{R}^\Gamma$, then by definition

$$\begin{aligned} Z_U^{\mathbf{R}} &= \{\, x \in \mathbf{R}^n \mid \exists \xi \in U \text{ such that } \xi \cdot b(x) = 0 \,\} \\ &= \pi_2\big(\pi_1^{-1}(U) \cap Y\big), \end{aligned}$$

If U is semialgebraic, then so is $\pi^{-1}(U) \cap Y$. Hence, by Tarski's theorem, $Z_U^{\mathbf{R}}$ is semialgebraic. □

In our main example (section 2.1), we had $U = \{\, f + g : g \in V \ \&\ \|g\|_B \le \varepsilon \,\}$, and the pseudozero set $Z^{\mathbf{R}}(f, B, \varepsilon)$ is, in the notation of the present section, $Z_U^{\mathbf{R}}$. (Of course, the real version of Proposition 2.1 shows directly that $Z_U^{\mathbf{R}}$ is semialgebraic, without invoking Tarski's principle.)

We discuss briefly some alternatives to the pseudozero set described in section 2.1. First observe that it is not necessary to choose a *linear* deformation space. Any semialgebraic domain of variation S can be used to produce semialgebraic pseudozero sets—in particular, we could choose to restrict variation to any semialgebraic subset of any finite-dimensional $\mathbf{R}^\Gamma$. (We might choose either a "deformation space", in which case U would be $f + V_0$ for some family V_0 of small deformations, or a "domain of variation," i.e., some set containing f, in which we want to consider neighborhoods of f. Either approach can be transformed into the other.) After specifying a domain of variation S, we typically choose a measure of distance in S. If d is any semialgebraic metric, then $U := \{\, t \in S \mid d(f,t) \le \varepsilon \,\}$ is semialgebraic. In our main example (2.1, above) we used the metric induced by the sup-norm:

$$d(f,t) := \sup_{i\alpha} |t_{i\alpha} - f_{i\alpha}|.$$

The usual Euclidean metric

$$d(f,t) := \left(\sum (t_{i\alpha} - f_{i\alpha})^2\right)^{1/2}$$

is also semialgebraic. Since the zeroes of $f(\xi, x)$ and of $f(\lambda\xi, x)$ are the same provided $\lambda \ne 0$, it is also reasonable to consider a "pseudometric" that is constant on lines through the origin. For example, if $\theta(f,t)$ is the angle between t and f, then

$$\sin\theta(f,t) = \left(1 - \frac{(\sum t_{i\alpha} f_{i\alpha})^2}{(\sum t_{i\alpha}^2)(\sum f_{i\alpha}^2)}\right)^{1/2}$$

is a semialgebraic pseudometric—and a true metric on the projective space $\mathbf{P}^{kN-1}(\mathbf{R})$. We do not know explicit quantifier-free representations (similar to Proposition 2.1) for pseudozero sets defined by the latter two norms—though of course by Proposition 2.3 they exist.

The knowledge that a pseudozero set is semialgebraic can be very useful. For example, it is well-known that a semialgebraic set has only finitely many connected components. The same is true, therefore, of a pseudozero set defined over a linear space of deformations by any of the metrics just listed. Another application of this knowledge is given in section 3.

While we have focused in this section on real pseudozero sets, we note that complex pseudozero sets may be treated within the framework we set up, since as we remarked previously we can always regard the complex zeros as the real zeros of the system built from the real and imaginary parts of the original system.

2.3. **Projective pseudozero sets and projectivizing.** Pseudozero sets in several variables behave in ways that are significantly different from pseudozero sets in one variable. For example, if $f(z)$ is a polynomial of exact degree n in the single variable z, then any small deformation $\widehat{f}$ of f in the space P_n of polynomials of degree n will have n roots counted with multiplicity. In fact, Mosier showed (see [22, Theorem 2]) that for all small ε, if $g \in N(f, B, \varepsilon)$, then both f and g have the same number of roots in each connected component of $Z(f, B, \varepsilon)$, and also there is at least one root of f in each connected component. This is false for systems $f = (f_1, \dots , f_k)$ of polynomials $f_i : \mathbf{C}^n \to \mathbf{C}$, because f may have "solutions at infinity." In this case, the pseudozero set $Z(f, B, \varepsilon)$ will contain points in $\mathbf{C}^n$ that are close to any solutions at infinity, but these points may be very far from any points of $Z(f)$; see the example in section 2.4. Difficulties such as this can be avoided by passing to projective completions, as we describe forthwith.

Let $F = (F_1, \dots , F_n)$ be a system of homogeneous forms in the variables $z_0, z_1, \dots , z_n$ of degrees $d_1, \dots , d_n$. Then $Z(F)$, as subset of $\mathbf{C}^{n+1}$, is a (generally infinite) union of lines through the origin. More commonly, we view $Z(F)$ as a subset of projective space $\mathbf{P}^n(\mathbf{C})$; see 1.3. We wish to define the projective pseudozero set. Let $\tilde{V}$ be a deformation space as in 2.1, but assume that all non-zero elements of $\tilde{V}$ are systems of homogeneous forms of multidegree $(d_1, \dots , d_n)$. (The tilde is to serve as a reminder that the systems in $\tilde{V}$ are homogeneous.) Let $\tilde{B}$ be a basis for $\tilde{V}$. Then $Z(F, \tilde{B}, \varepsilon)$, as a subset of $\mathbf{C}^{n+1}$, is also a union of lines through the origin, so it, too, may be—and generally *will* be—viewed as a subset of $\mathbf{P}^n(\mathbf{C})$. Since $\mathbf{P}^n(\mathbf{C})$ is compact and the pseudozero set is closed, $Z(F, \tilde{B}, \varepsilon)$ is also compact. In this setting, we can prove analogues of the theorems of Mosier referred to above.

Proposition 2.4. *Let $F = (F_1, \dots , F_n)$ be a system of homogeneous forms of degrees $d_1, \dots , d_n$ in the variables $z_0, z_1, \dots , z_n$, and assume $Z(F) \subseteq \mathbf{P}^n(\mathbf{C})$ has finitely many points (so, by Bézout's theorem, it has exactly $d_1 \cdots d_n$ points, counted with multiplicity). Let $\tilde{V}$ be a deformation space contained in the space of all homogeneous systems of degree $(d_1, \dots , d_n)$ and let $\tilde{B}$ be a basis for $\tilde{V}$. Then the following hold:*

1. *Let $G \in \tilde{V}$, and assume $\|G\|_{\tilde{B}} = 1$. Fix $\varepsilon > 0$. If $\delta \in (0, \varepsilon)$ is sufficiently small, $Z(F)$ and $Z(F + \delta G)$ have the same number of points, counted with multiplicity, in each connected component of $Z(F, \tilde{B}, \varepsilon)$.*

2. *For sufficiently small $\varepsilon > 0$, each connected component of $Z(F, \tilde{B}, \varepsilon)$ contains at least one point of $Z(F)$.*

Proof. This is an immediate consequence of [1, p. 199]. We present a second proof at the end of section 3.2, as an application of our Theorem 3.3. □

We now describe the "projectivization" of an affine pseudozero set. Let $f = (f_1, \ldots, f_k)$ be a system of polynomial equations of multidegree $d = (d_1, \ldots, d_k)$ in variables $z_1, \ldots, z_n$. The corresponding system of homogeneous equations F in $(z_0, z_1, \ldots, z_n)$ was described in section 1.3. Let B be the basis for a deformation space, as in 2.1, but we require that the systems in B have multidegree bounded by d. Let $\tilde{B} := \{ \tilde{b} : b \in B \}$, where each $\tilde{b}$ is the modification of the homogenization of b obtained by multiplying each component by whatever power of z_0 is required to raise the multidegree of $\tilde{b}$ to d. Then $Z(F, \tilde{B}, \varepsilon) \subseteq \mathbf{P}^n(\mathbf{C})$, and it is the union of $Z(F, \tilde{B}, \varepsilon) \cap U_0$ (which is isomorphic to $Z(f, B, \varepsilon)$) and $Z(F, \tilde{B}, \varepsilon) \cap \{z_0 = 0\}$ (the part "at infinity").

2.4. **Example.** Let $f = \{f_1 = z_2 + z_1^2 - 1, f_2 = z_2 - z_1^2 + 1\}$. Then the zero set $Z(f)$ is the intersection of these two parabolas, which consists of the two points $(1, 0)$, $(-1, 0)$. The homogeneous system is $F = \{F_1 = z_0 z_2 + z_1^2 - z_0^2, F_2 = z_0 z_2 - z_1^2 + z_0^2\}$. Its zero set contains in addition to the two points $Z(f)$, namely $[1, 1, 0]$, $[1, -1, 0]$, a point at infinity $[0, 0, 1]$. This is a tangential intersection of F_1 and F_2, of intersection multiplicity 2. Thus we have a total intersection multiplicity of 4 in agreement with Bézout's theorem: $(\deg F_1)(\deg F_2)$. Now let B be any basis of the homogeneous polynomials of degree 2 in (z_0, z_1, z_2). A deformation of the system F within the space of degree 2 homogeneous polynomials will be a pair of projective plane conics. For small ε, $Z(F, B, \varepsilon) \subset \mathbf{P}^2(\mathbf{C})$ will be a compact neighborhood of the three indicated roots. By dehomogenizing, we can view B as a basis of the polynomials of degree ≤ 2 in (z_1, z_2). Then the pseudozero set $Z(f, B, \varepsilon) \subset \mathbf{C}^2$ will consist not only of a compact neighborhood of the visible roots $(1, 0)$, $(-1, 0)$, but it will also have a noncompact component that is a neighborhood of the invisible root at infinity.

Now let $B' = \{z_0^2, z_0 z_2, z_1^2\}$. Then the deformations permitted by B' are all of the form

$$a z_0 z_2 + b z_1^2 + c z_0^2,$$

and for $b \neq 0$ these will all have a unique tangential intersection with the line at infinity at $[0, 0, 1]$. Thus, the corresponding affine system g deformed in this way will have two roots in the affine plane, which will be distinct for small perturbations around f. Thus, by restricting to B' as the permitted space of deformations, we do not see the "pathology" of the previous example. $Z(f, B', \varepsilon)$ will consist of compact neighborhoods of the visible roots $(1, 0)$, $(-1, 0)$ and nothing else, for small ε. The well-behavedness of this example is that the family of 0-dimensional subvarieties $Z(g)$ of $\mathbf{C}^2$ obtained from f by deformations within B' is a *flat* family of varieties. Flatness here amounts to the condition that

$$\dim \mathbf{C}[z_1, z_2]/(g_1, g_2)$$

is constant (here 2) as $g = (g_1, g_2)$ ranges over the (small) deformations of f lying in B' (i.e., $g \in N(f, B', \varepsilon)$; see [24, Corollary of Proposition 2, III.10, pp. 300-301]).

If B again represents the basis of all polynomials of degree ≤ 2 as before, the 0-dimensional subvarieties $g = 0$ of $\mathbf{C}^2$ given by the $g \in N(f, B, \varepsilon)$ no longer form a flat family; but if we consider the family of subvarieties of the projective plane defined by $G = 0$ for $G \in N(F, B, \varepsilon)$ we do get a flat family, because the dimensions of the affine algebras are constant in this family, equal to 4 by Bézout's theorem. However, if we consider a system in which the number of equations is greater than the number of variables, we will not get a flat family in the projective space, as Bézout's theorem not longer applies, and we have no guaranteed constancy of the dimensions of the affine algebras.

3. Diameters and projections of pseudozero sets

In this section, we are concerned with estimating the "size" of pseudozero sets, or their projections. We describe two ways of approaching this topic. The first is based entirely on properties of semialgebraic sets, and provides a surprising amount of information but does not lead immediately to practical algorithms. The second approach uses classical methods of algebraic geometry. This may potentially suggest some algorithms, but we leave the further development of this to a future work.

3.1. **Semialgebraic theory.** Problems concerning the "size" of pseudozero sets come in many variants. To give some illustrations, let f be a system of polynomial equations, and assume a_0 is a zero of f. Let $C_{a_0}(\varepsilon)$ be the intersection of the connected component of a_0 in $Z(f, B, \varepsilon)$ with the unit ball about a_0. Consider the following functions of ε:

i) the maximum distance between two points in $C_{a_0}(\varepsilon)$,
ii) the diameter of the largest closed ball that is contained entirely within $C_{a_0}(\varepsilon)$,
iii) the maximum distance from a point of $C_{a_0}(\varepsilon)$ to a true zero in $C_{a_0}(\varepsilon)$,
iv) the diameter of a projection of $C_{a_0}(\varepsilon)$ onto a given one-dimensional subspace.

Each of these functions is semialgebraic. One way to see this is to note that each can be defined explicitly from systems of polynomial inequalities using existential and universal quantifiers. For example, the value of *i*) at ε is that nonnegative real number y such that for no smaller y' are there points in $C_{a_0}(\varepsilon)$ at a distance y' from one another. For fixed f and a_0, the description of $C_{a_0}(\varepsilon)$ depends semialgebraically on ε; moreover, the distance itself (whether it be the usual Euclidean distance, or the distance induced by the sup-norm) is also semialgebraic. Thus, the remaining parts of this description can be made explicit in the required way.

The Hörmander-Łojasiewicz inequality shows that the order of vanishing of two semialgebraic functions can be compared in terms of exponents. The following theorem contains a statement. For background, see [2].

Theorem 3.1. *Let $K \subseteq \mathbf{R}^n$ be a compact semialgebraic set and let $g, h : K \to \mathbf{R}$ be two continuous semialgebraic functions such that $Z(g) \subseteq Z(h)$. Then there exist positive integers c and m such that*

$$|h|^m \leq c|g| \quad \text{on} \quad K.$$

A similar statement applies to complex functions with semialgebraic real components. Recently, estimates for the size of m have been determined. For example, [17] contains a result that implies the following: Let $f = (f_1, \dots, f_k)$, $f_i \in \mathbf{C}[z_1, \dots, z_n]$, $\deg(f_i) = d_i > 2$ and $k \leq n$. Assume $Z(f)$ is not empty. Then there are an integer $m \leq d_1 \cdots d_k$ and a constant c such that

$$\operatorname{dist}(z, Z(f))^m \leq c \max\{\, |f_i| \,\}(1 + \|z\|)^{d_1 \cdots d_k}$$

for all $z \in \mathbf{C}^n$.

Here is a sketch of how these ideas could be applied in the present context. Let us assume that a_0 is the only zero of f in $C_{a_0}(\varepsilon)$. Let $h(\varepsilon)$ be any of the four functions listed above. Then $h(0) = 0$; thus, applying the Hörmander-Łojasiewicz inequality to h and $g(\varepsilon) := \varepsilon$, we see that there are a positive rational exponent ν and a positive constant c such that

$$h(\varepsilon) \le c\,\varepsilon^{\nu}\,, \quad \text{for all } \varepsilon \in [0, 1].$$

What is missing in the above sketch is a proof that the corresponding h is a continuous function. This seems to be a subtle matter for some of the h's on this list, and will be addressed in more detail in a future publication. Here we will provide more details for the case of interest to us: that of projections. Assume that $f = (f_1, \ldots, f_k)$ is a system of equations, say in $\mathbf{C}[z_1, \ldots , z_n]$, and let U be a semialgebraic subset of the set of these polynomials (of some bounded degree). Let d be a semialgebraic metric function on the set U and let

$$U(f, \varepsilon) = \{g \in U : d(f, g) \le \varepsilon\}.$$

The corresponding pseudozero set $Z_{U(f,\varepsilon)}$ then is a semialgebraic subset of $\mathbf{C}^n = \mathbf{R}^{2n}$. Let $||\;\;||$ be any semialgebraic norm on the vector space $\mathbf{C}^k$. Finally, fix a compact semialgebraic $K \subset \mathbf{C}^n$ and define

$$K_\varepsilon = K \cap Z_{U(f,\varepsilon)}$$

which is a family of semialgebraic subsets depending on the parameter ε, with $K_0 = K \cap Z(f)$, and let

$$h(\varepsilon) = \sup_{z \in K_\varepsilon} ||f(z)||.$$

Lemma 3.2. *$h(\varepsilon)$ is a continuous semialgebraic function of ε for all ε near 0.*

Proof. That it is semialgebraic can be seen from Tarski's elimination of quantifiers theorem: the conditions defining the graph of h are in the first order language involving equalities and inequalities and involving existential and universal quantifiers over the real field. By Tarski's theorem this is then a semialgebraic set. As a semialgebraic function on the reals is in any case piecewise continuous, continuity of h in a neighborhood of 0 comes down to showing that

$$\lim_{\varepsilon \to 0^+} h(\varepsilon) = h(0) = 0.$$

First we note that

$$\bigcap_{\varepsilon > 0} K_\varepsilon = K_0 = K \cap Z(f). \tag{3.1}$$

To see this, let z belong to the left-hand side above. By definition of pseudozero sets, we can find $g_n \in U(f, 1/n)$ such that $g_n(z) = 0$ for integers n going to infinity. Clearly, g_n converges to f in the space of polynomials of the appropriate bounded degrees, and hence $f(z) = \lim_{n\to\infty} g_n(z) = 0$, showing that $z \in K \cap Z(f)$. Now let $\delta > 0$ be given. Then the set

$$V := K \cap \{z : ||f(z)|| < \delta\}$$

is an open neighborhood of $K \cap Z(f)$. Each of the sets K_ε is compact, and an easy argument using compactness and property (3.1) above shows that there is an $\varepsilon_0 > 0$ such that $K_\varepsilon \subset V$ for all $\varepsilon \le \varepsilon_0$, and this shows that $h(\varepsilon) < \delta$ for all ε in this range, verifying continuity. □

By the application of the Hörmander-Łojasiewicz inequality outlined above, we get an estimate of the form $h(\varepsilon) \leq c\varepsilon^{\nu}$. In the special case of linear deformations considered in section 2.1, this estimate follows easily, with the exponent $\nu = 1$, from the inequality in equation (2.2) of that section.

Now suppose that a_0 is an isolated zero of the system f, and suppose that a_0 is the only zero of f in K. For any index i the zero-set of the continuous semialgebraic function $||f(z)||$ on K (namely a_0) is contained in the zero-set of the continuous semialgebraic function $|z_i - z_i(a_0)|$. Hence we get an estimate of the form $|z_i(a) - z_i(a_0)|^m \leq c_1||f(a)||$ for all $a \in K$. If $a \in K \cap Z_{U(f,\varepsilon)}$, then, combining with the estimate for h, we get

$$|z_i(a) - z_i(a_0)| \leq c_2\varepsilon^{\nu}$$

for some constants $c_2 > 0$ and rational $\nu > 0$. Thus, the projection of this pseudozero set in the i^{th} coordinate direction has order of magnitude $\mathrm{O}(\varepsilon^{\nu})$. In the next section we will give another proof of this fact based on principles from local analytic geometry. One reason for doing this is that it gives an effective method for actually calculating the exponent ν.

3.2. **Local theory.** We give a more precise analysis of the projections of pseudozero sets in the case of an isolated zero. In fact, suppose that $Z(F)$ is a finite set of points. For this to be true, the number of equations, r, must be greater than or equal to the number n, the dimension of $\mathbf{P}^n$. If $r = n$, then the generalized form of Bézout's theorem states that the number of zeros of F, counted with the appropriate multiplicities, is equal to $(\deg F_1)\cdots(\deg F_n)$. Also, by making a linear change of variables if necessary, we can assume that all the zeros of F lie in the affine open set $U_0 = \mathbf{C}^n$. We will let $(z_1, \dots, z_n)$ be a set of affine coordinates on this.

To analyze the situation around any one zero $s = (s_1, \dots, s_n)$, three local rings are useful:

$$\mathbf{C}[z_1, \dots, z_n]_s \subset \mathbf{C}\langle\langle z_1 - s_1, \dots, z_n - s_n\rangle\rangle \subset \mathbf{C}[[z_1 - s_1, \dots, z_n - s_n]].$$

The first is the set of fractions of polynomials $h(z)/g(z)$ with $g(s) \neq 0$. The second is the set of power series in the indicated variables that converge in some neighborhood of s. The third is the set of formal power series in the indicated variables. The inclusion of the first into the second is via the Taylor expansion. Each of these is a Noetherian regular local ring of Krull dimension n. In each case, we will let $\mathfrak{m}$ denote the maximal ideal, which consists of those functions h such that $h(s) = 0$, which is generated as an ideal by $z_1 - s_1, \dots, z_n - s_n$. The third ring is the completion of the other two with respect to the $\mathfrak{m}$-adic topology. By shifting the origin, we may assume that $s = (0, \dots, 0)$; we make this simplifying assumption, which entails no loss of generality.

Therefore, assume that $f(z)$ is a system of polynomials with $(0, \dots, 0)$ as a common zero. Then $w = f(z)$ defines a polynomial map $\mathbf{C}^n_z \to \mathbf{C}^r_w$ which carries the origin 0_z in the z-space to the origin 0_w in the w-space. This also defines a map of the local ring at 0_w to the local ring at 0_z. To be definite, we will work with the convergent power series. We obtain a local homomorphism (i.e., one that carries the maximal ideal to the maximal ideal):

$$f^* : \mathbf{C}\langle\langle w_1, \dots, w_r\rangle\rangle \longrightarrow \mathbf{C}\langle\langle z_1, \dots, z_n\rangle\rangle, \quad f^*(w_i) = f_i(z).$$

In particular, $\mathbf{C}\langle\langle z\rangle\rangle$ becomes a module over the ring $\mathbf{C}\langle\langle w\rangle\rangle$ via f^*. Let us denote by

$$\mathbf{C}\langle\langle f\rangle\rangle \subset \mathbf{C}\langle\langle z\rangle\rangle$$

the image of f^*. Then, (0) is an isolated zero of the system f if and only if $\mathbf{C}\langle\langle z\rangle\rangle$ is a $\mathbf{C}\langle\langle w\rangle\rangle$-module of finite type. Equivalently, (0) is an isolated zero of f if and only if $\mathbf{C}\langle\langle z\rangle\rangle$ is a module of finite type over the subring $\mathbf{C}\langle\langle f\rangle\rangle$. In fact, we have the following fundamental theorem from local analytic geometry ([10, 1.11, p. 57; 3.2, pp. 132-134]):

Theorem 3.3. *Let $\varphi : X \to Y$ be a holomorphic map of analytic spaces, $x \in X$, and let $y = \varphi(x)$. We get a canonical local homomorphism of analytic local rings*

$$\varphi^* : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}.$$

Then the following conditions are equivalent:

1. *$\mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{Y,y}$-module via φ^*.*
2. *$\dim_{\mathbf{C}} \mathcal{O}_{X,x}/\varphi^*(\mathfrak{m}_{Y,y})\mathcal{O}_{X,x} < \infty$.*
3. *x is an isolated point in the fiber over y, $\varphi^{-1}(y)$.*

Suppose that any one of these conditions is satisfied. Then there are arbitrarily small open neighborhoods U of $x \in X$ such that:

1. *$\varphi(U)$ is an analytic subset of Y at y.*
2. *$\dim_y \varphi(U) = \dim_x X$.*
3. *$\varphi|U : U \to \varphi(U)$ is a finite morphism, and is open in x.*

(A morphism $\varphi : X \to Y$ of topological spaces is open in x if for every neighborhood V of x, $\varphi(V)$ is a neighborhood of $\varphi(x)$.)

In our situation $X = \mathbf{C}^n$, $Y = \mathbf{C}^r$, $\varphi = f$, $x = 0_z$, $y = 0_w$, $\mathcal{O}_{X,x} = \mathbf{C}\langle\langle z\rangle\rangle$, $\mathcal{O}_{Y,y} = \mathbf{C}\langle\langle w\rangle\rangle$, and $\varphi^*(\mathfrak{m}_{Y,y})$ is the ideal generated by the polynomials $f(z)$. Now $\mathbf{C}\langle\langle z\rangle\rangle/\mathfrak{m}_f\mathbf{C}\langle\langle z\rangle\rangle$, being finite-dimensional over $\mathbf{C}$ by Theorem 3.3, is an Artin local ring. Thus its maximal ideal is nilpotent. In other words, there is an integer $c > 0$ such that $(z_1, \ldots, z_n)^c \subset (f_1, \ldots, f_r)\mathbf{C}\langle\langle z\rangle\rangle$. Assuming an isolated zero at 0 as before, we have that the Krull dimension of $\mathbf{C}\langle\langle f\rangle\rangle$ is n by [20, Theorem 20, p. 81]. When the number of equations r equals the number of variables n, one can say more: $\mathbf{C}\langle\langle f\rangle\rangle$ is isomorphic with $\mathbf{C}\langle\langle w\rangle\rangle$, for if there were a kernel in the epimorphism $\mathbf{C}\langle\langle w\rangle\rangle \to \mathbf{C}\langle\langle f\rangle\rangle$, the Krull dimension would drop down from n. It then follows that $\mathbf{C}\langle\langle z\rangle\rangle$ is a *free* $\mathbf{C}\langle\langle f\rangle\rangle$-module of finite rank ([20, Theorem 46, p. 140]).

Condition 1 of the theorem implies that every element of $\mathcal{O}_{X,x}$ is *integrally dependent* on $\mathcal{O}_{Y,y}$. In our situation, this means that every element $h \in \mathbf{C}\langle\langle z\rangle\rangle$ satisfies a monic equation:

$$h^N + \Phi_{N-1}h^{N-1} + \ldots + \Phi_1 h + \Phi_0 = 0, \tag{3.2}$$

where $\Phi_i \in \mathbf{C}\langle\langle f\rangle\rangle$ for all i. Note that if $h \in \mathfrak{m}_z$, in other words, $h(0) = 0$, then $\Phi_0 = 0$, i.e., $\Phi_0 \in \mathfrak{m}_f$. Since it is important for our method, let us elaborate on this. Let

$$N = \dim_{\mathbf{C}} \mathbf{C}\langle\langle z\rangle\rangle/(f_1(z), \ldots, f_r(z))$$

and let $e_1, \ldots, e_N$ be a basis of this quotient. Lift these in any way to elements $e_1(z), \ldots, e_N(z)$ in $\mathbf{C}\langle\langle z\rangle\rangle$. Then by Nakayama's lemma, $e_1(z), \ldots, e_N(z)$ is a set of $\mathbf{C}\langle\langle w\rangle\rangle$-module generators of $\mathbf{C}\langle\langle z\rangle\rangle$. If $h \in \mathbf{C}\langle\langle z\rangle\rangle$, there are equations

$$h(z)\cdot e_i(z) = \sum_{j=1}^{N} a_{i,j}(f)e_j(z), \quad j = 1, \dots, N,$$

where $A = (a_{i,j}(w))$ is a matrix of elements of $\mathbf{C}\langle\langle w\rangle\rangle$. Since $h \cdot 1 - A$ has a nonzero element in its kernel, namely the vector of generators $(e_1(z), \dots, e_N(z))$, we have

$$\det(h\cdot 1 - A) = h^N + \Phi_{N-1}h^{N-1} + \ldots + \Phi_1 h + \Phi_0 = 0,$$

which shows in particular that every element of $\mathbf{C}\langle\langle z\rangle\rangle$ satisfies such an equation of integral dependence of degree N (and possibly less).

Let $H(z) = \sum_{|\alpha|=s} c_\alpha z^\alpha$ be a nonzero homogeneous polynomial of degree d. If $||\cdot||$ is any norm on $\mathbf{C}^n$, then there are constants $c_1, c_2 > 0$ with the following properties:

1. $|H(z)| < c_1||z||^d$ for all $z \in \mathbf{C}^n$.
2. For any $\varepsilon > 0$ there exists z with $||z|| = \varepsilon$ such that $|H(z)| \geq c_2\varepsilon^d$.

Informally, if z has size ε, $H(z)$ will have size ε^d. Let $h \in \mathfrak{m}_z^d - \mathfrak{m}_z^{d+1}$. Then $h(z) = H(z)u(z)$, where $H(z)$ is a homogeneous form of degree d, and $u \in \mathbf{C}\langle\langle z\rangle\rangle$ is a holomorphic unit, i.e., $u(0) \neq 0$, or equivalently, $1/u$ belongs to $\mathbf{C}\langle\langle z\rangle\rangle$. Since u defines a continuous function on some neighborhood of 0, and is nonvanishing at 0, we have the elementary estimates:

Lemma 3.4. *For any $h \in \mathfrak{m}_z^d - \mathfrak{m}_z^{d+1}$ there exist a neighborhood U of 0 in $\mathbf{C}^n$ and constants $c_1, c_2 > 0$ such that*

1. $|h(z)| < c_1||z||^d$ *for all* $z \in U$.
2. *For any $\varepsilon > 0$ sufficiently small there exists $z \in U$ with $||z|| = \varepsilon$ such that $|h(z)| \geq c_2\varepsilon^d$.*

Informally, if z has size ε, $h(z)$ will have size ε^d as $\varepsilon \to 0$.

Proposition 3.5. *Let $f_1, \dots, f_r, h$ be a collection of nonzero elements of $\mathbf{C}\langle\langle z\rangle\rangle$ with $f_i(0) = h(0) = 0$ for all i.*

1. *Then there are a neighborhood U of 0 in $\mathbf{C}^n$, a rational number $\mu > 0$, and a constant $c_1 > 0$ such that the following holds: For all $\varepsilon > 0$ sufficiently small, there exists $z \in U$ with $||f(z)|| = \varepsilon$,*

$$|h(z)| \geq c_1\varepsilon^\mu.$$

2. *Suppose in addition that 0 is an isolated zero of the system $f = 0$. Then there are a neighborhood U of $0 \in \mathbf{C}^n$, a rational number $\nu > 0$, and a constant $c_2 > 0$ such that the following holds: For all $\varepsilon > 0$ sufficiently small, and all $z \in U$ with $||f(z)|| < \varepsilon$,*

$$|h(z)| < c_2\varepsilon^\nu.$$

Proof. 1) Let U be a small enough neighborhood of 0 so that h and all the f_i converge in U. Now let $\lambda : \Delta \to U$ be a nonconstant holomorphic map, with $\lambda(0) = 0$, where $\Delta \subset \mathbf{C}$ is a small disk neighborhood of 0, and suppose that $\lambda(\Delta)$ is not contained in the union of the analytic subsets $h = 0$, $f_i = 0$. Let t be a coordinate on Δ, and consider the composite map $t \mapsto w(t) : \Delta \to \mathbf{C}^r$ given by $f \circ \lambda$. We can write (up to a change of coordinates)

$$w(t) = (t^{b_1}u_1(t), \dots, t^{b_r}u_r(t))$$

for integers $b_i > 0$, and holomorphic functions $u_i(t)$ with $u_i(0) \neq 0$. Then we clearly have a constant $c > 0$ and an integer $b > 0$ such that $||w(t)|| = ||f(\lambda(t))|| = \varepsilon \Rightarrow |t| \geq c\varepsilon^{1/b}$ for all sufficiently small ε. Now $h(\lambda(t))$ is a nonconstant holomorphic function of t, equal therefore to $t^a u(t)$ for a holomorphic function $u(t)$ with $u(0) \neq 0$ and an integer $a > 0$. Therefore we have a constant $c' > 0$ and an integer $a > 0$ such that $||h(\lambda(t))|| \geq c'|t|^a$ for all sufficiently small t. Combining with the previous estimate, we get $||f(\lambda(t))|| = \varepsilon \Rightarrow ||h(\lambda(t))|| \geq c^a c' \varepsilon^{a/b}$, for all sufficiently small ε, which is what we wanted to prove.

2) In case 0 is an isolated zero, h will be integrally dependent on the f, as we have discussed. Write equation (3.2) in the form

$$h^N + \sum_{i \in I} \Phi_i h^i = -\Phi_0 - \sum_{j \in J} \Phi_j h^j = X,$$

where $I, J \subset \{1, \dots, N-1\}$ are the indices with $i \in I \Rightarrow \Phi_i(0) \neq 0$, $j \in J \Rightarrow \Phi_j(0) = 0$. Let a be the smallest index in I. Then we can write the above equation as $Yh^a = X$, where

$$Y = h^{N-a} + \sum_{i \in I-a} \Phi_i h^{i-a} + \Phi_a.$$

Since $Y(0) = \Phi_a(0) \neq 0$, Y is an invertible element of the ring $\mathbf{C}\langle\langle z \rangle\rangle$. Therefore, we have an equation $h^a = X/Y$ in that ring. For $j = 0$ or $j \in J$, we have $\Phi_j \in \mathfrak{m}_f^{b_j}$, for an integer $b_j > 0$. Therefore, there is a constant $c_j > 0$ such that $||f(z)|| < \varepsilon \Rightarrow |\Phi_j(f)| < c_j \varepsilon^{b_j}$ for all sufficiently small ε. Then, h being a continuous function in a neighborhood of 0 and Y being a continuous function nonvanishing at 0, it is clear that there are a neighborhood U of 0, an integer $b > 0$ and a constant $c > 0$ such that $z \in U$ and $||f(z)|| < \varepsilon \Rightarrow |X/Y| < c\varepsilon^b$. We then obtain part 1 of the lemma from the equation $h^a = X/Y$, with the constant $\nu = b/a$. □

Corollary 3.6. *Let f be a system of polynomials and suppose that a_0 is an isolated zero of this system. Let K be any compact neighborhood of $a_0 \in \mathbf{C}^n$. With the notation of Proposition 2.1, let $a \in Z(f, B_w, \varepsilon) \cap K$. Then for each $i = 1, \dots, n$ there are a constant $c > 0$ and a rational number $\nu > 0$ such that for all sufficiently small ε,*

$$|z_i(a) - z_i(a_0)| < c\varepsilon^\nu.$$

In other words, the projection of the ε-pseudozero set of a_0 onto the i-th coordinate axis is of size $\mathrm{O}(\varepsilon^\nu)$.

Proof. There is a constant $c_1 > 0$ so that $||w^{-1}B(a)||_1 < c_1$ for all a in the compact set K. Then Proposition 2.1 shows that for $a \in Z(f, B_w, \varepsilon) \cap K$, $||f(a)||_\infty < c_1\varepsilon$. The corollary then follows from part 2 of the proposition, applied to the function $h = z_i$. □

In the previous proof, we obtained estimates for the size of $|h|$ based on an equation of integral dependence on the f. Of course, such an equation is not unique. In any case the ideal in $\mathbf{C}\langle\langle f \rangle\rangle[T]$ of polynomials annihilating h is finitely generated because that ring is Noetherian. It seems to be a nontrivial problem to give optimal estimates for h by this method. In general, the exponent ν that appears in the above corollary depends on the direction of projection. This is related to tangency conditions of the defining equations with the direction of projection. This will be seen in some examples to be discussed in section 3.4.

We finish this section by including the following, which can be used to give an alternate proof of Proposition 2.4.

Proposition 3.7. *Let $f_1, \ldots, f_n \in \mathbf{C}\langle\langle z_1, \ldots, z_n\rangle\rangle$ be convergent power series such that 0 is an isolated zero of the system $f_1 = 0, \ldots, f_n = 0$. Let $g_1, \ldots, g_n \in \mathbf{C}\langle\langle z_1, \ldots, z_n\rangle\rangle$ be arbitrary elements. There is a neighborhood U of 0 in $\mathbf{C}^n$ such that for all t sufficiently near 0, the zero-set*

$$Z_t = \{f_1 + tg_1 = 0, \ldots, f_n + tg_n = 0\} \cap U$$

is a finite set of points, and

$$\sum_{z \in Z_t} mult(\mathcal{O}_{Z_t,z}) = mult(\mathcal{O}_{Z_0,0}) = \dim_{\mathbf{C}} \mathbf{C}\langle\langle z_1, \ldots, z_n\rangle\rangle / (f_1, \ldots, f_n)\,.$$

Proof. By assumption that we have an isolated zero at the origin, the dimension of this algebra is finite, according to theorem 3.3. Consider a small polydisk neighborhood Δ of the origin in $(t, z_1, \ldots, z_n)$-space in which all of $f_1, \ldots, g_n$ converge and such that 0 is the only zero of the system $t = 0$, $f_1 = 0, \ldots, f_n = 0$ in Δ. Let Δ_1 be the projection of Δ onto the t-coordinate. Let $Z \subset \Delta$ be the analytic space defined by the ideal $(f_1 + tg_1, \ldots, f_n + tg_n)$, and let $\varphi : Z \to \Delta_1$ be the projection onto the t-coordinate. For fixed t, let $Z_t \subset \Delta$ be the analytic space defined by the ideal $(f_1 + tg_1, \ldots, f_n + tg_n)$. By our assumption, $0 \in Z$ is an isolated point in the fiber $\varphi^{-1}(0)$. Therefore by theorem 3.3, there is a neighborhood U of 0 in $(t, z_1, \ldots, z_n)$-space such that $\varphi : Z \cap U \to \Delta_1$ is a finite morphism. In particular, it is a proper morphism, and by Grauert's theorem, $\varphi_* \mathcal{O}_Z$ is a coherent analytic sheaf in a neighborhood of 0 in Δ_1. We will show that it is a flat $\mathcal{O}_{\Delta_1}$-module in a neighborhood of 0, which implies that is locally free. This proves the claim because the fiber of this vector bundle in t is

$$(\varphi_* \mathcal{O}_Z)_t \otimes (\mathcal{O}_{\Delta_1,t}/\mathfrak{m}_{\Delta_1,t}) = \bigoplus_{z \in Z_t} \mathcal{O}_{Z_t,z},$$

which has a constant dimension for all small t, and

$$\dim_{\mathbf{C}} \mathcal{O}_{Z_t,z} = \text{mult}\,(\mathcal{O}_{Z_t,z})\,.$$

By coherence, it is sufficient to verify this flatness (or local freeness) at $t = 0$. This amounts to showing that

$$R = \mathbf{C}\langle\langle t, z_1, \ldots, z_n\rangle\rangle / (f_1 + tg_1, \ldots, f_n + tg_n)$$

is a flat $A = \mathbf{C}\langle\langle t\rangle\rangle$-module. We will use the local criterion for flatness [20, Theorem 49, pp. 145-147]. In Matsumura's notation, $M = B = R$, the ideal I is the maximal ideal $\mathfrak{m}$ of A generated by t. Note that R is then idealwise separated for I by example 1 on p. 145 of loc. cit. We need only check that

$$\text{Tor}_1^A\,(A/\mathfrak{m}, R) = 0.$$

Using the exact sequence $0 \longrightarrow A \xrightarrow{t} A \longrightarrow A/\mathfrak{m} \longrightarrow 0$ as a free resolution of $A/\mathfrak{m}$ for the computation of Tor, we see that we must show that multiplication by t is injective on R. Since R/t has finite $\mathbf{C}$-dimension, the elements $(t, f_1 + tg_1, \ldots, f_n + tg_n)$ form a system of parameters in the local ring $\mathbf{C}\langle\langle t, z_1, \ldots, z_n\rangle\rangle$. But that ring is regular, hence Cohen-Macaulay, so this system of parameters is a regular sequence. In particular, t is not a zero-divisor in R, as claimed. ∎

3.3. **Visualization of projected pseudozeros.** The description of $Z(f, B, \varepsilon)$ given by Proposition 2.1 enables us to compute, plot and visualize pseudozeros of multivariate polynomials. The pseudozeros of a multivariate polynomial system form a set in a high dimensional space $\mathbf{C}^n$, $n \geq 2$, which can only be seen from its "shadows" on low dimensional spaces, for example, from its projections onto the various coordinate axes. Since a multivariate system is generally solved through reductions to univariate equations in practice, it is often desirable to visualize one-dimensional coordinate projections for quantitative information about the numerical conditioning of the system.

For a given $v \in \mathbf{C}^n$, let $Z_j(f, B, \varepsilon, v)$ denote the projections of $Z(f, B, \varepsilon)$ onto the z_j-space around v. It is necessary to rescale ε by $\varepsilon := \varepsilon||f||$ to be compatible with standard numerical error analysis. Then

$$
\begin{aligned}
&Z_j(f, B, \varepsilon||f||, v) \\
&\quad = \{z \in \mathbf{C}^n : z_i = v_i \text{ for } i \neq j, \text{ and } ||f(z)||_\infty \leq \varepsilon||f|| \cdot ||w^{-1}B(z)||_1\}.
\end{aligned}
\tag{3.3}
$$

One way of viewing $Z_j(f, B, \varepsilon||f||, v)$ is to plot the values of the projection of

$$
ps(z) := \log_{10}\left(\frac{||f(z)||_\infty}{||f|| \cdot ||w^{-1}B(z)||_1}\right)
\tag{3.4}
$$

over a set of grid points around v in z_j-space. The negative value of ps at a point $z = \hat{\xi}$ illustrates proportionally the number of significant digits that might be lost in the numerical zero $\hat{\xi}$ when a stable zero-finding algorithm is used for obtaining $\hat{\xi}$.

3.4. **Examples.** We examine the following systems:

1. Two unit balls intersect at $\xi = (2, 2)$:

$$
f^{(1)} = \begin{cases} f_1 &= (z_1 - 1)^2 + (z_2 - 2)^2 - 1, \\ f_2 &= (z_1 - 3)^2 + (z_2 - 2)^2 - 1. \end{cases}
$$

2. Three unit balls intersect at $\xi = (2, 2, 2)$:

$$
f^{(2)} = \begin{cases} f_1 &= (z_1 - 1)^2 + (z_2 - 2)^2 + (z_3 - 2)^2 - 1, \\ f_2 &= (z_1 - 3)^2 + (z_2 - 2)^2 + (z_3 - 2)^2 - 1, \\ f_3 &= (z_1 - 2)^2 + (z_2 - 1)^2 + (z_3 - 2)^2 - 1. \end{cases}
$$

3. The polynomial taken from [21]:

$$
f^{(3)} = \begin{cases} f_1 &= 1.6 \times 10^{-3}(z_1^2 + z_2^2) - 1, \\ f_2 &= 5.3 \times 10^{-4}(z_1^2 + z_2^2 + z_3^2) + 2.7 \times 10^{-2} z_1 - 1, \\ f_3 &= -1.4 \times 10^{-4} z_1 + 10^{-4} z_2 + z_3 - 3.4 \times 10^{-3}. \end{cases}
$$

Figures 1, 2, and 3 show numerically computed $Z_j(f, B, \varepsilon||f||, v)$ for these polynomial systems under the Taylor bases with different shifts s.

We can link these figures to the theoretical analysis of section 3.2. Consider $f^{(1)}$. To make the formulas a little nicer, shift the origin from the root $\xi = (2, 2)$ to $(0, 0)$. The equations are then

$$
\begin{aligned}
f_1 &= (z_1 + 1)^2 + z_2^2 - 1, \\
f_2 &= (z_1 - 1)^2 + z_2^2 - 1.
\end{aligned}
$$

From Proposition 2.1, if $(z_1, z_2) \in Z(f^{(1)}, B, \varepsilon)$, where B is any basis of the polynomials of degree ≤ 2 in (z_1, z_2), there is a constant c such that $|f_1| \leq c\varepsilon$ and $|f_2| \leq c\varepsilon$ for all sufficiently small ε. From the above equations it is easy to see that

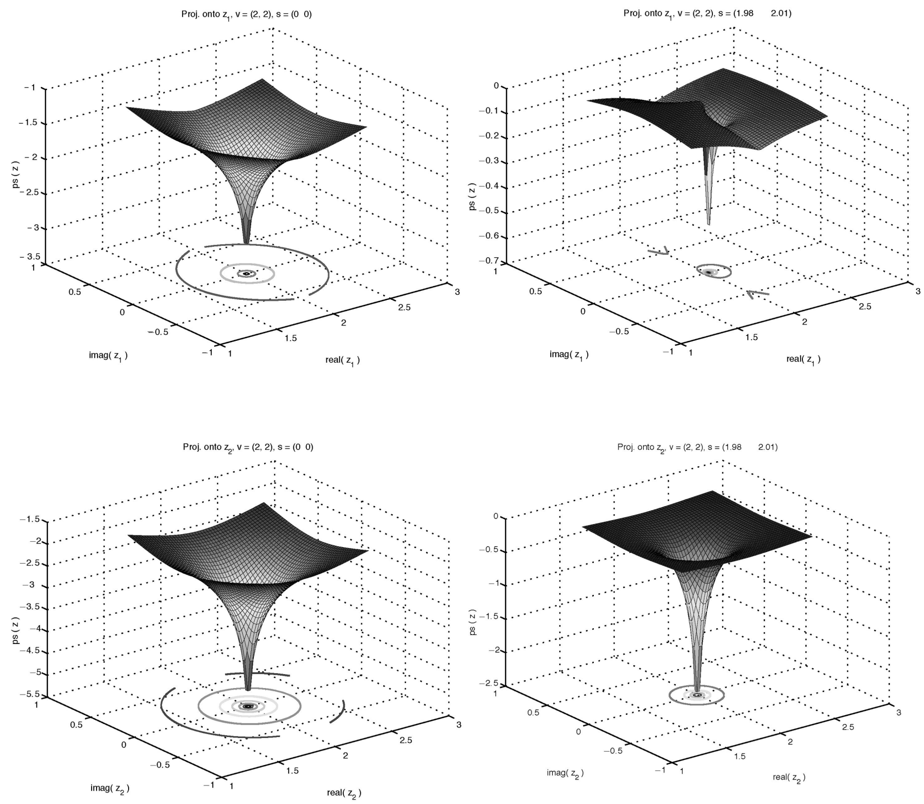

FIGURE 1. Projections of pseudozeros for $f^{(1)}$: two unit balls intersect at $\xi = (2, 2)$. Left column: $s = (0, 0)$; right column: $s = (1.98, 2.01)$.

$f_1 - f_2 = (z_1 + 1)^2 - (z_1 - 1)^2 = 4z_1$, so $z_1 = (f_1 - f_2)/4$. From this it follows that there is a constant c_1 so that if $(z_1, z_2) \in Z(f^{(1)}, B, \varepsilon)$, then $|z_1| \leq c_1\varepsilon$. We will symbolize this by $z_1 = \mathrm{O}(\varepsilon)$. This is the size of the projection of the pseudozero set of the root $(0, 0)$ in the above system, or $\xi = (2, 2)$ of the original system, onto the z_1-axis. Putting the found equation for z_1 back into the system $f^{(1)}$, we get

$$z_2^2 + (f_1 - f_2)^2/16 - (f_1 + f_2)/2 = 0,$$

which is an equation of integral dependence as guaranteed by the general theory above. From this we see that $z_2 = \mathrm{O}(\varepsilon^{1/2})$. These estimates are in agreement with what was found by computer experiments shown by Figure 1: during numerical processing, the projection along the z_2-axis likely loses twice as many significant digits as the one along the z_1-axis (see *ps*-scale: -5.5 vs. -3.5). In Table 1 we see that the z_1-coordinate of the computed zero of $f^{(1)} = 0$ coincides with the exact zero $(2, 2)$ up to machine accuracy $\varepsilon = \mathrm{O}(10^{-16})$, whereas the z_2-coordinate only agrees up to an accuracy $\varepsilon^{1/2} = \mathrm{O}(10^{-8})$. This differing behavior of the two projections is visible from the picture of the system $f^{(1)}$: two circles meeting tangentially at ξ with a tangent line parallel to the z_2-axis. The pseudozero set can

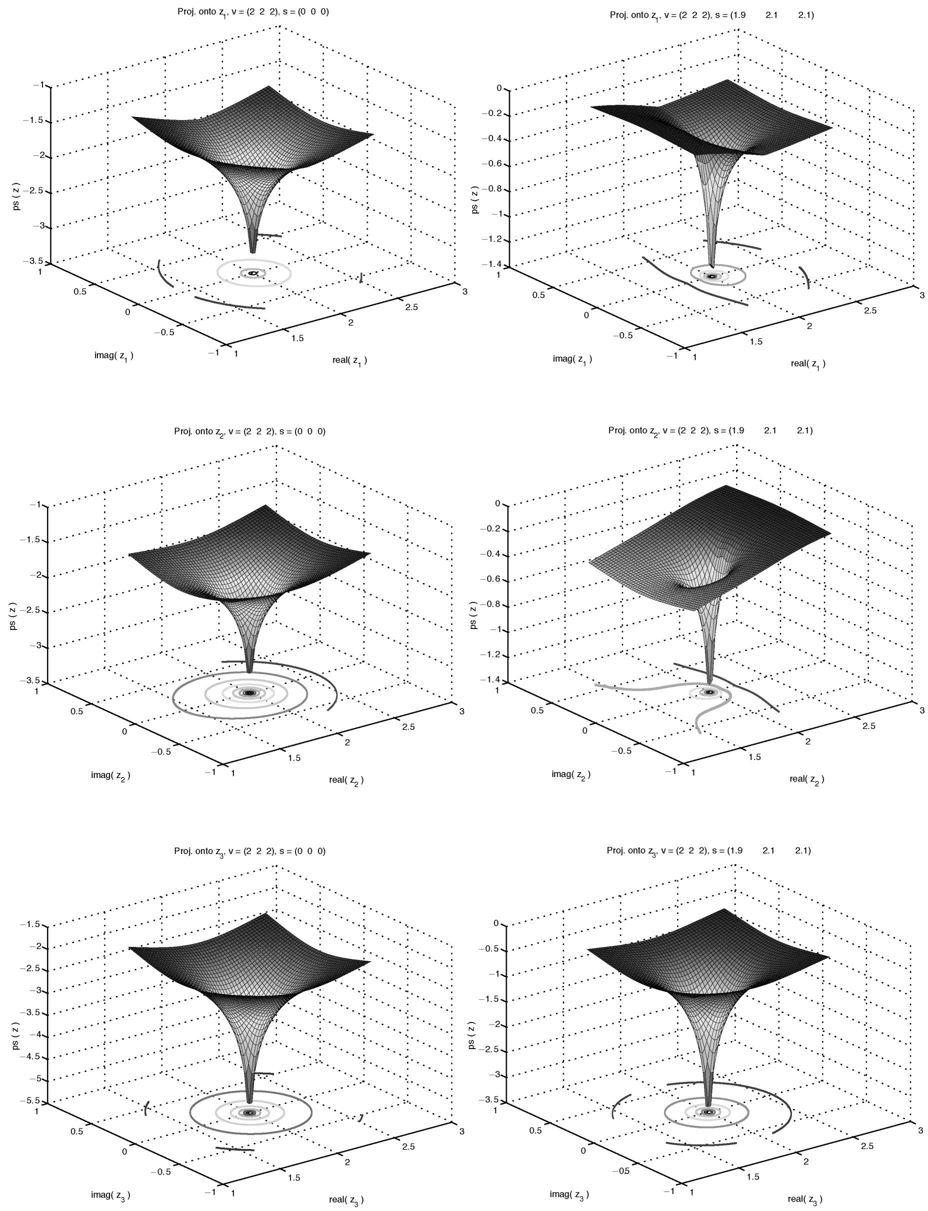

FIGURE 2. Projections of pseudozeros for $f^{(2)}$: three unit balls intersect at $\xi = (2, 2, 2)$. Left column: $s = (0, 0, 0)$; right column: $s = (1.99, 2.01, 2.01)$.

be visualized approximately as the intersection of the tubular neighborhoods of the circles $f_1 = 0$ and $f_2 = 0$. It is clear from this picture that there is a qualitative difference between the projections onto the first and second axis.

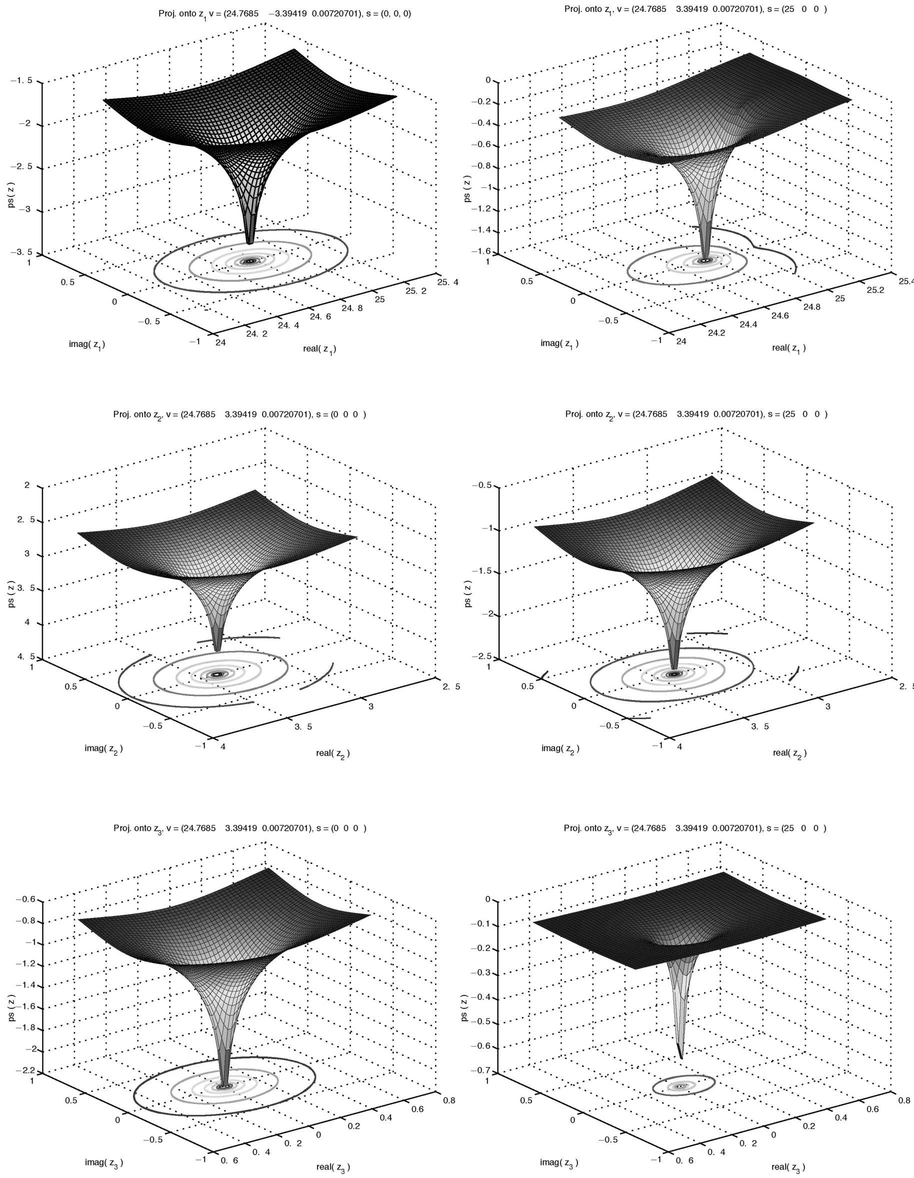

FIGURE 3. Projections of pseudozeros for $f^{(3)}$ around the zero $\xi \approx (24.7685, -3.39416, 0.00720701)$. Left column: $s = (0, 0, 0)$; right column: $s = (25, 0, 0)$.

A similar analysis of the system $f^{(2)} = 0$ with the zero shifted to $(0, 0, 0)$ gives equations of integral dependence:

$$\begin{aligned}
&z_1 - (f_1 - f_2)/4 &&= 0,\\
&z_2 + (f_1 + f_2 - 2f_3)/4 &&= 0,\\
&z_3^2 - (f_1 + f_2)/2 + (f_1 - f_2)^2/16 + (f_1 + f_2 - 2f_3)^2/16 &&= 0.
\end{aligned}$$

TABLE 1. Numerical results

Polynomial System	Direction of Proj. z_j	$\lvert\xi_j - \hat{\xi}_j\rvert$	$\lvert\xi_j - \hat{\xi}_j/\lvert\xi_j\rvert$
$f^{(1)} = 0$	$j = 1$	0.0	0.0
$\xi = (2, 2)$	$j = 2$	3.2×10^{-8}	1.6×10^{-8}
$f^{(2)} = 0$	$j = 1$	4.4×10^{-16}	2.2×10^{-16}
$\xi = (2, 2, 2)$	$j = 2$	8.9×10^{-16}	4.4×10^{-16}
	$j = 3$	7.7×10^{-8}	3.8×10^{-8}
$f^{(3)} = 0$	$j = 1$	2.5×10^{-14}	1.0×10^{-15}
$\xi = (24.7685, -3.39416,$	$j = 2$	3.8×10^{-14}	1.1×10^{-14}
$0.00720701)$	$j = 3$	4.3×10^{-18}	6.0×10^{-16}

This leads to the estimates for the projection of the pseudozero neighborhood of the root $\xi = (2, 2, 2)$ onto the coordinate axes: $z_1 = \mathrm{O}(\varepsilon)$, $z_2 = \mathrm{O}(\varepsilon)$, $z_3 = \mathrm{O}(\varepsilon^{1/2})$, in agreement with Figure 2 and Table 1.

For arbitrary polynomial systems, it it impossible to derive optimal estimate analytically as we did for $f^{(1)}$ and $f^{(2)}$ (proved by Abel in 1824). In this situation, we can obtain local information on the size of pseudozero sets through numerical computation and visualization. For example, Figure 3 shows that, for $f^{(3)}$ in the neighborhood of $v = (24.7685, -3.39419, 0.00720701)$, if $\hat{\xi}$ is a numerical approximation to a zero ξ, the coordinate projections of $\hat{\xi}$ along z_1, z_2 and z_3 will proportionally lose 3, 4, and 2 (left column) or 2, 3, and 1 (right column) significant digits respectively.

4. Applications

Both analysis and visualization in the previous section reveal that the numerical sensitivity of the polynomial zeros depends on

1. direction of projection, and
2. polynomial basis.

Based on these results, we propose in this section algorithmic ideas that would improve numerical conditioning for zeros of multivariate polynomial systems.

4.1. **Direction of projection.**

In Figures 1-3, the projected ps values (see (3.4)) along different z_j-axes illustrate that the numerical sensitivity of polynomial zeros also depends on the direction of projection. To verify this observation, we transformed the multivariate polynomial systems discussed in the previous section into univariate equations using multivariate resultants, then numerically solved the resulting algebraic eigenvalue problems [19]. Table 1 shows the experimental results. We use $\hat{\xi}_j$ to denote the computed j-th component of ξ. It is interesting to notice that the absolute errors $\lvert\xi_j - \hat{\xi}_j\rvert$ listed in Table 1 are 100% consistent with the sizes of the projected pseudozero sets shown in Figures 1-3.

The dependency of numerical sensitivity on the direction of projections suggests that the numerical conditioning of polynomial zeros could be improved by projecting the system along the direction that leads to less sensitive univariate polynomial

TABLE 2. Numerical results

Ordering	$\vert\xi_1 - \hat{\xi}_1\vert$	$\vert\xi_2 - \hat{\xi}_2\vert$
(1)	0.0	0.0
(2)	4.4×10^{-16}	3.2×10^{-8}

equations first. A numerical experiment was implemented on the polynomial system $f^{(1)} = 0$. The system was solved by two different orderings:

(1) Project the system onto z_1-space and compute $\hat{\xi}_1$; then compute $\hat{\xi}_2$, a numerical zero of $f_1^{(1)}(\hat{\xi}_1, z_2) = 0$ (or $f_2^{(1)}(\hat{\xi}_1, z_2) = 0$).

(2) Project the system onto z_2-space and compute $\hat{\xi}_2$; then compute $\hat{\xi}_1$, a numerical zero of $f_1^{(1)}(z_1, \hat{\xi}_2) = 0$ (or $f_2^{(1)}(z_1, \hat{\xi}_2) = 0$).

The numerical results are shown in Table 2.

4.2. **Polynomial basis.** For univariate polynomials, the dependency of their numerical conditioning on the basis has been investigated using the condition numbers [9, 12] and polynomial pseudozeros [33]. The analytical and experimental results from the previous sections suggest that the numerical conditioning of multivariate polynomials is also basis-dependent. Since the numerical zero-finding procedure for polynomials represented by the power basis can be adapted to the Taylor basis formula through variable exchange, we will focus on the Taylor basis $B^{(s)}(z) := \{(z - s)^\alpha\}$ in this section.

Let $f(z)$ be represented in the Taylor bases $B^{(s)}(z)$ and $f(\xi) = 0$. Figures 1-3 show that as the shift s moves closer to ξ, the size of the connected component of $Z(f, B^{(s)}, \epsilon)$ that contains ξ decreases, indicating that the zero ξ is becoming less numerically sensitive to the coefficient perturbation. In [33], for univariate polynomial equations, this observation was analyzed, and an algorithm that combines symbolic formulation and numerical computation was proposed and implemented. The algorithm iteratively improves the accuracy of numerical zeros through a sequence of Taylor bases. Extending the analysis and the computational technique to general multivariate polynomial systems is currently under our investigation. In fact, the computational technique that uses change of Taylor basis has been implicitly used in practice on a class of special multivariate polynomial systems, *linear system of equations*:

$$Az = b, \tag{4.1}$$

where A is an $n \times n$ nonsingular matrix with complex coefficients and b is a vector in $\mathbf{C}^n$. The solution to (4.1) is the zero for the residual polynomial

$$r(z) := b - Az, \qquad z \in \mathbf{C}^n. \tag{4.2}$$

When (4.1) is ill-conditioned, e.g., the condition number of A,

$$cond(A) := ||A^{-1}||||A||,$$

is large, a computed solution $\hat{z}$ to (4.1) is expected to lose at least $\log_{10} cond(A)$ digits in accuracy [30, p. 120]. In this situation, an established algorithm, called *Iterative Refinement* [14], can be used to improve the computed solution $\hat{z}$.

Algorithm: ITERATIVE REFINEMENT

Input:

A := *nonsingular matrix in* $\mathbf{C}^{n\times n}$;

b := *vector in* $\mathbf{C}^n$;

$x^{(1)}$:= *an approximation to the solution of* $Az = b$.

Output:

$x^{(k)}$:= *a new approximation to the solution of* $Az = b$.

For k=1,2, $\cdots$

1. *Symbolically evaluate or numerically compute with high-precision* $r(x^{(k)}) = b - Ax^{(k)}$;
2. *Numerically solve* $Ad = r(x^{(k)})$ *by a stable method;*
3. *Update* $x^{(k+1)} = x^{(k)} + d$;

Until $||d|| < tol$ *or* $k = k_{max}$.

The convergence of the algorithm has been proven through roundoff error analysis (e.g., [14, 32]). Here, using the concept of pseudozeros for multivariate polynomials developed in previous sections, we give a new proof which provides further algorithmic insight for the Iterative Refinement Algorithm: the essence of the algorithm is to symbolically reformulate the residual polynomial (4.2) by the new Taylor basis $B^{(k+1)}(z) = [1, z - x^{(k+1)}]$ iteratively (see Step 1 of the algorithm).

For the proof, we assume that we have an explicit "solver" $\widehat{A}$ for which we have an estimate of the form

$$||A - \widehat{A}|| = \max_{i,j}|a_{ij} - \widehat{a}_{ij}| < \eta.$$

For example, we may have $\widehat{A} = LU$, as in [32, pp. 107ff.]. We assume that $\widehat{A}$ is also invertible, which will be the case for sufficiently small η. Then we define $x^{(1)}$ via

$$x^{(1)} = \widehat{A}^{-1}b$$

and inductively

$$x^{(k+1)} = x^{(k)} + \widehat{A}^{-1}r^{(k)},$$

where $r^{(k)} = r(x^{(k)})$ for the residual polynomial. In what follows, we assume (as in [32, p. 121]) that there are no further rounding errors in determining the sequence $x^{(k)}$. A more precise analysis can be given that includes these roundoff errors.

Theorem 4.1. *Under the above hypotheses, if* η *is small enough, there is a constant* $c > 0$ *such that*

$$||x - x^{(k+1)}||_\infty < c\eta||x - x^{(k)}||_\infty,$$

where x *is the true solution to* $Ax = b$. *The constant* c *and the conditions on* η *are explicitly computable from the matrix* A.

Proof. Let $B^{(k)}(z)$ be the basis of the polynomials of degree ≤ 1 in the variables z given by $[1, z - x^{(k)}]$. We consider a system of weights $w = [0, 1, \ldots, 1]$; that is, we assign 0 to the constant term and 1 to every linear term. Since we are assigning the weight 0 to the constant term, we are going to consider deformations of the system

$$r(z) = b - Az = r^{(k)} - A(z - x^{(k)})$$

that leave the constant term $r^{(k)}$ unchanged (see the discussion following Proposition 2.1). Define

$$\widehat{r}^{(k)}(z) = r^{(k)} - \widehat{A}(z - x^{(k)}).$$

We have $||r - \widehat{r}^{(k)}||_{B_w} = ||A - \widehat{A}|| < \eta$. Note that $x^{(k+1)}$ is a solution to $\widehat{r}^{(k)}(z) = 0$. Thus,

$$x^{(k+1)} \in Z(r,\, B_w^{(k)},\, \eta).$$

From equation (2.2) following Proposition 2.1 we get

$$||r^{(k+1)}||_\infty \leq \eta ||w^{-1} B^{(k)}(x^{(k+1)})||_1 = \eta ||x^{(k+1)} - x^{(k)}||_1 \leq n\eta ||x^{(k+1)} - x^{(k)}||_\infty.$$

We can estimate the left-hand side as follows:

$$\begin{aligned} ||r^{(k+1)}||_\infty = ||r(x) - r(x^{(k+1)})||_\infty &= ||A(x - x^{(k+1)})||_\infty \\ &\geq \frac{1}{\sqrt{n}} ||A(x - x^{(k+1)})||_2 \\ &\geq \frac{\sigma_{\inf}}{\sqrt{n}} ||x - x^{(k+1)}||_2 \\ &\geq \frac{\sigma_{\inf}}{\sqrt{n}} ||x - x^{(k+1)}||_\infty, \end{aligned}$$

where $\sigma_{\inf}$ is the smallest singular value of the matrix A. We get

$$\begin{aligned} \frac{\sigma_{\inf}}{\sqrt{n}} ||x - x^{(k+1)}||_\infty &\leq n\eta ||x^{(k+1)} - x^{(k)}||_\infty \\ &\leq n\eta \left(||x - x^{(k+1)}||_\infty + ||x - x^{(k)}||_\infty \right), \end{aligned}$$

or

$$\left(\frac{\sigma_{\inf}}{\sqrt{n}} - n\eta \right) ||x - x^{(k+1)}||_\infty \leq n\eta ||x - x^{(k)}||_\infty.$$

Assuming that $\sigma_{\inf} - n^{3/2}\eta > 0$, we obtain the proposition by taking any constant

$$c \geq \frac{n^{3/2}\eta}{\sigma_{\inf} - n^{3/2}\eta}.$$

Convergence is guaranteed by this proposition provided that $c\eta < 1$. □

In applications, one often has an expression for η in the form $f(n) cond(A) \epsilon$, where $f(n)$ is some simple explicit function, and ϵ is the unit roundoff (or machine precision). Thus we get the usual estimates using the pseudozero of the multivariate polynomial.

5. Conclusions

Pseudozero sets offer a potentially useful tool for considering computational problems that arise in solving polynomial systems. Results well-known in the one-dimensional case have non-trivial generalizations to higher dimensions. Several powerful mathematical theories on semialgebraic sets and local algebraic geometry offer substantial insight. Numerical experiments demonstrate the practical value of the concept. Algorithmic ideas for solving multivariate polynomial systems that are based on the notion of pseudozero are proposed.

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