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THE CLASS NUMBER ONE PROBLEM FOR SOME NON-ABELIAN NORMAL CM-FIELDS OF DEGREE 48

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Abstract. We prove that there is precisely one normal CM-field of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of $\mathbb{Q}(\sqrt{5},\sqrt{101},\theta)$ with $\theta^3-\theta^2-5\theta-1=0$.

1. Introduction

According to [O] and [H], there exist only finitely many normal CM-fields with class number one, and their degrees are less than or equal to 436. All imaginary abelian number fields with class number one are known in [Y]: their degrees are less than or equal to 24. All normal CM-fields of degree less than 48 with class number one are known by many authors ([LO1], [LO2], [Lef], [LLO], [LP1], [LP2], [LOO], [Lou3], [P], [YPK], [PsK], [CK2], and [CK3]). In the following table we sum up the numbers of the non-abelian normal CM-fields N with class number one according to their degrees.

$[N:\mathbb{Q}]$	nb	$[N:\mathbb{Q}]$	nb	$[N:\mathbb{Q}]$	nb	$[N:\mathbb{Q}]$	nb
8	17	20	1	32	6	42	0
12	9	24	7	36	3	44	0
16	12	28	0	40	1		

In this paper we study the non-abelian normal CM-fields that contain a normal CM-subfield of degree 16, and will prove the following:

Theorem 1. *There exists one and only one normal CM-field N of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of the real dihedral number field $K_{12}=\mathbb{Q}(\sqrt{5},\sqrt{101},\theta)$ of degree 12 with $\theta^3-\theta^2-5\theta-1=0$, narrow class number 4 and class number 2. The extension N/K_{12} is cyclic quartic, $d_N=d_{K_{12}}^4=2^{32}\cdot 5^{24}\cdot 101^{24}$, the maximal totally real subfield of N is the Hilbert class field of K_{12}, and the Galois group $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to the semi-direct product $C_3\rtimes D_{16}$.*

2. Prerequisite and notation

We use the following notation. For a number field K, we let h_K, d_K, ω_K, and ζ_K denote the class number, the absolute value of the discriminant, the number of roots of unity in K, and the Dedekind zeta function of K, respectively. If K

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is a CM-field, we let h_K^-, K^+ and $Q_K \in \{1,2\}$ be the relative class number, the maximal real subfield and the Hasse unit index of K, respectively. For an abelian extension F/K we denote by $\mathfrak{F}_{F/K}$ the finite part of its conductor. For a positive integer n we let $\zeta_n = e^{2i\pi/n}$. Before starting the proof of Theorem 1 we recall the well-known results which will be used later in this paper.

Proposition 1. (1) ([LOO, Lemma 2]) *If K is a normal CM-field, then the complex conjugation is in the center of its Galois group.*

(2) ([CH, Lemma 13.5]) *Let K be a CM-field. If there is at least one ramified prime ideal in K/K^+ which is lying above an odd prime, then $Q_K = 1$.*

(3) ([Ho, Theorem 5], [Ok], or [LOO, Theorem 5]) *Let $k \subset K$ be two CM-fields. Then h_k^- divides $4h_K^-$. Moreover, if $[K:k]$ is odd, then h_k^- divides h_K^- and $Q_k = Q_K$.*

(4) ([LO2] and [Lou1, Proposition 6]) *Let K be a CM-field and let t be the number of prime ideals of K^+ that are ramified in the quadratic extension K/K^+. Then 2^{t-1} divides h_K^-. Moreover, if $Q_K = 2$, then 2^t divides h_K^-.*

(5) ([Lou1, Proposition 13] and [LO2, Proposition 2]) *Let $K = L_1L_2$ be a CM-field which is a compositum of two CM-fields L_1 and L_2 with the same maximal totally real subfield. Then*

$$h_K^- = \frac{Q_K}{Q_{L_1}Q_{L_2}}\frac{\omega_K}{\omega_{L_1}\omega_{L_2}}h_{L_1}^-h_{L_2}^-$$

and $h_{L_1}^-h_{L_2}^-$ divides $4h_K^-$. In particular, if L_1 and L_2 are isomorphic, then $\omega_K = \omega_{L_1} = \omega_{L_2} = 2$ and $h_K^- = (Q_K/2)(h_{L_1}^-/Q_{L_1})^2$.

(6) ([M, Corollary 2.2 and 2.3]) *Let E/F be an extension of number fields. Then h_F divides $[E:F]h_E$. Moreover, if no nontrivial abelian subextension of E/F is unramified over F, then h_F divides h_E.*

(7) *Let $K = L_1L_2$ be a CM-field which is a compositum of two CM-fields L_1 and L_2 with the same maximal totally real subfield $L_1^+ = L_2^+$. If $h_K = 1$, then h_{L_1} and h_{L_2} are 1 or 2. If $h_K = 1$ and $h_{L_1} = h_{L_2} = 2$, then $h_{L_1^+} = h_{L_2^+} = 2$.*

Proof. We only need to prove the last statement of (7). If $h_K = 1$ and $h_{L_1} = h_{L_2} = 2$, then K is the Hilbert class field of L_1 and is at the same time that of L_2. Hence, K^+ is the Hilbert class field of $L_1^+ = L_2^+$ (see [P, Lemma 6.2]). □

Proposition 2. *Let K be a CM-field of degree $2n$.*

(1) ([W]) $h_K^- = Q_K\omega_K/(2\pi)^n \cdot \sqrt{d_K/d_{K^+}} \cdot \mathrm{Res}_{s=1}(\zeta_K)/\mathrm{Res}_{s=1}(\zeta_{K^+})$

(2) ([LO2, Proposition 9]) *Let $\beta_K = 1 - (2/\log d_K)$ and*

$$\varepsilon_K = \max(1 - 2\pi n e^{1/n}/d_K^{1/2n}, 2/5\exp(-2\pi n/d_K^{1/2n})).$$

If $\zeta_K(\beta_K) \le 0$, then $\mathrm{Res}_{s=1}(\zeta_K) \ge 2\varepsilon_K/(e\log d_K)$.

(3) ([Lou2]) *There exists a computable constant $\mu_k > 0$ such that for any abelian extension K/k of degree m unramified at all the infinite places we have*

$$\mathrm{Res}_{s=1}(\zeta_K) \le (\mathrm{Res}_{s=1}(\zeta_k)\ )^m\left(\frac{1}{2(m-1)}\log(d_K/d_k^m) + 2\mu_k\right)^{m-1}.$$

Let C_m denote the cyclic group of order m, D_m the dihedral group of order m, Q_m the quaternion group of order m and set $G_6 = \langle b,c,z | b^4 = c^2 = z^2 = 1, c^{-1}bc = bz, bz = zb, cz = zc\rangle$ and $G_9 = \langle b,c,z | b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = zc\rangle$ (in the notation of [JL]). Throughout this paper, N denotes a non-abelian normal CM-field of degree 48. We assume that the 3-Sylow subgroup of its Galois group $\mathrm{Gal}(N/Q)$ is normal, and we let M denote the normal CM-subfield of degree 16 of N. According to Proposition 1, if $h_N = 1$, then $h_M^- = 1$ (moreover, either N/M is ramified at least one finite place and $h_M = 1$ or N/M is unramified at all places, $h_M = 3$, and N is the Hilbert class field of M). Now, there are 26 normal CM-fields of degree 16 with relative class number one (see [LO2], [Lou3], [CK1], [PK], and Theorem 2 below). If $\mathrm{Gal}(M/Q)$ is non-abelian, then it is equal to $Q_8 \times C_2, G_6, D_{16}, G_9, D_8 \times C_2$. For proving Theorem 1, we first prove that if $\mathrm{Gal}(M/Q) \neq D_{16}, G_9, D_8 \times C_2$, then we can use Proposition 1 and the known solutions to various (relative) class number problems for suitable CM-subfields of N to prove that $h_N > 1$. Now, assume that $\mathrm{Gal}(M/Q) = D_{16}, G_9$, or $D_8 \times C_2$. We will show that we can find a subfield L of M^+ such that N/L is abelian and such that the use of abelian L-functions to factorize ζ_N/ζ_L readily yields $(\zeta_N/\zeta_L)(s) \geq 0$ for $0 < s < 1$. Since M is known, L also is known, we will check that $\zeta_L(s) \leq 0$ for $0 < s < 1$ and we will therefore deduce that $\zeta_N(s) \leq 0$ for $0 < s < 1$. Using Proposition 2, we will obtain explicit lower bounds for h_N^-, according to which we will be able to compute explicit upper bounds on d_N when $h_N = 1$ and to construct a short list of number fields N containing all such N's with $h_N = 1$. We will finally explain how one can use the method expounded in [Lou5] and [Lou6] to compute the relative class numbers of these finitely many CM-fields N that remain, thus completing the proof of Theorem 1.

3. Case 1: M is abelian

We will show the following.

Proposition 3. *If N contains an abelian number field M of degree 16, then $h_N > 1$.*

Proof. Let K_3 be any cubic subfield of N. Since N is non-abelian, K_3 is not normal, its normal closure K_6 is a dihedral real sextic field, and we let k_2 denote the (real) quadratic subfield of K_6. The Galois group $\mathrm{Gal}(M/\mathbb{Q})$ is isomorphic to C_{16}, $C_8 \times C_2$, $C_4 \times C_4$, $C_4 \times C_2 \times C_2$, or $C_2 \times C_2 \times C_2 \times C_2$.

(i) If $\mathrm{Gal}(M/\mathbb{Q}) = C_{16}$, then $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to $C_3 \rtimes C_{16} = \langle a, b | a^3 = b^{16} = 1, b^{-1}ab = a^{-1}\rangle$, and N is a compositum of M and the real dihedral field of degree 6 that is fixed by $\langle b^2\rangle$. According to [Lou4, Theorem 5] we have $h_N^- > 1$.

(ii) If $\mathrm{Gal}(M/\mathbb{Q}) = C_8 \times C_2$ with $\mathrm{Gal}(M^+/\mathbb{Q}) = C_8$, then $\mathrm{Gal}(N/\mathbb{Q}) = \langle a, b, c | a^3 = b^8 = c^2 = 1, b^{-1}ab = a^{-1}, ac = ca, bc = cb\rangle$ with $\mathrm{Gal}(N/N^+) = \langle b^4 c\rangle$. The subfield K_{12} fixed by $\langle b^2 c\rangle$ is a normal CM-field with Galois group isomorphic to Q_{12}. By [LP1] $h_{K_{12}}^- > 4$, whence $h_N^- > 1$ by Proposition 1(3). If $\mathrm{Gal}(M/\mathbb{Q}) = C_8 \times C_2$ with $\mathrm{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$, then $h_N^- > 1$ by [CK1]. Hence $h_N^- > 1$ by Proposition 1(3).

(iii) If $\mathrm{Gal}(M/\mathbb{Q}) = C_4 \times C_4$, then $\mathrm{Gal}(N/\mathbb{Q}) = Q_{12} \times C_4$ and $\mathrm{Gal}(N^+/\mathbb{Q})$ is isomorphic to either $S_3 \times C_4$ or $Q_{12} \times C_2$. Let ψ_1 and ψ_2 be two odd primitive characters of order 4 such that M is associated with the group $\langle \psi_1, \psi_2\rangle$. If k_2 is associated with $\langle \psi_1^2\rangle$ or $\langle \psi_2^2\rangle$, then $\mathrm{Gal}(N^+/\mathbb{Q}) = S_3 \times C_4$. Assume that k_2 is associated with $\langle \psi_1^2\rangle$. Let $M_{12,1}$ be the compositum of K_6 and the quartic field associated with $\langle \psi_1\rangle$, and $M_{12,2}$ the compositum of K_6 and the quartic field associated with $\langle \psi_1\psi_2^2\rangle$. Then $M_{12,1}$ and $M_{12,2}$ are quaternion CM-fields of degree 12 with the same maximal real subfield K_6. According to [LP1, Theorem 1], there is no pair of $(M_{12,1}, M_{12,2})$ such that $h^-_{M_{12,1}}|4$, $h^-_{M_{12,2}}|4$, and at the same time $M^+_{12,1} = M^+_{12,2}$, whence $h^-_N > 1$. By symmetry, if k_2 is associated with $\langle\psi_2^2\rangle$, then $h^-_N > 1$. Assume now that k_2 is associated with $\langle\psi_1^2\psi_2^2\rangle$. Let $M_{24,1}$ be the compositum of K_6 and the imaginary cyclic quartic field associated with $\langle\psi_1\rangle$, and $M_{24,2}$ the compositum of K_6 and the imaginary cyclic quartic field associated with $\langle\psi_2\rangle$. Then $M_{24,1}$ and $M_{24,2}$ are normal CM-fields with Galois group isomorphic to $S_3 \times C_4$ which have the same maximal real subfield. Using Proposition 1(5) we verify that $h^-_N = h^-_{M_{24,1}} h^-_{M_{24,2}}$. By [P, Theorem 1] there is only on CM-field of relative class number one with Galois group isomorphic to $S_3 \times C_4$, whence $h^-_N > 1$.

(iv) If $\mathrm{Gal}(M/\mathbb{Q}) = C_4 \times C_2 \times C_2$ with $\mathrm{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$, then $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to either $Q_{12} \times C_2 \times C_2$ or $S_3 \times C_2 \times C_4$. Let ψ be the odd primitive Dirichlet character of order 4, and let χ_1 and χ_2 be two quadratic odd characters such that M is associated with the group $\langle\psi, \chi_1, \chi_2\rangle$. If k_2 is associated with $\langle\psi^2\rangle$, then the compositum $M_{12,1}$ of K_6 and the field associated with $\langle\psi\rangle$, and the compositum $M_{12,2}$ of K_6 and the field associated with $\langle\psi\chi_1\chi_2\rangle$ are normal CM-fields with Galois group Q_{12} and $M^+_{12,1} = M^+_{12,2}$. By [LP1, Theorem 1], $h^-_N > 1$. If k_2 is associated with $\langle\psi^2\chi_1\chi_2\rangle$ or $\langle\chi_1\chi_2\rangle$, then we let $M_{24,1}$ be the compositum of K_6 and the field associated with $\langle\psi\rangle$, and $M_{24,2}$ the compositum of K_6 and the field associated with $\langle\psi^2, \chi_1, \chi_2\rangle$. Then $\mathrm{Gal}(M_{24,1}/\mathbb{Q}) = S_3 \times C_4$, $\mathrm{Gal}(M_{24,2}/\mathbb{Q}) = S_3 \times C_2 \times C_2$, $M^+_{24,1} = M^+_{24,2}$, and $N = M_{24,1}M_{24,2}$. By [P, Theorem 1] $h^-_{M_{24,1}} > 1$ and $h^-_{M_{24,2}} > 1$, whence according to Proposition 1(7) we have $h_N > 1$.

(v) If $\mathrm{Gal}(M/\mathbb{Q}) = C_4 \times C_2 \times C_2$ with $\mathrm{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, then $h^-_M > 1$ by [CK1]. Hence $h^-_N > 1$.

(vi) If $\mathrm{Gal}(M/\mathbb{Q}) = C_2 \times C_2 \times C_2 \times C_2$, then $h^-_M > 1$ by [CK1]. Hence $h^-_N > 1$.

□

4. Case 2: $\mathrm{Gal}(M/\mathbb{Q}) \in \{D_{16}, Q_8 \times C_2\}$

In this section we assume that $\mathrm{Gal}(M/\mathbb{Q}) \in \{D_{16}, Q_8 \times C_2\}$ and $h^-_M = 1$. We will prove that there is exactly one field N with $h_N = 1$. In subsection 4.1 we assume that $G(M/\mathbb{Q}) = D_{16}$, and in subsection 4.2 we assume that $G(M/\mathbb{Q}) = Q_8 \times C_2$.

4.1. $G(M/\mathbb{Q}) = D_{16}$. There are five dihedral CM-fields M of degree 16 with relative class number one [LO2, Theorem 10]: the narrow Hilbert class fields of $\mathbb{Q}(\sqrt{pq})$ with $(p, q) \in \{(2, 257), (5, 101), (5, 181), (13, 53), (13, 61)\}$. The narrow Hilbert class field of $\mathbb{Q}(\sqrt{2 \cdot 257})$ has class number three and the remaining four M's have class number one. We set $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ and $k = \mathbb{Q}(\sqrt{pq})$. The field M has three quadratic subfields L_1, L_2, and k with $\mathrm{Gal}(M/L_1) = \mathrm{Gal}(M/L_2) = D_8$, and $\mathrm{Gal}(M/k) = C_8$. Therefore, the Galois group $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to $D_{16} \times C_3$ if N contains only one cubic cyclic subfield, D_{48} or $(D_8 \times C_3) \rtimes^1 C_2 = C_3 \rtimes D_{16}$ $= \langle a, b, c \mid a^3 = b^8 = c^2 = 1, c^{-1}bc = b^{-1}, b^{-1}ab = a^{-1}, ac = ca\rangle$, otherwise. In [Lef] it is proved that if $\mathrm{Gal}(N/\mathbb{Q}) = D_{48}$, then $h^-_N > 1$. We deal with the fields N with $\mathrm{Gal}(N/\mathbb{Q}) = D_{16} \times C_3$ in 4.1.1 and the fields N with $\mathrm{Gal}(N/\mathbb{Q}) = (D_8 \times C_3) \rtimes^1 C_2$ in 4.1.2, respectively.

4.1.1. $\mathrm{Gal}(N/\mathbb{Q}) = D_{16} \times C_3$. Let K_3 denote the cyclic cubic subfield of N. Since $K_3/\mathbb{Q}$ is ramified, N/M is ramified and if $h_N = 1$, then $h_M = 1$ by point (6) of Proposition 1. Therefore, M cannot be equal to the narrow Hilbert class field of $\mathbb{Q}(\sqrt{2 \cdot 257})$. Note that N/k is cyclic of degree 24.

Lemma 1. *Let χ be any one of the eight characters of order 24 associated with the cyclic extension N/k.*

(1) *We have $(\zeta_N/\zeta_K)(s) \geq 0$ in the range $0 < s < 1$.*
(2) *For each given M with $h_M = 1$ we can compute a bound $N_{k/\mathbb{Q}}(\mathfrak{F}) \leq C$ on the norms of the conductors $\mathfrak{F}$ of the cyclic cubic extensions kK_3/k for the N's such that $h_N^- = 1$. These bounds are listed in Table 1.*
(3) *Assume that $h_N = 1$. Then $\mathfrak{F} = (l)$ for some prime l which splits in k, or $\mathfrak{F} = \mathfrak{B}_l$ for some prime ideal $\mathfrak{B}_l$ of k above a prime l ramified in k.*
(4) *h_M^- divides h_N^-, $L(0,\chi) \in \mathbb{Q}(\sqrt{2}, \sqrt{-3})$, and $h_N^-/h_M^- = N_{\mathbb{Q}(\sqrt{2},\sqrt{-3})}(\frac{1}{4}L(0,\chi))^2$ is a perfect square which can be computed using the techniques developed in* [Lou5] *and* [Lou6].

Proof. (1) It follows from $(\zeta_N/\zeta_K)(s) = \prod_{i=1}^{11} |L(s,\chi^i)|^2$.

(2) We have verified that for the above four M's, $\zeta_K(s) \leq 0$ in the range $0 < s < 1$. Hence, $\zeta_N(s) \leq 0$ for $0 < s < 1$. Using [Lou2, Lemma 12 and Proposition 13] we compute explicitly $\mu_k \operatorname{Res}_{s=1}(\zeta_k)$ and apply Proposition 2 to get lower bound for h_N^-. Since M/M^+ is unramified at all finite places and $Q_M = \omega_M = 2$, N/N^+ is unramified at all finite places, $d_{N^+} = d_k^{12} N_{k/\mathbb{Q}}(\mathfrak{F})^8$, and $Q_N = \omega_N = 2$. From this lower bound for h_N^- we obtain the upper bounds C on $N_{k/\mathbb{Q}}(\mathfrak{F})$ such that $h_N^- = 1$ implies $N_{k/\mathbb{Q}}(\mathfrak{F}) \leq C$.

(3) If the number of ramified primes in $K_3/\mathbb{Q}$ is greater than one, then 3 divides h_{K_3}, whence $3|h_{N^+}$ by Proposition 1(6). If there is a prime divisor l of $N_{k/\mathbb{Q}}(\mathfrak{F})$ which is inert in k, then 3^4 divides h_N^-. Since M is the narrow Hilbert class field of k, (l) splits completely in M/k, whence there are at least 4 prime ideals ramified in N^+/M^+ which split at the same time in M/M^+. Hence, $3^4|h_N^-$ by [LOO, Proposition 8].

(4) According to [Lou6], the value $L(0,\chi)$ is an algebraic integer of $\mathbb{Q}(\zeta_{24})$ and

$$h_N^-/h_M^- = N_{\mathbb{Q}(\zeta_{24})/\mathbb{Q}}(\frac{1}{4}L(0,\chi)).$$

Let $\mathrm{Gal}(N/\mathbb{Q}) = \langle a,b,c | a^3 = b^8 = c^2 = 1, b^{-1}ab = a, c^{-1}ac = a, c^{-1}bc = b^{-1}\rangle$, where $\mathrm{Gal}(N/k) = \langle a,b\rangle$. The restriction of c to k generates $\mathrm{Gal}(k/\mathbb{Q})$ and using Artin's reciprocity theorem we obtain that $\chi \circ c = \chi^7$ and

$$\sigma_7(L(0,\chi)) = L(0,\chi^7) = L(0,\chi \circ c) = L(0,\chi).$$

Therefore, $L(0,\chi) \in \mathbb{Q}(\sqrt{2},\sqrt{-3})$, the subfield of $\mathbb{Q}(\zeta_{24})$ fixed by σ_7. It follows that

$$h_N^-/h_M^- = (N_{\mathbb{Q}(\sqrt{2},\sqrt{-3})/\mathbb{Q}}(\frac{1}{4}L(0,\chi)))^2$$

is a perfect square. □

We verify that $h_N^- > 1$ for all fields N satisfying points (2) and (3) in Lemma 1, and sum up the computational results in Table 1.

TABLE 1.

k	$\mu_k \operatorname{Res}_{s=1}(\zeta_k) \leq$	$N_{k/\mathbb{Q}}(\mathfrak{F}) \leq$	$\mathfrak{F}$	h_N^-
$\mathbb{Q}(\sqrt{5 \cdot 101})$	2.160235	36000	(7)	193^2
			(9)	463^2
			(19)	9337^2
			(31)	49252^2
			(67)	4540144^2
			(73)	1987279^2
			(79)	3111696^2
			(103)	8589721^2
			(127)	51439012^2
			(163)	180487972^2
			(181)	330579292^2
$\mathbb{Q}(\sqrt{5 \cdot 181})$	2.154802	4700	(7)	964^2
			$\mathfrak{P}_{181}$	4032^2
$\mathbb{Q}(\sqrt{13 \cdot 53})$	1.976174	6000	$\mathfrak{P}_{13}$	28^2
			(19)	23863^2
			(31)	130729^2
			(43)	538657^2
			(67)	8539888^2
			(73)	11762791^2
$\mathbb{Q}(\sqrt{13 \cdot 61})$	2.531191	18000	$\mathfrak{P}_{13}$	63^2
			(7)	964^2
			(9)	1519^2
			(31)	189724^2
			(37)	540127^2
			$\mathfrak{P}_{61}$	487^2
			(67)	4314289^2
			(103)	22194844^2
			(127)	73351108^2

4.1.2. $\operatorname{Gal}(N/\mathbb{Q}) = (D_8 \times C_3) \rtimes^1 C_2$. The field N has three non-normal cubic subfields. Let K_3 be any one of them, K_6 its normal closure, and k_2 the quadratic subfield of K_6. Since if $\operatorname{Gal}(M/k_2) = C_8$, then $\operatorname{Gal}(N/\mathbb{Q}) = D_{48}$ and $h_N^- > 1$. It follows that $k_2 = \mathbb{Q}\,(\sqrt{p})$ or $\mathbb{Q}\,(\sqrt{q})$, and $\operatorname{Gal}(M/k_2) = D_8$.

Lemma 2. (1) *We have* $(\zeta_N/\zeta_K)(s) \geq 0$ *in the range* $0 < s < 1$.

(2) *There exists some positive integer* $f \geq 1$ *such that* $\mathfrak{F}_{K_6/k_2} = (f)$. *For each given* M *with* $h_M^- = 1$ *we can compute a bound* $f \leq C$ *on the conductors* (f) *of the cyclic cubic extensions* K_6/k_2 *for the* N*'s such that* $h_N^- = 1$. *These bounds and the possible* f*'s are given in Table* 2.

(3) *Let* χ *be any one of the four characters of order* 12 *associated with the cyclic extension* N/K. *Then* h_M^- *divides* h_N^-, $L(0,\chi) \in \mathbb{Q}$, *and* $h_N^-/h_M^- = (L(0,\chi)/16)^4$ *is a perfect fourth power which can be computed by using the techniques developed in* [Lou5] *and* [Lou6].

Table 2.

K	k_2	$\mu_{k_2}\operatorname{Res}_{s=1}(\zeta_{k_2}) <$	$f <$	f	h_N^-
$\mathbb{Q}(\sqrt{2},\sqrt{257})$	$\mathbb{Q}(\sqrt{257})$			1	2^4
$\mathbb{Q}(\sqrt{5},\sqrt{101})$	$\mathbb{Q}(\sqrt{5})$	0.043633	13	None	
	$\mathbb{Q}(\sqrt{101})$	0.562340	22	2	1
$\mathbb{Q}(\sqrt{5},\sqrt{181})$	$\mathbb{Q}(\sqrt{5})$		105	None	
	$\mathbb{Q}(\sqrt{181})$	0.855096	30	17	166^4
$\mathbb{Q}(\sqrt{13},\sqrt{53})$	$\mathbb{Q}(\sqrt{13})$	0.146745	19	$2{\cdot}5$	35^4
	$\mathbb{Q}(\sqrt{53})$	0.407728	52	$2{\cdot}5$	50^4
				$2{\cdot}13$	31^4
$\mathbb{Q}(\sqrt{13},\sqrt{61})$	$\mathbb{Q}(\sqrt{13})$		130	11	85^4
				$2{\cdot}61$	175^4
	$\mathbb{Q}(\sqrt{61})$	0.487910	95	13	6^4
				$2{\cdot}9$	142^4
				$2{\cdot}11$	233^4

Proof. (1) Since N/K and M^+/k are cyclic of degree 12 and 4, respectively, then as in point (1) of Lemma 1 we obtain $(\zeta_N/\zeta_{M^+})(s) \geq 0$ and $(\zeta_{M^+}/\zeta_K)(s) \geq 0$ for $0 < s < 1$.

(2) The first part follows from [Mar, Theorem III.1] or [LPL, Theorem 4]. For $K = \mathbb{Q}(\sqrt{2},\sqrt{257})$ we have verified that $\zeta_K(s) \leq 0$ in the range $0 < s < 1$. Hence, $\zeta_N(s) \leq 0$ for $0 < s < 1$ for every M with $h_M^- = 1$. Since N^+/k_2 is abelian and $d_{N^+}/d_{k_2}^{12} = f^{16}$, using Proposition 2 we get upper bound C on f such that $h_N^- = 1$ implies $f \leq C$. To alleviate the list of possible conductors f we use the same argument as in point (3) of Lemma 1: If there is a prime divisor l of f which is inert in $\mathbb{Q}(\sqrt{pq})$, then 3^4 divides h_N^-.

(3) Let K_{12} be the compositum of K and K_6. We have

$$h_N^-/h_M^- = N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(L(0,\chi)/2^4).$$

Assume $\operatorname{Gal}(N/K) = \langle a, b^2\rangle$. Let χ_- be any one of two quartic characters associated with the cyclic extension M/K and χ_+ any one of two cubic characters associated with the cyclic extension K_{12}/K such that $\chi = \chi_-\chi_+$. Using the Artin reciprocity theorem, it can be easily verified that $\chi_- \circ b = \chi_-$, $\chi_- \circ c = \chi_-^{-1}$, $\chi_+ \circ b = \chi_+^{-1}$, and $\chi_+ \circ c = \chi_+$, whence $\chi \circ b = \chi^5$ and $\chi \circ c = \chi^7$. For a positive integer n let $\sigma_n \in \operatorname{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$ with $\sigma_n(\zeta_{12}) = \zeta_{12}^n$. We have

$$\sigma_5(L(0,\chi)) = L(0,\chi^5) = L(0,\chi \circ b) = L(0,\chi)$$

and $\sigma_7(L(0,\chi)) = L(0,\chi)$. Since $\langle\sigma_5,\sigma_7\rangle = \operatorname{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$ we have $L(0,\chi) \in \mathbb{Q}$, whence h_N^-/h_M^- is the 4-th power of some integer.

□

Our computational results are given in Table 2. When $K = \mathbb{Q}(\sqrt{2}, \sqrt{257})$, if $h_N = 1$, then N/M, N^+/M^+, and K_6/k_2 are unramified. Otherwise, $h_N \equiv 0 \bmod 3$. Since $\mathbb{Q}(\sqrt{2})$ has class number one, we must have $k_2 = \mathbb{Q}\,(\sqrt{257})$ and $(f) = 1$. Note that when $K = \mathbb{Q}(\sqrt{5}, \sqrt{101})$, $k_2 = \mathbb{Q}(\sqrt{101})$, and $f = 2$, we have $K_6 = \mathbb{Q}(\sqrt{101}, \theta)$ with $\theta^3 - \theta^2 - 5\theta - 1 = 0$. Using KASH ([K]) we verify that the class group of $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$ is isomorphic to C_2 and the narrow class group of this field is isomorphic to C_4. It follows that N^+ is the Hilbert class field of $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$ and N is the narrow Hilbert class field of this field. In addition, thanks to KASH we verify that the class number of N^+ is equal to 1.

4.2. $\mathrm{Gal}(M/\mathbb{Q}) = Q_8 \times C_2$**.** By [Lou3, Theorem 1],

$$M = \mathbb{Q}\left(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{-(2+\sqrt{2})(3+\sqrt{3})}\right)$$

is the only normal CM-field of relative class number one with Galois group isomorphic to $Q_8 \times C_2$. This field has class number one and $Q_M = 2$. In this subsection we assume that N contains this field M and will prove that $h_N > 1$. The Galois group $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to either $Q_8 \times C_2 \times C_3$ or $Q_{24} \times C_2$ according to whether N has a cyclic cubic subfield or not.

4.2.1. $\mathrm{Gal}(N/\mathbb{Q}) = Q_8 \times C_2 \times C_3$. The field N has only one cyclic cubic subfield K_3. The composita

$$N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-(2+\sqrt{2})(3+\sqrt{3})}) \quad \text{and} \quad N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1})$$

have the same maximal real subfield $K_3(\sqrt{2}, \sqrt{3})$. Suppose that $h_N = 1$. By Proposition 1(7) we would have $h^-_{N_1} = 1$ or $h^-_{N_2} = 1$. Since every octic quaternion CM-field has an even relative class number, $h^-_{N_1}$ is even. Using [CK1] we verify that there is no imaginary abelian number field with Galois group isomorphic to $C_2 \times C_2 \times C_2 \times C_3$ of relative class number one which contains the field $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-1})$. Hence $h_N > 1$.

4.2.2. $\mathrm{Gal}(N/\mathbb{Q}) = Q_{24} \times C_2$. The field N contains a non-normal cubic subfield K_3. The compositum

$$N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-(2+\sqrt{2})(3+\sqrt{3})})$$

is a normal CM-field with Galois group isomorphic to Q_{24}. The compositum $N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1})$ is a normal CM-field with Galois group isomorphic to $D_{12} \times C_2$. We have $N = N_1N_2$ with $N_1^+ = N_2^+ = K_3(\sqrt{2}, \sqrt{3})$. Note that $h^-_{N_1}$ is even. According to [P, Theorem 1], $h_{N_2} > 1$. By Proposition 1(7) it follows that $h_N > 1$.

5. Case 3: $\mathrm{Gal}(M/\mathbb{Q}) \in \{G_9, G_6\}$

In subsection 5.1 we assume that $G(M/\mathbb{Q}) = G_9$, and in subsection 5.2 we assume that $G(M/\mathbb{Q}) = G_6$.

5.1. $\mathrm{Gal}(M/\mathbb{Q}) = G_9$**.** There is only one normal CM-field M of relative class number one with Galois group isomorphic to G_9 ([LO2, Theorem 20]):

$$M = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{37}, \sqrt{-(2\sqrt{2}+3\sqrt{5})(2+\sqrt{5})}).$$

Assume that N contains M. The aim of this subsection is to prove that $h_N > 1$. Note that $\mathrm{Gal}(M/\mathbb{Q}(\sqrt{2\cdot 5\cdot 37})) = Q_8$, $\mathrm{Gal}(M/\mathbb{Q}(\sqrt{2})) = \mathrm{Gal}(M/\mathbb{Q}(\sqrt{5})) = \mathrm{Gal}(M/\mathbb{Q}(\sqrt{37})) = D_8$, and $\mathrm{Gal}(M/\mathbb{Q}(\sqrt{2\cdot 5})) = \mathrm{Gal}(M/\mathbb{Q}(\sqrt{2\cdot 37})) = \mathrm{Gal}(M/\mathbb{Q}(\sqrt{5\cdot 37})) = C_4\times C_2$. Therefore, $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to $G_9\times C_3$ if N contains only one cubic cyclic subfield K_3, $(Q_8\times C_3)\rtimes C_2$, $(D_8\times C_3)\rtimes^2 C_2 = \langle a,b,c \mid a^3=b^2=c^2=z^4=1, c^{-1}bc=bz^2, bz=zb, cz=zc, b^{-1}ab=a, c^{-1}ac=a, z^{-1}az=a^{-1}\rangle$, or $(C_4\times C_2\times C_3)\rtimes C_2$ otherwise. We divide this subsection into four parts according to $\mathrm{Gal}(N/\mathbb{Q})$.

5.1.1. $\mathrm{Gal}(N/\mathbb{Q}) = G_9\times C_3$. We will show that $h_N > 1$. We first get an upper bound C on the conductor of $K_3/\mathbb{Q}$ such that if $h_N^- = 1$, then the conductor is less than or equal to C. Let $K = \mathbb{Q}(\sqrt{2},\sqrt{5\cdot 37})$. Since N/K is cyclic of degree 12, then as in point (1) of Lemma 1 we obtain $(\zeta_N/\zeta_{M^+})(s)\geq 0$ for $0<s<1$. We verify that $\zeta_{M^+}(s)\leq 0$ for $s\in]0,1[$, which implies $\zeta_N(s)\leq 0$ for $s\in]0,1[$. Let $F=\mathbb{Q}(\sqrt{2\cdot 5\cdot 37})$. The extension N^+/F is abelian of degree 12 and $d_{N^+}/d_F^{12} = N_{F/\mathbb{Q}}(\mathfrak{F}_{FK_3/F})^8$. Using $\mu_F\,\mathrm{Res}_{s=1}(\zeta_F)\leq 2.227842$ and Proposition 2, we obtain that if $h_N^-\leq 1$, then $N_{F/\mathbb{Q}}(\mathfrak{F}_{FK_3/F})\leq 1300$, whence $\mathfrak{F}_{FK_3/F}\in\{(7),(3)^2,(13),(19),(31),\mathfrak{P}_{37}\}$ with $(37)=\mathfrak{P}_{37}^2$ in F.

- (i) When $\mathfrak{F}_{FK_3/F}=(7)$, K_3 is of conductor 7. There are four non-normal octic CM-fields containing $\mathbb{Q}(\sqrt{2})$ with relative class number one. Let $M_{8,1} = \mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{-(2\sqrt{2}+3\sqrt{5})(2+\sqrt{5})})$ and $M_{8.2}$ its conjugate over $\mathbb{Q}$. Set $N_{24,1}=M_{8,1}K_3$ and $N_{24,2}=M_{8,2}K_3$. By Proposition 1(5) $h_N^- = (h_{N_{24,1}}^-)^2$. The two prime ideals lying above 7 in $M_{8,1}^+=\mathbb{Q}(\sqrt{2},\sqrt{5})$ split in $M_{8,1}$. Hence, $3^2|h_{N_{24,1}}^-$ by [LOO, Proposition 8] and $3^4|h_N^-$.
- (ii) When $\mathfrak{F}_{FK_3/F}=(3)^2$, K_3 is of conductor 9. Let $M_{8,1},M_{8,2},N_{24,1}$, and $N_{24,2}$ be as in (i). One prime ideal lying above 37 in $\mathbb{Q}(\sqrt{2},\sqrt{5})$ is ramified in $M_{8,1}$ and 37 splits in K_3, whence $2^2|h_{N_{24,1}}^-$ and $2^4|h_N^-$. By the same argument we prove that if $\mathfrak{F}_{FK_3/F}=(19)$, then $2^4|h_N^-$.
- (iii) When $\mathfrak{F}_{FK_3/F}=(13)$, K_3 is of conductor 13. Let $M_{8,1}$ be any one of two non-normal octic CM-fields of M containing $\mathbb{Q}(\sqrt{2},\sqrt{37})$ and $M_{8,2}$ its conjugate over $\mathbb{Q}$. Let $N_{24,1}=M_{8,1}K_3$ and $N_{24,2}=M_{8,2}K_3$. One prime ideal lying above 5 in $\mathbb{Q}(\sqrt{2},\sqrt{37})$ is ramified in $M_{8,1}$ and 5 splits in K_3, which implies $2^2|h_{N_{24,1}}^-$ and $2^4|h_N^-$.
- (iv) When $\mathfrak{F}_{FK_3/F}=(31)$, K_3 is of conductor 31. We let $M_{8,1}$ be any one of two non-normal octic subfields of M containing $\mathbb{Q}(\sqrt{5},\sqrt{37})$ and $M_{8,2}$ its conjugate over $\mathbb{Q}$. One prime ideal lying above 2 is ramified in $M_{8,1}$ and 2 splits in K_3, whence $2^2|h_{N_{24,1}}^-$ and $2^4|h_N^-$.
- (v) When $\mathfrak{F}_{FK_3/F}=\mathfrak{P}_{37}$, K_3 is of conductor 37. According to [Ma], the cyclic sextic subfield $K_3(\sqrt{2\cdot 5})$ of N has class number 6, whence by Proposition 1(6) $3|h_N$.

5.1.2. $\mathrm{Gal}(N/\mathbb{Q}) = (D_8\times C_3)\rtimes^2 C_2$. Let K_3 be any one of three non-normal cubic subfields of N, K_6 its normal closure, and k_2 the quadratic subfield of K_6. We have $\mathrm{Gal}(M/k_2)=D_8$. Let K be the intermediate field between M^+ and k_2 such that $G(M/K)=C_4$. Then M/K is unramified at all finite primes, $\mathrm{Gal}(M^+/k_2) = C_2\times C_2$, $\mathrm{Gal}(N/K)=C_{12}$, and $\mathrm{Gal}(N^+/k_2)=C_2\times C_2\times C_3$. As in point (1) of

Table 3.

k	$f \leq$	f	h_N^-
$\mathbb{Q}(\sqrt{2})$	22	None	
$\mathbb{Q}(\sqrt{5})$	8	None	
$\mathbb{Q}(\sqrt{37})$	44	2	2^4

Table 4.

k_2	C	f	h_N^-
$\mathbb{Q}(\sqrt{2\cdot 5})$	140	37	1021^2
$\mathbb{Q}(\sqrt{2\cdot 37})$	21	9	5044^2
$\mathbb{Q}(\sqrt{5\cdot 37})$	35	None	

Lemma 1 we obtain $(\zeta_N/\zeta_{M^+})(s) \geq 0$ and $\zeta_N(s) \leq 0$ for $s \in]0,1[$. Since N^+/k_2 is abelian of degree 12 and $d_{N^+}/d_{k_2}^{12} = f^{16} N_{K_6/\mathbb{Q}}(\mathfrak{D}_{N^+/K_6})$, where $(f) = \mathfrak{F}_{K_6/k_2}$ and $\mathfrak{D}_{N^+/K_6}$ denotes the discriminant of the extension N^+/K_6, using Proposition 2 we obtain upper bound C on f such that if $h_N^- = 1$, then $f \leq C$. Our computational results are given in Table 3. As in point (3) of Lemma 2 we can easily verify that h_N^-/h_M^- is the 4-th power of some rational integer.

5.1.3. $\mathrm{Gal}(N/\mathbb{Q}) = (C_4 \times C_2 \times C_3) \rtimes C_2 = \langle a,b,c,z \mid a^3 = b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = zc, b^{-1}ab = a, c^{-1}ac = a^{-1}, z^{-1}az = a\rangle$. In this case N has three non-normal cubic subfields and its associated quadratic subfield k_2 has $\mathrm{Gal}(M/k_2) = C_4 \times C_2$. For a fixed field k_2 there are two intermediate fields K between k_2 and M^+ such that $\mathrm{Gal}(M/K) = C_4$. Let K be any one of these two fields. Then M/K is unramified at all finite primes, $\mathrm{Gal}(N/K) = C_{12}$, $\mathrm{Gal}(M^+/k_2) = C_2 \times C_2$, and $\mathrm{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3$. Analogously as in subsection 5.1.2 we get upper bound C on f such that if $h_N^- = 1$, then $f \leq C$. Since $\mathrm{Gal}(N/k_2) = C_4 \times C_2 \times C_3$, we compute h_N^- using Hecke L-functions over k_2 (see Table 4).

5.1.4. $\mathrm{Gal}(N/\mathbb{Q}) = (Q_8 \times C_3) \rtimes C_2$. In this case N has three non-normal cubic subfields such that its associated quadratic subfield k_2 is equal to $\mathbb{Q}(\sqrt{2\cdot 5\cdot 37})$. Let K be any one of three quartic fields containing $\mathbb{Q}(\sqrt{2\cdot 5\cdot 37})$. Then $\mathrm{Gal}(N/K) = C_{12}$ and $\mathrm{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3$. As in subsection 5.1.2 we verify that if $h_N^- = 1$, then $f \leq 36$ with $\mathfrak{F}_{K_6/k_2} = (f)$. There is no sextic field K_6 containing $\mathbb{Q}(\sqrt{2\cdot 5\cdot 37})$ with $f \leq 36$. Therefore, $h_N^- > 1$.

5.2. $\mathrm{Gal}(M/\mathbb{Q}) = G_6$**.** In [Lou3] it is proved that there are exactly two such fields M with $h_M = 1$: the composita $M = M_1M_2$ listed in Table 5. In fact, those are the only fields with Galois group isomorphic to G_6 of relative class number one. The Galois group $\mathrm{Gal}(M^+/\mathbb{Q})$ is isomorphic to D_8 or $C_4 \times C_2$. When $\mathrm{Gal}(M^+/\mathbb{Q}) = D_8$, M is a compositum of an octic dihedral CM-field $M_{8,1}$ and an imaginary abelian number field $M_{8,2}$ with $\mathrm{Gal}(M_{8,2}/\mathbb{Q}) = C_4 \times C_2$. Using [YK] and [CK1], we verify

Table 5.

$M_{8,1}$	$M_{8,2}$	$h^-_{M_{8,1}}$	$h^-_{M_{8,2}}$
$\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5+\sqrt{17})/2})$ $\mathrm{Gal}(M_{8.1}/\mathbb{Q}) = D_8$ $\mathrm{Gal}(M_{8,1}/\mathbb{Q}(\sqrt{34})) = C_4$	$\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(17+3\sqrt{17})/2})$ $\mathrm{Gal}(M_{8.2}/\mathbb{Q}) = D_8$ $\mathrm{Gal}(M_{8,2}/\mathbb{Q}(\sqrt{2})) = C_4$	1	2
$\mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-(9+\sqrt{13})/2})$ $\mathrm{Gal}(M_{8.1}/\mathbb{Q}) = D_8$ $\mathrm{Gal}(M_{8,1}/\mathbb{Q}(\sqrt{221})) = C_4$	$\mathbb{Q}(\sqrt{17}, \sqrt{-(13+2\sqrt{13})})$ $\mathrm{Gal}(M_{8.2}/\mathbb{Q}) = C_4 \times C_2$ $\mathrm{Gal}(\mathbb{Q}(\sqrt{-(13+2\sqrt{13})}/\mathbb{Q}) = C_4$	1	2

that there is only one such field M with $h^-_M = 1$: the second field in Table 5. When $\mathrm{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$, M is a compositum of two octic dihedral CM-fields $M_{8,1}$ and $M_{8,2}$ with $M^+_{8,1} = M^+_{8,2}$. According to [Lou3, Theorem 2] there is only one such field M with $h^-_M = 1$: the first field in Table 5. We will prove that $h_N > 1$. If $K_3/\mathbb{Q}$ is normal, then $\mathrm{Gal}(N/\mathbb{Q}) = G_6 \times C_3$. Otherwise, $\mathrm{Gal}(N/\mathbb{Q}) = C_3 \rtimes G_6$. Let $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$. Then $N = N_1N_2$ with $N_1^+ = N_2^+ = K_3K_4$.

5.2.1. $\mathrm{Gal}(N/\mathbb{Q}) = G_6 \times C_3$. If $\mathrm{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = D_8 \times C_3 = \mathrm{Gal}(N_2/\mathbb{Q})$. According to [P, Theorem 1], $h^-_{N_1} > 1$ and $h^-_{N_2} > 1$, whence $h_N > 1$. If $\mathrm{Gal}(M^+/\mathbb{Q}) = D_8$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = D_8 \times C_3$, and $\mathrm{Gal}(N_2/\mathbb{Q}) = C_4 \times C_2 \times C_3$. Using [CK1], we verify that $h^-_{N_2} > 4$, whence $h^-_N > 1$.

5.2.2. $\mathrm{Gal}(N/\mathbb{Q}) = C_3 \rtimes G_6$.

(a) Let M be the first field in Table 5. The quadratic field k_2 associated with K_3 is either $\mathbb{Q}(\sqrt{34})$, $\mathbb{Q}(\sqrt{2})$, or $\mathbb{Q}(\sqrt{17})$. If $k_2 = \mathbb{Q}(\sqrt{34})$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = D_{24}$, and $\mathrm{Gal}(N_2/\mathbb{Q}) = C_3 \rtimes D_8$. If $k_2 = \mathbb{Q}(\sqrt{2})$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8$, and $\mathrm{Gal}(N_2/\mathbb{Q}) = D_{24}$. If $k_2 = \mathbb{Q}(\sqrt{17})$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8 = \mathrm{Gal}(N_2/\mathbb{Q})$. According to [P, Theorem 1] and [Lef, Theorem 4.1], we have $h^-_{N_1} > 1$ and $h^-_{N_2} > 1$, whence $h_N > 1$.

(b) Let M be the second field in Table 5. The quadratic field k_2 associated with K_3 is either $\mathbb{Q}(\sqrt{221})$, $\mathbb{Q}(\sqrt{13})$, or $\mathbb{Q}(\sqrt{17})$. If $k_2 = \mathbb{Q}(\sqrt{221})$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = D_{24}$, and $\mathrm{Gal}(N_2/\mathbb{Q}) = S_3 \times C_4$. If $k_2 = \mathbb{Q}(\sqrt{13})$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8$ and $\mathrm{Gal}(N_2/\mathbb{Q}) = Q_{12} \times C_2$. If $k_2 = \mathbb{Q}(\sqrt{17})$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8$, and $\mathrm{Gal}(N_2/\mathbb{Q}) = S_3 \times C_4$. Using [Lef, Theorem 4.1] and [P, Theorem 1], we have $h^-_{N_1} > 1$ and $h^-_{N_2} > 1$, whence $h_N > 1$.

6. Case 4: $\mathrm{Gal}(M/\mathbb{Q}) = D_8 \times C_2$

To begin with, we prove the following:

Theorem 2 (Compare with [Lou3, Theorems 2 and 3])**.** *There are four normal CM-fields M of degree 16 and Galois group $D_8 \times C_2$ with relative class number one, those given in the following Table 6.*

Table 6.

M^+	$\alpha : M = M^+(\sqrt{-\alpha})$	Q_M	ω_M
$\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17})$	$5(5+\sqrt{17})/2$	2	2
$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{3+\sqrt{3}})$	1	2	24
$\mathbb{Q}(\sqrt{3}, \sqrt{11}, \sqrt{15+8\sqrt{3}})$	1	2	12
$\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{3(5+\sqrt{17})/2})$	3	2	6

Proof. The Galois group $\mathrm{Gal}(M^+/\mathbb{Q})$ is isomorphic to either D_8 or $C_2 \times C_2 \times C_2$. When $\mathrm{Gal}(M^+/\mathbb{Q}) = D_8$, M is a compositum of an octic dihedral CM-field $M_{8,1}$ and an imaginary abelian number field $M_{8,2}$ with $\mathrm{Gal}(M_{8,2}/\mathbb{Q}) = C_2 \times C_2 \times C_2$. Let L be any one of four non-normal quartic CM-subfields of $M_{8,1}$. According to [Lou3, Proposition 16], $h_M^- = 1$ if and only if $(h_L^-, h_{M_{8,2}}^-) \in \{(1,4),(1,2),(2,1)\}$. In the case that $(h_L^-, h_{M_{8,2}}^-) = (1,4)$ or $(1,2)$, $M_{8,1}^+ = M_{8,2}^+$ is of the form $\mathbb{Q}(\sqrt{p}, \sqrt{q})$, where (p,q) is one of the 19 pairs given in [LO1, Theorem 8]. Then $M_{8,2} = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-m})$, where $\mathbb{Q}(\sqrt{-m})$ is one of $81 (= 9 + 18 + 54\ )$ imaginary quadratic fields of class number one, two or four. For these $1539 (= 19 \times 81)$ CM-fields $M_{8,2}$ we compute $h_{M_{8,2}}^-$ and verify that there is only one field M with $h_M^- = 1$: the fourth field in Table 6. In the case that $(h_L^-, h_{M_{8,2}}^-) = (2,1)$, using [YK] and [CK1] we verify that there are exactly two fields M with $h_M^- = 1$: the second and third fields in Table 6. When $\mathrm{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, M is a compositum of two octic dihedral CM-fields $M_{8,1}$ and $M_{8,2}$. According to [Lou3, Theorem 2], there is only one such M with $h_M^- = 1$: the first field in Table 6. □

From now on we assume that M is one of these four fields and we will prove that $h_N > 1$. We classify the Galois group $\mathrm{Gal}(N/\mathbb{Q})$. Let K_3 be any cubic subfield of N. If K_3 is not normal, then its normal closure K_6 is a dihedral real sextic field and we let k_2 denote the (real) quadratic subfield of K_6. If K_3 is normal, then $\mathrm{Gal}(N/\mathbb{Q}) = D_8 \times C_6$. If K_3 is not normal over $\mathbb{Q}$, then $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to either $D_8 \times S_3$, $D_{24} \times C_2$, or $(C_3 \rtimes D_8) \times C_2$. Let $K_4 = M_{8,1}^+ = M_{8,2}^+$. If $K_4 \cap k_2 = \mathbb{Q}$, then $\mathrm{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, and $\mathrm{Gal}(N/\mathbb{Q}) = D_8 \times S_3$. If $K_4 \cap k_2 \supsetneqq \mathbb{Q}$, then $M_{8,1}/k_2$ is either cyclic or biquadratic bicyclic. If $M_{8,1}/k_2$ is cyclic quartic, then $\mathrm{Gal}(N/\mathbb{Q}) = D_{24} \times C_2$. If $M_{8,1}/k_2$ is biquadratic bicyclic, then $\mathrm{Gal}(K_3M_{8,1}/\mathbb{Q}) = C_3 \rtimes D_8 = \langle a,b,c | a^3 = b^4 = c^2 = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = b^{-1} \rangle$, and $\mathrm{Gal}(N/\mathbb{Q}) = (C_3 \rtimes D_8) \times C_2$.

6.1. $\mathrm{Gal}(N/\mathbb{Q}) = D_8 \times C_6$. If $\mathrm{Gal}(M^+/\mathbb{Q}) = D_8$, then the composita $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$ are normal CM-subfields of N with the same maximal real subfields K_3K_4. By [P, Theorem 1], the compositum of the dihedral octic CM-field $\mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-(9+\sqrt{13})/2})$ and of the cyclic cubic field of conductor 13 is the only normal CM-field of relative class number one with Galois group isomorphic to $D_8 \times C_3$. In Table 6 there is no field M containing $\mathbb{Q}(\sqrt{13}, \sqrt{17})$. It follows then that $h_{N_1}^- > 1$. According to [CK1], there are exactly two imaginary abelian number fields with Galois group isomorphic to $C_2 \times C_2 \times C_2 \times C_3$ of relative class number one: $F_7(\sqrt{-1}, \sqrt{-3}, \sqrt{-7})$ and $F_7(\sqrt{-3}, \sqrt{-7}, \sqrt{-15})$, where F_7 denotes the cyclic cubic field of conductor 7. In Table 6 there is no field M containing $\mathbb{Q}(\sqrt{3},\sqrt{7})$, or $\mathbb{Q}(\sqrt{5},\sqrt{21})$, whence $h_{N_2}^- > 1$. It follows that if $\mathrm{Gal}(M^+/\mathbb{Q}) = D_8$, then $h_N > 1$. If $\mathrm{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, then $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$ are normal CM-fields with Galois group isomorphic to $D_8 \times C_3$. Note that $M_{8,1}^+ = M_{8,2}^+ = \mathbb{Q}(\sqrt{2},\sqrt{17})$. According to [P, Theorem 1], $h_{N_1}^- > 1$ and $h_{N_2}^- > 1$, which implies $h_N > 1$.

6.2. $\mathrm{Gal}(N/\mathbb{Q}) = D_8 \times S_3$**.** In this case N has three non-normal real cubic subfields. Let K_3, K_6, k_2 and K_4 be as above. We have that $K_4 \cap k_2 = \mathbb{Q}$, $\mathrm{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, M is the first field in Table 6. In addition, we have $K_4 = \mathbb{Q}(\sqrt{2},\sqrt{17})$, and $k_2 = \mathbb{Q}(\sqrt{m})$ with $m \in \{5, 2\cdot 5, 5\cdot 17, 2\cdot 5\cdot 17\}$. Let (f) be the conductor of the extension K_6/k_2 with f a positive integer.

Lemma 3. 1. *We have $\zeta_N(s) \le 0$ for $0 < s < 1$.*

2. *For each given k_2 in the above we can compute a bound of $f \le C$ on the conductor (f) for N's such that $h_N^- = 1$. These bounds and the possible f's are compiled in Table 7.*
3. *The quotient h_N^-/h_M^- is the perfect fourth power of some rational integer.*

Proof. (1) Let χ_{N/M^+} be any one of two characters associated with the cyclic sextic extension N/M^+. We have

$$\frac{\zeta_N(s)}{\zeta_{M^+}(s)} = \frac{\zeta_M(s)}{\zeta_{M^+}(s)} |L(s,\chi_{N/M^+})L(s,\chi_{N/M^+}^2)|^2$$

and

$$\frac{\zeta_M(s)}{\zeta_{M^+}(s)} = \frac{\zeta_{M_{8,1}}(s)}{\zeta_{K_4}(s)}\frac{\zeta_{M_{8,2}}(s)}{\zeta_{K_4}(s)} = L(s,\psi_1)^2 L(s,\psi_2)^2,$$

where ψ_i is the unique irreducible character of degree 2 of $\mathrm{Gal}(M_{8.i}/\mathbb{Q})$ the dihedral group of order 8, and $L(s,\psi_i)$ denotes the Artin L-function associated with ψ_i for $i = 1, 2$. Since ψ_i is real valued, $L(s,\psi_i)$ is on the real axis and $L(s,\psi_i)^2 \ge 0$. For $M^+ = \mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{17})$ we have verified that $\zeta_{M^+}(s) \le 0$ for $s \in]0,1[$, whence $\zeta_N(s) \le 0$.

(2) Since M/M^+ is unramified at all finite primes, N/N^+ is unramified at all finite primes and $d_N/d_{N^+} = d_{N^+} = d_{k_2}^{12} f^{16} N_{K_6/\mathbb{Q}}(\mathfrak{D}_{N^+/K_6})$. Using Proposition 2, we get an upper bound on f. Since the prime ideals lying above 2 and those above 17 split in M/M^+, if $(f,2) > 1$ or $(f,17) > 1$, then 3^2 divides h_N^- by [LOO, Proposition 8]. Note that the prime ideals lying above 13 split in M/M^+, whence the relative class number of the fourth field N in Table 7 is divisible by 3^4.

TABLE 7.

k_2	$\mu_{k_2}\,\mathrm{Res}_{s=1}(\zeta_{k_2}) \le$	$f \le$	f	h_N^-
$\mathbb{Q}(\sqrt{5})$	0.0436324	10	NONE	
$\mathbb{Q}(\sqrt{2\cdot 5})$	0.4276490	41	37	$(920)^4$
$\mathbb{Q}(\sqrt{5\cdot 17})$	0.5861712	31	9	$(44)^4$
$\mathbb{Q}(\sqrt{2\cdot 5\cdot 17})$	1.4062136	38	13	$3^4 \mid h_N^-$

(3) Note that $M_{8,1}$ and $M_{8,2}$ are cyclic over $\mathbb{Q}(\sqrt{34})$. Let K be the compositum of $\mathbb{Q}(\sqrt{34})$ and k_2. Then $\mathrm{Gal}(N/K) = C_{12}$. Let χ be any one of the four characters of order 12 associated with the cyclic extension N/K. Similarly to point (3) of Lemma 2, we verify that $L(0,\chi) \in \mathbb{Q}$ and $h_N^-/h_M^- = (L(0,\chi)/2^4)^4$. □

In conclusion, we have proved that every normal CM-field with Galois group isomorphic to $D_8 \times S_3$ has class number greater than one. Our computational results are given in Table 7.

6.3. $\mathrm{Gal}(N/\mathbb{Q}) = D_{24} \times C_2$. In this case N has three non-normal cubic fields and $M_{8,1}/k_2$ is cyclic. Let $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$. Then we have $N = N_1N_2$ with $N_1^+ = N_2^+ = K_3K_4$. If $\mathrm{Gal}(M^+/\mathbb{Q}) = D_8$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = D_{24}$, and $\mathrm{Gal}(N_2/\mathbb{Q}) = S_3 \times C_2 \times C_2$. If $\mathrm{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, then $\mathrm{Gal}(N_1/\mathbb{Q}) = D_{24} = \mathrm{Gal}(N_2/\mathbb{Q})$. Using [Lef, Theorem 4.1] and [P, Theorem 1], we verify that in both cases $h_{N_1}^- > 1$ and $h_{N_2}^- > 1$. It follows that the class number of a normal CM-field with Galois group isomorphic to $D_{24} \times C_2$ is greater than one.

6.4. $\mathrm{Gal}(N/\mathbb{Q}) = (C_3 \rtimes D_8) \times C_2$. In this case N has three non-normal cubic fields and $M_{8,1}/k_2$ is biquadratic bicyclic. Then the Galois group of the compositum $N_1 = M_{8,1}K_3$ over $\mathbb{Q}$ is isomorphic to $C_3 \rtimes D_8$, whence $h_{N_1}^- > 1$ ([P, Theorem 13]). If $\mathrm{Gal}(M^+/\mathbb{Q}) = D_8$, then $\mathrm{Gal}(N_2/\mathbb{Q}) = S_3 \times C_2 \times C_2$. If $\mathrm{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, then $\mathrm{Gal}(N_2/\mathbb{Q}) = C_3 \rtimes D_8$. By [P, Theorems 1 and 13] $h_{N_2}^- > 1$. Consequently, if $\mathrm{Gal}(N/\mathbb{Q}) = (C_3 \rtimes D_8) \times C_2$, then $h_N > 1$.

To conclude, Theorem 1 is now proved with completion.

All computations were carried out using Pari-Gp ([Pa]) and KASH ([K]).

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