THE CLASS NUMBER ONE PROBLEM
FOR SOME NON-ABELIAN NORMAL CM-FIELDS
OF DEGREE 48

KU-YOUNG CHANG AND SOUN-HI KWON

ABSTRACT. We prove that there is precisely one normal CM-field of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of \( \mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta) \) with \( \theta^3 - \theta^2 - 5\theta - 1 = 0 \).

1. Introduction

According to [O] and [H], there exist only finitely many normal CM-fields with class number one, and their degrees are less than or equal to 436. All imaginary abelian number fields with class number one are known in [Y]: their degrees are less than or equal to 24. All normal CM-fields of degree less than 48 with class number one are known by many authors ([LO1], [LO2], [Lef], [LLO], [LP1], [LP2], [LOO], [Lou3], [P], [YPK], [PsK], [CK2], and [CK3]). In the following table we sum up the numbers of the non-abelian normal CM-fields \( N \) with class number one according to their degrees.

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In this paper we study the non-abelian normal CM-fields that contain a normal CM-subfield of degree 16, and will prove the following:

Theorem 1. There exists one and only one normal CM-field \( N \) of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of the real dihedral number field \( K_{12} = \mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta) \) of degree 12 with \( \theta^3 - \theta^2 - 5\theta - 1 = 0 \), narrow class number 4 and class number 2. The extension \( N/K_{12} \) is cyclic quartic, \( d_N = d_{K_{12}}^4 = 2^{32} \cdot 5^{24} \cdot 101^{24} \), the maximal totally real subfield of \( N \) is the Hilbert class field of \( K_{12} \), and the Galois group \( \text{Gal}(N/\mathbb{Q}) \) is isomorphic to the semi-direct product \( C_3 \rtimes D_{16} \).

2. Prerequisite and notation

We use the following notation. For a number field \( K \), we let \( h_K \), \( d_K \), \( \omega_K \), and \( \zeta_K \) denote the class number, the absolute value of the discriminant, the number of roots of unity in \( K \), and the Dedekind zeta function of \( K \), respectively. If \( K \)
is a CM-field, we let $h_K^-$, $K^+$ and $Q_K \in \{1, 2\}$ be the relative class number, the maximal real subfield and the Hasse unit index of $K$, respectively. For an abelian extension $F/K$ we denote by $\mathfrak{F}_{F/K}$ the finite part of its conductor. For a positive integer $n$ we let $\zeta_n = e^{2i\pi/n}$. Before starting the proof of Theorem 1 we recall the well-known results which will be used later in this paper.

**Proposition 1.**  
(1) ([LOO] Lemma 2) If $K$ is a normal CM-field, then the complex conjugation is in the center of its Galois group.

(2) ([CH Lemma 13.5]) Let $K$ be a CM-field. If there is at least one ramified prime ideal in $K/K^+$ which is lying above an odd prime, then $Q_K = 1$.

(3) ([H2 Theorem 5], [R], or [LOO Theorem 5]) Let $k \subset K$ be two CM-fields. Then $h_K^-$ divides $4h_{K^+}^-$. Moreover, if $[K : k]$ is odd, then $h_k^-$ divides $h_K^-$ and $Q_k = Q_K$.

(4) ([LO2 and LO1 Proposition 6]) Let $K$ be a CM-field and let $t$ be the number of prime ideals of $K^+$ that are ramified in the quadratic extension $K/K^+$. Then $2^{t-1}$ divides $h_K^-$. Moreover, if $Q_K = 2$, then $2^{t}$ divides $h_K^-$. 

(5) ([LO1 Proposition 13] and [LO2 Proposition 2]) Let $L_1L_2$ be a CM-field which is a compositum of two CM-fields $L_1$ and $L_2$ with the same maximal totally real subfield. Then

$$h_K^- = \frac{Q_K}{Q_{L_1}Q_{L_2}} \frac{\omega_K}{\omega_{L_1}\omega_{L_2}} h_{L_1}^- h_{L_2}^-$$

and $h_{L_1}^- h_{L_2}^- = 4h_{K^+}^-$. In particular, if $L_1$ and $L_2$ are isomorphic, then $\omega_K = \omega_{L_1} = \omega_{L_2} = 2$ and $h_K^- = (Q_K/2)(h_{L_1}^-/Q_{L_1})^2$.

(6) ([M Corollary 2.2 and 2.3]) Let $E/F$ be an extension of number fields. Then $h_E$ divides $[E:F]h_F$. Moreover, if no nontrivial abelian subextension of $E/F$ is unramified over $F$, then $h_E$ divides $h_F$.

(7) Let $K = L_1L_2$ be a CM-field which is a compositum of two CM-fields $L_1$ and $L_2$ with the same maximal totally real subfield $L_1^+ = L_2^+$. If $h_K = 1$, then $h_{L_1}$ and $h_{L_2}$ are 1 or 2. If $h_K = 1$ and $h_{L_1} = h_{L_2} = 2$, then $h_{L_1^+} = h_{L_2^+} = 2$.

Proof. We only need to prove the last statement of (7). If $h_K = 1$ and $h_{L_1} = h_{L_2} = 2$, then $K$ is the Hilbert class field of $L_1$ and is at the same time that of $L_2$. Hence, $K^+$ is the Hilbert class field of $L_1^+ = L_2^+$ (see [P Lemma 6.2]).

**Proposition 2.** Let $K$ be a CM-field of degree $2n$.

(1) ([LO] $h_K^+ = Q_K\omega_K/(2\pi)^n \cdot \sqrt{d_K/d_{K^+}} \cdot \text{Res}_{s=1}(\zeta_K)/\text{Res}_{s=1}(\zeta_{K^+})$

(2) ([LO2 Proposition 9]) Let $\beta_K = 1 - (2/\log d_K)$ and

$$\varepsilon_K = \max(1 - 2\pi n/\sqrt{d_K}, 2/5\exp(-2\pi n/d_K^{1/2n})).$$

If $\zeta_K(\beta_K) \leq 0$, then $\text{Res}_{s=1}(\zeta_K) \geq 2\varepsilon_K / (\varepsilon \log d_K)$.

(3) ([LO2]) There exists a computable constant $\mu_k > 0$ such that for any abelian extension $K/k$ of degree $m$ unramified at all the infinite places we have

$$\text{Res}_{s=1}(\zeta_K) \leq (\text{Res}_{s=1}(\zeta_K))^m \left( \frac{1}{2(m-1)} \log(d_K/d_{K^+}) + 2\mu_k \right)^{m-1}.$$ 

Let $C_m$ denote the cyclic group of order $m$, $D_m$ the dihedral group of order $m$, $Q_m$ the quaternion group of order $m$ and set $G_6 = \langle b, c, z | b^4 = c^2 = z^2 = 1, c^{-1}bc = bz, bz = zb, cz = cz \rangle$ and $G_7 = \langle b, c, z | b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = cz \rangle$ (in the notation of [JJ]). Throughout this paper, $N$ denotes a non-abelian
normal CM-field of degree 48. We assume that the 3-Sylow subgroup of its Galois group $\text{Gal}(N/Q)$ is normal, and we let $M$ denote the normal CM-subfield of degree 16 of $N$. According to Proposition 1, if $h_N = 1$, then $h^3_N = 1$ (moreover, either $N/M$ is ramified at least one finite place and $h_M = 1$ or $N/M$ is unramified at all places, $h_M = 3$, and $N$ is the Hilbert class field of $M$). Now, there are 26 normal CM-fields of degree 16 with relative class number one (see [LO2], [Lou3], [CK1], [PK], and Theorem 2 below). If $\text{Gal}(M/Q)$ is non-abelian, then it is equal to $Q_8 \times C_2, G_6, D_{16}, G_9, D_8 \times C_2$. For proving Theorem 1, we first prove that if $\text{Gal}(M/Q) \neq D_{16}, G_9, D_8 \times C_2$, then we can use Proposition 1 and the known solutions to various (relative) class number problems for suitable CM-subfields of $N$ to prove that $h_N > 1$. Now, assume that $\text{Gal}(M/Q) = D_{16}, G_9$, or $D_8 \times C_2$. We will show that we can find a subfield $L$ of $M^+$ such that $N/L$ is abelian and such that the use of abelian $L$-functions to factorize $\zeta_N/\zeta_L$ readily yields $(\zeta_N/\zeta_L)(s) \geq 0$ for $0 < s < 1$. Since $M$ is known, $L$ also is known, we will check that $\zeta_L(s) \leq 0$ for $0 < s < 1$ and we will therefore deduce that $\zeta_N(s) \leq 0$ for $0 < s < 1$. Using Proposition 2, we will obtain explicit lower bounds for $h_N^−$, according to which we will be able to compute explicit upper bounds on $d_N$ when $h_N = 1$ and to construct a short list of number fields $N$ containing all such $N$’s with $h_N = 1$. We will finally explain how one can use the method expounded in [Lou5] and [Lou6] to compute the relative class numbers of these finitely many CM-fields $N$ that remain, thus completing the proof of Theorem 1.

3. Case 1: $M$ is abelian

We will show the following.

**Proposition 3.** If $N$ contains an abelian number field $M$ of degree 16, then $h_N > 1$.

**Proof.** Let $K_3$ be any cubic subfield of $N$. Since $N$ is non-abelian, $K_3$ is not normal, its normal closure $K_6$ is a dihedral real sextic field, and we let $k_2$ denote the (real) quadratic subfield of $K_6$. The Galois group $\text{Gal}(M/Q)$ is isomorphic to $C_{16}, C_8 \times C_2, C_4 \times C_4, C_4 \times C_2 \times C_2$, or $C_2 \times C_2 \times C_2 \times C_2$.

(i) If $\text{Gal}(M/Q) = C_{16}$, then $\text{Gal}(N/Q)$ is isomorphic to $C_3 \times C_{16} = \langle a, b | a^3 = b^{16} = 1, b^{-1}ab = a^{-1} \rangle$, and $N$ is a compositum of $M$ and the real dihedral field of degree 6 that is fixed by $\langle b^2 \rangle$. According to [Lou4] Theorem 5 we have $h_N^− > 1$.

(ii) If $\text{Gal}(M/Q) = C_8 \times C_2$ with $\text{Gal}(M^+/Q) = C_8$, then $\text{Gal}(N/Q) = \langle a, b, c | a^3 = b^8 = c^2 = 1, b^{-1}ab = a^{-1}, ac = ca, bc = cb \rangle$ with $\text{Gal}(N/N^+) = \langle b^2c \rangle$. The subfield $K_{12}$ fixed by $\langle b^2c \rangle$ is a normal CM-field with Galois group isomorphic to $Q_{12}$. By [LP1] $h_{K_{12}} > 4$, whence $h_N^− > 1$ by Proposition 1(3). If $\text{Gal}(M/Q) = C_8 \times C_2$ with $\text{Gal}(M^+/Q) = C_4 \times C_2$, then $h_N^− > 1$ by [CK1]. Hence $h_N^− > 1$ by Proposition 1(3).

(iii) If $\text{Gal}(M/Q) = C_4 \times C_4$, then $\text{Gal}(N/Q) = Q_{12} \times C_4$ and $\text{Gal}(N^+/Q)$ is isomorphic to either $S_3 \times C_4$ or $Q_{12} \times C_2$. Let $\psi_1$ and $\psi_2$ be two odd primitive characters of order 4 such that $M$ is associated with the group $\langle \psi_1, \psi_2 \rangle$. If $k_2$ is associated with $\langle \psi_1^2 \rangle$, then $\text{Gal}(N^+/Q) = S_3 \times C_4$. Assume that $k_2$ is associated with $\langle \psi_2^2 \rangle$. Let $M_{12,1}$ be the compositum of $K_6$ and the quartic field associated with $\langle \psi_1 \rangle$, and $M_{12,2}$ the compositum of $K_6$ and the quartic field associated with $\langle \psi_1 \psi_2 \rangle$. Then $M_{12,1}$ and $M_{12,2}$ are quaternion CM-fields.
of degree 12 with the same maximal real subfield $K_6$. According to [LP] Theorem 1, there is no pair of $(M_{12,1}, M_{12,2})$ such that $h_{M_{12,1}} = 1$, $h_{M_{12,2}} = 1$, and at the same time $M_{12,1}^+ = M_{12,2}^+$. By symmetry, if $k_2$ is associated with $\langle \psi_2^2 \rangle$, then $h_N > 1$. Assume now that $k_2$ is associated with $\langle \psi_1^2 \psi_3 \rangle$. Let $M_{24,1}$ be the compositum of $K_6$ and the imaginary cyclic quartic field associated with $\langle \psi_1 \rangle$, and $M_{24,2}$ the compositum of $K_6$ and the imaginary cyclic quartic field associated with $\langle \psi_2 \rangle$. Then $M_{24,1}$ and $M_{24,2}$ are normal CM-fields with Galois group isomorphic to $S_3 \times C_4$ which have the same maximal real subfield. Using Proposition 1(5) we verify that $h_N = h_{M_{24,1}} h_{M_{24,2}}$. By [P] Theorem 1 there is only on CM-field of relative class number one with Galois group isomorphic to $S_3 \times C_4$, whence $h_N > 1$.

(iv) If $\text{Gal}(M/Q) = C_4 \times C_2 \times C_2$ with $\text{Gal}(M^+/Q) = C_4 \times C_2$, then $\text{Gal}(N/Q)$ is isomorphic to either $Q_{12} \times C_2 \times C_2$ or $S_3 \times C_2 \times C_4$. Let $\psi$ be the odd primitive Dirichlet character of order 4, and let $\chi_1$ and $\chi_2$ be the quadratic odd characters such that $M$ is associated with the group $\langle \psi, \chi_1, \chi_2 \rangle$. If $k_2$ is associated with $\langle \psi^2 \rangle$, then the compositum $M_{12,1}$ of $K_6$ and the field associated with $\langle \psi \rangle$, and the compositum $M_{12,2}$ of $K_6$ and the field associated with $\langle \psi \chi_1 \chi_2 \rangle$ are normal CM-fields with Galois group $Q_{12}$ and $M_{12,1}^+ = M_{12,2}^+$. By [LP] Theorem 1, $h_N > 1$. If $k_2$ is associated with $\langle \psi^2 \chi_1 \chi_2 \rangle$ or $\langle \chi_1 \chi_2 \rangle$, then we let $M_{24,1}$ be the compositum of $K_6$ and the field associated with $\langle \psi \rangle$, and $M_{24,2}$ the compositum of $K_6$ and the field associated with $\langle \psi^2, \chi_1, \chi_2 \rangle$. Then $\text{Gal}(M_{24,1}/Q) = S_3 \times C_4$, $\text{Gal}(M_{24,2}/Q) = S_3 \times C_2 \times C_2$, $M_{24,1}^+ = M_{24,2}^+$, and $N = M_{24,1} M_{24,2}$. By [P] Theorem 1 $h_{M_{24,1}} > 1$ and $h_{M_{24,2}} > 1$, whence according to Proposition 1(7) we have $h_N > 1$.

(v) If $\text{Gal}(M/Q) = C_4 \times C_2 \times C_2$ with $\text{Gal}(M^+/Q) = C_2 \times C_2 \times C_2$, then $h_M > 1$ by [CK]. Hence $h_N > 1$.

(vi) If $\text{Gal}(M/Q) = C_2 \times C_2 \times C_2 \times C_2$, then $h_M > 1$ by [CK]. Hence $h_N > 1$.

4. Case 2: $\text{Gal}(M/Q) \in \{D_{16}, Q_8 \times C_2\}$

In this section we assume that $\text{Gal}(M/Q) \in \{D_{16}, Q_8 \times C_2\}$ and $h_M = 1$. We will prove that there is exactly one field $N$ with $h_N = 1$. In subsection 4.1 we assume that $G(M/Q) = D_{16}$, and in subsection 4.2 we assume that $G(M/Q) = Q_8 \times C_2$.

4.1. $G(M/Q) = D_{16}$. There are five dihedral CM-fields $M$ of degree 16 with relative class number one [LO] Theorem 10: the narrow Hilbert class fields of $Q(\sqrt{pq})$ with $(p, q) \in \{(2, 257), (5, 101), (5, 181), (13, 53), (13, 61)\}$. The narrow Hilbert class field of $Q(\sqrt{2 \cdot 257})$ has class number three and the remaining four $M$’s have class number one. We set $K = Q(\sqrt{p}, \sqrt{q})$ and $k = Q(\sqrt{pq})$. The field $M$ has three quadratic subfields $L_1$, $L_2$, and $k$ with $\text{Gal}(M/L_1) = \text{Gal}(M/L_2) = D_8$, and $\text{Gal}(M/k) = C_8$. Therefore, the Galois group $\text{Gal}(N/Q)$ is isomorphic to $D_{16} \times C_3$ if $N$ contains only one cubic cyclic subfield, $D_{48}$ or $(D_8 \times C_3)^\times = C_3 \times D_{16} = \langle a, b, c \mid a^3 = b^8 = c^2 = 1, c^{-1}bc = b^{-1}, b^{-1}ab = a^{-1}, ac = ca \rangle$, otherwise. In [LO] it is proved that if $\text{Gal}(N/Q) = D_{48}$, then $h_N > 1$. We deal with the fields $N$ with $\text{Gal}(N/Q) = D_{16} \times C_3$ and the fields $N$ with $\text{Gal}(N/Q) = (D_8 \times C_3)^\times = C_3$ in 4.1.1 and the fields $N$ with $\text{Gal}(N/Q) = (D_8 \times C_3)^\times = C_2$ in 4.1.2, respectively.
4.1.1. \( \text{Gal}(N/Q) = D_{16} \times C_4 \). Let \( K_3 \) denote the cyclic cubic subfield of \( N \). Since \( K_3/Q \) is ramified, \( N/M \) is ramified and if \( h_N = 1 \), then \( h_M = 1 \) by point (6) of Proposition 1. Therefore, \( M \) cannot be equal to the narrow Hilbert class field of \( \mathbb{Q}(\sqrt{2} \cdot 257) \). Note that \( N/k \) is cyclic of degree 24.

**Lemma 1.** Let \( \chi \) be any one of the eight characters of order 24 associated with the cyclic extension \( N/k \).

1. We have \((\zeta_N/\zeta_K)(s) \geq 0 \) in the range \( 0 < s < 1 \).
2. For each given \( M \) with \( h_M = 1 \) we can compute a bound \( N_{k/Q}(\mathfrak{f}) \leq C \) on the norms of the conductors \( \mathfrak{f} \) of the cyclic cubic extensions \( kK_3/k \) for the \( N \)’s such that \( h_N = 1 \). These bounds are listed in Table 1.
3. Assume that \( h_N = 1 \). Then \( \mathfrak{f} = (l) \) for some prime \( l \) which splits in \( K \), or \( \mathfrak{f} = \mathfrak{B}_l \) for some prime ideal \( \mathfrak{B}_l \) of \( k \) above a prime \( l \) ramified in \( k \).
4. \( h_M \) divides \( h_N \), \( L(0, \chi) \in \mathbb{Q}(\sqrt{2}, \sqrt{-3}) \), and \( h_N/h_M = N_{\mathbb{Q}(\sqrt{2} \cdot 257)}(\frac{1}{4}L(0, \chi))^2 \) is a perfect square which can be computed using the techniques developed in [Lou5] and [Lou6].

**Proof.** (1) It follows from \((\zeta_N/\zeta_K)(s) = \prod_{i=1}^{11} |L(s, \chi^i)|^2 \).

(2) We have verified that for the above four \( M \)’s, \( \zeta_K(s) \leq 0 \) in the range \( 0 < s < 1 \). Hence, \( \zeta_N(s) \leq 0 \) for \( 0 < s < 1 \). Using [Lou2] Lemma 12 and Proposition 13 we compute explicitly \( \mu_k \text{ Res}_{s=1}(\zeta_k) \) and apply Proposition 2 to get lower bound for \( h_N \). Since \( M/M^+ \) is unramified at all finite places and \( Q_M = \omega_M = 2 \), \( N/N^+ \) is unramified at all finite places, \( d_{N^+} = d_k^2 N_{k/Q}(\mathfrak{f})^8 \), and \( Q_N = \omega_N = 2 \). From this lower bound for \( h_N \) we obtain the upper bounds \( C \) on \( N_{k/Q}(\mathfrak{f}) \) such that \( h_N = 1 \) implies \( N_{k/Q}(\mathfrak{f}) \leq C \).

(3) If the number of ramified primes in \( K_3/Q \) is greater than one, then 3 divides \( h_{K_3} \), whence 3 divides \( h_{N_{K_3}} \) by Proposition 1(6). If there is a prime divisor \( l \) of \( N_{k/Q}(\mathfrak{f}) \) which is inert in \( k \), then 3\(^4 \) divides \( h_N \). Since \( M \) is the narrow Hilbert class field of \( k \), \( (l) \) splits completely in \( M/k \), whence there are at least 4 prime ideals ramified in \( N^+/M^+ \) which split at the same time in \( M/M^+ \). Hence, \( 3^4 h_N \) by [LOO] Proposition 8.

(4) According to [Lou1], the value \( L(0, \chi) \) is an algebraic integer of \( \mathbb{Q}(\zeta_{24}) \) and

\[
h_N/h_M = N_{\mathbb{Q}(\zeta_{24})/\mathbb{Q}}(\frac{1}{4}L(0, \chi)).
\]

Let \( \text{Gal}(N/Q) = \langle a, b, c | a^3 = b^8 = c^2 = 1, b^{-1}ab = a, c^{-1}ac = a, c^{-1}bc = b^{-1} \rangle \), where \( \text{Gal}(N/k) = \langle a, b \rangle \). The restriction of \( c \) to \( k \) generates \( \text{Gal}(k/Q) \) and using Artin’s reciprocity theorem we obtain that \( \chi \circ c = \chi^7 \) and

\[
\sigma_7(L(0, \chi)) = L(0, \chi^7) = L(0, \chi \circ c) = L(0, \chi).
\]

Therefore, \( L(0, \chi) \in \mathbb{Q}(\sqrt{2}, \sqrt{-3}) \), the subfield of \( \mathbb{Q}(\zeta_{24}) \) fixed by \( \sigma_7 \). It follows that

\[
h_N/h_M = (N_{\mathbb{Q}(\sqrt{2}, \sqrt{-3})/\mathbb{Q}}(\frac{1}{4}L(0, \chi))^2
\]

is a perfect square.

We verify that \( h_N > 1 \) for all fields \( N \) satisfying points (2) and (3) in Lemma 1, and sum up the computational results in Table 1.
4.1.2. \( \text{Gal}(N/\mathbb{Q}) = (D_8 \times C_3) \rtimes C_2 \). The field \( N \) has three non-normal cubic subfields. Let \( K_3 \) be any one of them, \( K_6 \) its normal closure, and \( k_2 \) the quadratic subfield of \( K_6 \). Since if \( \text{Gal}(M/k_2) = C_8 \), then \( \text{Gal}(N/\mathbb{Q}) = D_4 \times C_2 \) and \( h^- > 1 \). It follows that \( k_2 = \mathbb{Q}(\sqrt{p}) \) or \( \mathbb{Q}(\sqrt{q}) \), and \( \text{Gal}(M/k_2) = D_8 \).

**Lemma 2.**

1. We have \( (\zeta_N/\zeta_K)(s) \geq 0 \) in the range \( 0 < s < 1 \).
2. There exists some positive integer \( f \geq 1 \) such that \( \mathfrak{h}_{K_6/k_2} = (f) \). For each given \( M \) with \( h^-_M = 1 \) we can compute a bound \( f \leq C \) on the conductors \( (f) \) of the cyclic cubic extensions \( K_6/k_2 \) for the \( N \)'s such that \( h^-_N = 1 \). These bounds and the possible \( f \)'s are given in Table 2.
3. Let \( \chi \) be any one of the four characters of order 12 associated with the cyclic extension \( N/K \). Then \( h^-_M \) divides \( h^-_N, L(0, \chi) \in \mathbb{Q}, \) and \( h^-_N/h^-_M = (L(0, \chi)/16)^4 \) is a perfect fourth power which can be computed by using the techniques developed in [Lou5] and [Lou6].
Table 2.

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<td>$\mathbb{Q}(\sqrt{13}, \sqrt{61})$</td>
<td>$\mathbb{Q}(\sqrt{61})$</td>
<td>0.487910</td>
<td>95</td>
<td>13</td>
<td>$6^4$</td>
</tr>
</tbody>
</table>

Proof. (1) Since $N/K$ and $M^+/k$ are cyclic of degree 12 and 4, respectively, then as in point (1) of Lemma 1 we obtain $(\zeta_N^t/\zeta_{M^+})(s) \geq 0$ and $(\zeta_{M^+}/\zeta_K)(s) \geq 0$ for $0 < s < 1$.

(2) The first part follows from [Mar, Theorem III.1] or [LPL, Theorem 4]. For $K = \mathbb{Q}(\sqrt{2}, \sqrt{257})$ we have verified that $\zeta_K(s) \leq 0$ in the range $0 < s < 1$. Hence, $\zeta_N(s) \leq 0$ for $0 < s < 1$ for every $M$ with $h_M^- = 1$. Since $N^+/k_2$ is abelian and $d_{M^+}/d_{k_2}^2 = f^1$, using Proposition 2 we get upper bound $C$ on $f$ such that $h_N^- = 1$ implies $f \leq C$. To alleviate the list of possible conductors $f$ we use the same argument as in point (3) of Lemma 1: If there is a prime divisor $l$ of $f$ which is inert in $\mathbb{Q}(\sqrt{pq})$, then $3^l$ divides $h_N^-$. (3) Let $K_{12}$ be the compositum of $K$ and $K_6$. We have

$$h_N^-/h_M^- = N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(L(0, \chi)/2^4).$$

Assume $\text{Gal}(N/K) = \langle a, b^2 \rangle$. Let $\chi_-$ be any one of two quartic characters associated with the cyclic extension $M/K$ and $\chi_+$ any one of two cubic characters associated with the cyclic extension $K_{12}/K$ such that $\chi = \chi_+\chi_-$. Using the Artin reciprocity theorem, it can be easily verified that $\chi_- \circ b = \chi_-$, $\chi_- \circ c = \chi_-^{-1}$, $\chi_+ \circ b = \chi^{-1}_+$, and $\chi_+ \circ c = \chi_+$, whence $\chi \circ b = \chi^3$ and $\chi \circ c = \chi^7$. For a positive integer $n$ let $\sigma_n \in \text{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$ with $\sigma_n(\zeta_{12}) = \zeta_{12}^n$. We have

$$\sigma_5(L(0, \chi)) = L(0, \chi^5) = L(0, \chi \circ b) = L(0, \chi)$$

and $\sigma_7(L(0, \chi)) = L(0, \chi)$. Since $\langle \sigma_5, \sigma_7 \rangle = \text{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$ we have $L(0, \chi) \in \mathbb{Q}$, whence $h_N^-/h_M^-$ is the 4-th power of some integer.
Our computational results are given in Table 2. When $K = \mathbb{Q}(\sqrt{2}, \sqrt{257})$, if $h_N = 1$, then $N/M$, $N^+/M^+$, and $K_6/k_2$ are unramified. Otherwise, $h_N \equiv 0 \mod 3$. Since $\mathbb{Q}(\sqrt{2})$ has class number one, we must have $k_2 = \mathbb{Q}(\sqrt{257})$ and $(f) = 1$. Note that when $K = \mathbb{Q}(\sqrt{5}, \sqrt{101})$, $k_2 = \mathbb{Q}(\sqrt{101})$, and $f = 2$, we have $K_6 = \mathbb{Q}(\sqrt{101}, \theta)$ with $\theta^3 - \theta^2 - 5\theta - 1 = 0$. Using KASH ([K]) we verify that the class group of $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$ is isomorphic to $C_2$ and the narrow class group of this field is isomorphic to $C_4$. It follows that $N^+$ is the Hilbert class field of $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$ and $N$ is the narrow Hilbert class field of this field. In addition, thanks to KASH we verify that the class number of $N^+$ is equal to 1.

4.2. $\text{Gal}(M/\mathbb{Q}) = Q_8 \times C_2$. By [Lou3] Theorem 1,

$$M = \mathbb{Q} \left( \sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{-(2 + \sqrt{2})(3 + \sqrt{3})} \right)$$

is the only normal CM-field of relative class number one with Galois group isomorphic to $Q_8 \times C_2$. This field has class number one and $Q_M = 2$. In this subsection we assume that $N$ contains this field $M$ and will prove that $h_N > 1$. The Galois group $\text{Gal}(N/\mathbb{Q})$ is isomorphic to either $Q_8 \times C_2 \times C_3$ or $Q_{24} \times C_2$ according to whether $N$ has a cyclic cubic subfield or not.

4.2.1. $\text{Gal}(N/\mathbb{Q}) = Q_8 \times C_2 \times C_3$. The field $N$ has only one cyclic cubic subfield $K_3$. The composita

$$N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-(2 + \sqrt{2})(3 + \sqrt{3})}) \quad \text{and} \quad N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1})$$

have the same maximal real subfield $K_3(\sqrt{2}, \sqrt{3})$. Suppose that $h_N = 1$. By Proposition 1(7) we would have $h_{N_1} = 1$ or $h_{N_2} = 1$. Since every octic quaternion CM-field has an even relative class number, $h_{N_i}$ is even. Using [CK1] we verify that there is no imaginary abelian number field with Galois group isomorphic to $C_2 \times C_2 \times C_2 \times C_3$ of relative class number one which contains the field $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-1})$. Hence $h_N > 1$.

4.2.2. $\text{Gal}(N/\mathbb{Q}) = Q_{24} \times C_2$. The field $N$ contains a non-normal cubic subfield $K_3$. The compositum

$$N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-(2 + \sqrt{2})(3 + \sqrt{3})})$$

is a normal CM-field with Galois group isomorphic to $Q_{24}$. The compositum $N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1})$ is a normal CM-field with Galois group isomorphic to $D_{12} \times C_2$. We have $N = N_1N_2$ with $N_1^+ = N_2^+ = K_3(\sqrt{2}, \sqrt{3})$. Note that $h_{N_1}$ is even. According to [P2] Theorem 1, $h_{N_2} > 1$. By Proposition 1(7) it follows that $h_N > 1$.

5. Case 3: $\text{Gal}(M/\mathbb{Q}) \in \{G_9, G_6\}$

In subsection 5.1 we assume that $G(M/\mathbb{Q}) = G_9$, and in subsection 5.2 we assume that $G(M/\mathbb{Q}) = G_6$.

5.1. $\text{Gal}(M/\mathbb{Q}) = G_9$. There is only one normal CM-field $M$ of relative class number one with Galois group isomorphic to $G_9$ ([LO2] Theorem 20):

$$M = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{37}, \sqrt{-(2\sqrt{2} + 3\sqrt{5})(2 + \sqrt{5})})$$.
Assume that \( N \) contains \( M \). The aim of this subsection is to prove that \( h_N > 1 \). Note that \( \text{Gal}(M/Q(\sqrt{2}, \sqrt{5}, 37)) = Q_8 \), \( \text{Gal}(M/Q(\sqrt{2})) = \text{Gal}(M/Q(\sqrt{5})) = \text{Gal}(M/Q(\sqrt{37})) = \mathcal{D}_8 \), and \( \text{Gal}(M/Q(\sqrt{2}, 5)) = \text{Gal}(M/Q(\sqrt{2}, 37)) = \mathcal{D}_8 \). Therefore, \( \text{Gal}(N/Q) \) is isomorphic to \( G_0 \times C_3 \) if \( N \) contains only one cubic cyclic subfield \( K_3 \), \((Q_8 \times C_3) \times C_2, (\mathcal{D}_8 \times C_2) \times C_2 = \langle a, b, c \mid a^3 = b^2 = c^2 = z^4 = 1, c^{-1}bc = b^2, bz = zb, cz = zc, b^{-1}ab = a, c^{-1}ac = a, z^{-1}az = a^{-1} \rangle, \) or \((C_4 \times C_2 \times C_3) \times C_2 \) otherwise. We divide this subsection into four parts according to \( \text{Gal}(N/Q) \).

5.1.1. \( \text{Gal}(N/Q) = G_9 \times C_3 \). We will show that \( h_N > 1 \). We first get an upper bound on \( C \) for the conductor of \( K_3/Q \) such that \( h_N = 1 \), then the conductor is less than or equal to \( C \). Let \( K = Q(\sqrt{2}, \sqrt{5}, 37) \). Since \( N/K \) is cyclic of degree 12, then as in point (1) of Lemma 1 we obtain \( (\zeta_N^s/\zeta_{N^+}) < 0 \) for \( 0 < s < 1 \). We verify that \( \zeta_{N^+}(s) \leq 0 \) for \( s \in ]0, 1[ \), which implies \( \zeta_N(s) \leq 0 \) for \( s \in ]0, 1[ \). Let \( F = Q(\sqrt{2}, \sqrt{5}, 37) \). The extension \( N^+/F \) is abelian of degree 12 and \( d_{N^+/F} = N_{F/Q}(\mathfrak{S}_{FK_3/F})^8 \). Using \( \mu_F \text{Res}_{s=1}(\zeta_F) \leq 2.227842 \) and Proposition 2, we obtain that if \( h^{-}_N \leq 1 \), then \( N_{F/Q}(\mathfrak{S}_{FK_3/F}) \leq 1300 \), whence \( \mathfrak{S}_{FK_3/F} \in \{ (7), (3^2), (13), (19), (31), 2^3 \} \) with \((37) = Q_{37}^2 \) in \( F \).

(i) When \( \mathfrak{S}_{FK_3/F} = (7) \), \( K_3 \) is of conductor 7. There are four non-normal octic CM-fields containing \( \mathbb{Q}(\sqrt{2}) \) with relative class number one. Let \( M_{8,1} = \mathbb{Q}(\sqrt{2}, \sqrt{5}, 37) \). The prime ideals 37 splits in \( M_{8,1} \), and 2 splits in \( M_8 \). The two prime ideals lying above 7 in \( M_{8,1} = \mathbb{Q}(\sqrt{2}, \sqrt{5}, 37) \) split in \( M_{8,1} \). Hence, \( 2^2|h_N^{-1} \).

(ii) When \( \mathfrak{S}_{FK_3/F} = (3^2) \), \( K_3 \) is of conductor 9. Let \( M_{8,1}, M_{8,2}, N_{24,1}, \) and \( N_{24,2} \) be as in (i). One prime ideal lying above 37 in \( \mathbb{Q}(\sqrt{2}, \sqrt{5}) \) is ramified in \( M_{8,1} \) and 37 splits in \( K_3 \), whence \( 2^2|h_N^{-1} \).

(iii) When \( \mathfrak{S}_{FK_3/F} = (13) \), \( K_3 \) is of conductor 13. Let \( M_{8,1} \) be any one of two non-normal octic CM-fields of \( M \) containing \( \mathbb{Q}(\sqrt{2}, \sqrt{5}, 37) \) and \( M_{8,2} \) its conjugate over \( \mathbb{Q} \). Let \( N_{24,1} = M_{8,1}K_3 \) and \( N_{24,2} = M_{8,2}K_3 \). One prime ideal lying above 5 in \( \mathbb{Q}(\sqrt{2}, \sqrt{5}, 37) \) is ramified in \( M_{8,1} \) and 5 splits in \( K_3 \), which implies \( 2^2|h_N^{-1} \).

(iv) When \( \mathfrak{S}_{FK_3/F} = (31) \), \( K_3 \) is of conductor 31. We let \( M_{8,1} \) be any one of two non-normal octic subfields of \( M \) containing \( \mathbb{Q}(\sqrt{5}, \sqrt{37}) \) and \( M_{8,2} \) its conjugate over \( \mathbb{Q} \). One prime ideal lying above 2 is ramified in \( M_{8,1} \) and 2 splits in \( K_3 \), whence \( 2^4|h_N^{-1} \).

(v) When \( \mathfrak{S}_{FK_3/F} = \mathfrak{S}_{37} \), \( K_3 \) is of conductor 37. According to \( \mathfrak{Ma} \), the cyclic sextic subfield \( K_3(\sqrt{2}, \sqrt{5}) \) of \( N \) has class number 6, whence by Proposition 1(6) \( 3|h_N \).

5.1.2. \( \text{Gal}(N/Q) = (D_8 \times C_3) \times C_2 \). Let \( K_3 \) be any one of three non-normal cubic subfields of \( N \), \( K_6 \) its normal closure, and \( k_2 \) the quadratic subfield of \( K_6 \). We have \( \text{Gal}(M/k_2) = D_8 \). Let \( B \) be the intermediate field between \( M^+/k_2 \) such that \( G(M/K) = C_4 \). Then \( M/K \) is unramified at all finite primes, \( \text{Gal}(M^+/k_2) = C_2 \times C_2 \), \( \text{Gal}(N/K) = C_{12} \), and \( \text{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3 \). As in point (1) of
Lemma 1 we obtain \((\zeta_N/\zeta_{M^+})(s) \geq 0\) and \(\zeta_N(s) \leq 0\) for \(s \in ]0,1[\). Since \(N^+/k_2\) is abelian of degree 12 and \(d_{N^+/k_2}^{12} = f^{16}N_{K_6/K_0}(\mathfrak{D}_{N^+/K_6})\), where \((f) = \mathfrak{f}_{K_6/k_2}\) and \(\mathfrak{D}_{N^+/K_6}\) denotes the discriminant of the extension \(N^+/K_6\), using Proposition 2 we obtain upper bound \(C\) on \(f\) such that if \(h_N^2 = 1\), then \(f \leq C\). Our computational results are given in Table 3. As in point (3) of Lemma 2 we can easily verify that \(h_N^2/h_M^2\) is the 4-th power of some rational integer.

Table 3.

<table>
<thead>
<tr>
<th>(k)</th>
<th>(f)</th>
<th>(f)</th>
<th>(h_N^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{Q}(\sqrt{2}))</td>
<td>22</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{5}))</td>
<td>8</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{37}))</td>
<td>44</td>
<td>2</td>
<td>(2^4)</td>
</tr>
</tbody>
</table>

Table 4.

<table>
<thead>
<tr>
<th>(k_2)</th>
<th>(C)</th>
<th>(f)</th>
<th>(h_N^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{Q}(\sqrt{2-5}))</td>
<td>140</td>
<td>37</td>
<td>1021(^2)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{2 \cdot 37}))</td>
<td>21</td>
<td>9</td>
<td>5044(^2)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{5 \cdot 37}))</td>
<td>35</td>
<td>None</td>
<td></td>
</tr>
</tbody>
</table>

5.1.3. \(\text{Gal}(N/\mathbb{Q}) = (C_4 \times C_2 \times C_3) \rtimes C_2 = \langle a, b, c, z \mid a^3 = b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = zc, b^{-1}ab = a, c^{-1}ac = a^{-1}, z^{-1}az = a \rangle\). In this case \(N\) has three non-normal cubic subfields and its associated quadratic subfield \(k_2\) has \(\text{Gal}(M/k_2) = C_4 \times C_2\). For a fixed field \(k_2\) there are two intermediate fields \(K\) between \(k_2\) and \(M^+\) such that \(\text{Gal}(M/K) = C_4\). Let \(K\) be any one of these two fields. Then \(M/K\) is unramified at all finite primes, \(\text{Gal}(N/K) = C_{12}\), \(\text{Gal}(M^+/k_2) = C_2 \times C_2\), and \(\text{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3\). Analogously as in subsection 5.1.2 we get upper bound \(C\) on \(f\) such that if \(h_N^2 = 1\), then \(f \leq C\). Since \(\text{Gal}(N/k_2) = C_4 \times C_2 \times C_3\), we compute \(h_N^2\) using Hecke \(L\)-functions over \(k_2\) (see Table 4).

5.1.4. \(\text{Gal}(N/\mathbb{Q}) = (Q_8 \times C_3) \rtimes C_3\). In this case \(N\) has three non-normal cubic subfields such that its associated quadratic subfield \(k_2\) is equal to \(\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})\). Let \(K\) be any one of three quartic fields containing \(\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})\). Then \(\text{Gal}(N/K) = C_{12}\) and \(\text{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3\). As in subsection 5.1.2 we verify that if \(h_N^2 = 1\), then \(f \leq 36\) with \(\mathfrak{f}_{K_6/k_2} = (f)\). There is no sextic field \(K_6\) containing \(\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})\) with \(f \leq 36\). Therefore, \(h_N^2 > 1\).

5.2. \(\text{Gal}(M/\mathbb{Q}) = G_6\). In \([\text{Lou3}]\) it is proved that there are exactly two such fields \(M\) with \(h_M = 1\): the composita \(M = M_1M_2\) listed in Table 5. In fact, those are the only fields with Galois group isomorphic to \(G_6\) of relative class number one. The Galois group \(\text{Gal}(M^+/\mathbb{Q})\) is isomorphic to \(D_8\) or \(C_4 \times C_2\). When \(\text{Gal}(M^+/\mathbb{Q}) = D_8\), \(M\) is a compositum of an octic dihedral CM-field \(M_{8,1}\) and an imaginary abelian number field \(M_{8,2}\) with \(\text{Gal}(M_{8,2}/\mathbb{Q}) = C_4 \times C_2\). Using [YK] and [CK1], we verify...
that there is only one such field $M$ with $h_M^- = 1$: the second field in Table 5. When $\text{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$, $M$ is a compositum of two octic dihedral CM-fields $M_{8,1}$ and $M_{8,2}$ with $M_{8,1}^+ = M_{8,2}^+$. According to [Lou3, Theorem 2] there is only one such field $M$ with $h_M^- = 1$: the first field in Table 5. We will prove that $h_N > 1$. If $K_3/\mathbb{Q}$ is normal, then $\text{Gal}(N/\mathbb{Q}) = G_6 \times C_3$. Otherwise, $\text{Gal}(N/\mathbb{Q}) = C_3 \times G_6$. Let $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$. Then $N = N_1N_2$ with $N_1^+ = N_2^+ = K_3K_4$.

5.2.1. $\text{Gal}(N/\mathbb{Q}) = G_6 \times C_3$. If $\text{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$, then $\text{Gal}(N_1/\mathbb{Q}) = D_8 \times C_3 = \text{Gal}(N_2/\mathbb{Q})$. According to [P] Theorem 1, $h_{N_1}^- > 1$ and $h_{N_2}^- > 1$, whence $h_N > 1$. If $\text{Gal}(M^+/\mathbb{Q}) = D_8$, then $\text{Gal}(N_1/\mathbb{Q}) = D_8 \times C_3$, and $\text{Gal}(N_2/\mathbb{Q}) = C_4 \times C_2 \times C_3$. Using [CK1], we verify that $h_{N_2}^- > 4$, whence $h_N > 1$.

5.2.2. $\text{Gal}(N/\mathbb{Q}) = C_3 \times G_6$.

(a) Let $M$ be the first field in Table 5. The quadratic field $k_2$ associated with $K_3$ is either $\mathbb{Q}(\sqrt{34})$, $\mathbb{Q}(\sqrt{2})$, or $\mathbb{Q}(\sqrt{17})$. If $k_2 = \mathbb{Q}(\sqrt{34})$, then $\text{Gal}(N_1/\mathbb{Q}) = D_{24}$, and $\text{Gal}(N_2/\mathbb{Q}) = C_3 \times D_8$. If $k_2 = \mathbb{Q}(\sqrt{2})$, then $\text{Gal}(N_1/\mathbb{Q}) = C_3 \times D_8$, and $\text{Gal}(N_2/\mathbb{Q}) = D_{24}$. If $k_2 = \mathbb{Q}(\sqrt{17})$, then $\text{Gal}(N_1/\mathbb{Q}) = C_3 \times D_8 = \text{Gal}(N_2/\mathbb{Q})$. According to [P] Theorem 1 and [CK1] Theorem 4.1, we have $h_{N_1}^- > 1$ and $h_{N_2}^- > 1$, whence $h_N > 1$.

(b) Let $M$ be the second field in Table 5. The quadratic field $k_2$ associated with $K_3$ is either $\mathbb{Q}(\sqrt{221})$, $\mathbb{Q}(\sqrt{173})$, or $\mathbb{Q}(\sqrt{17})$. If $k_2 = \mathbb{Q}(\sqrt{221})$, then $\text{Gal}(N_1/\mathbb{Q}) = D_{24}$, and $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_4$. If $k_2 = \mathbb{Q}(\sqrt{173})$, then $\text{Gal}(N_1/\mathbb{Q}) = C_3 \times D_8$ and $\text{Gal}(N_2/\mathbb{Q}) = Q_{12} \times C_2$. If $k_2 = \mathbb{Q}(\sqrt{17})$, then $\text{Gal}(N_1/\mathbb{Q}) = C_3 \times D_8$, and $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_4$. Using [CK1] Theorem 4.1] and [P] Theorem 1], we have $h_{N_1}^- > 1$ and $h_{N_2}^- > 1$, whence $h_N > 1$.

6. Case 4: $\text{Gal}(M/\mathbb{Q}) = D_8 \times C_2$

To begin with, we prove the following:

**Theorem 2** (Compare with [Lou3] Theorems 2 and 3]). There are four normal CM-fields $M$ of degree 16 and Galois group $D_8 \times C_2$ with relative class number one, those given in the following Table 6.
Table 6.

<table>
<thead>
<tr>
<th>$M^+$</th>
<th>$\alpha: M=M^+ (\sqrt{-\alpha})$</th>
<th>$Q_M$</th>
<th>$\omega_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17})$</td>
<td>$5(5 + \sqrt{17})/2$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{3+\sqrt{3}})$</td>
<td>1</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{3}, \sqrt{11}, \sqrt{15 + 8\sqrt{3}})$</td>
<td>1</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{3(5 + \sqrt{17})/2})$</td>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

Proof. The Galois group $\text{Gal}(M^+/\mathbb{Q})$ is isomorphic to either $D_8$ or $C_2 \times C_2 \times C_2$. When $\text{Gal}(M^+/\mathbb{Q}) = D_8$, $M$ is a compositum of an octic dihedral CM-field $M_{8,1}$ and an imaginary abelian number field $M_{8,2}$ with $\text{Gal}(M_{8,2}/\mathbb{Q}) = C_2 \times C_2 \times C_2$. Let $L$ be any one of four non-normal quartic CM-subfields of $M_{8,1}$. According to Proposition 16, $h^{-1}_M = 1$ if and only if $(h^{-1}_L, h^{-1}_{M_{8,2}}) \in \{(1, 4), (1, 2), (2, 1)\}$. In the case that $(h^{-1}_L, h^{-1}_{M_{8,2}}) = (1, 4)$ or $(1, 2)$, $M_{8,1}^+ = M_{8,2}^+$ is of the form $\mathbb{Q}(\sqrt{r}, \sqrt{t})$, where $(p, q)$ is one of the 19 pairs given in [LO1, Theorem 8]. Then $M_{8,2} = \mathbb{Q}(\sqrt{r}, \sqrt{t}, \sqrt{-m})$, where $\mathbb{Q}(\sqrt{t})$ is one of 81 (= 9 + 18 + 54 ) imaginary quadratic fields of class number one, two or four. For these 1539 ( = 19 x 81 ) CM-fields $M_{8,2}$ we compute $h^{-1}_{M_{8,2}}$ and verify that there is only one field $M$ with $h^{-1}_M = 1$: the fourth field in Table 6. In the case that $(h^{-1}_L, h^{-1}_{M_{8,2}}) = (2, 1)$, using [YK] and [CK1], we verify that there are exactly two fields $M$ with $h^{-1}_M = 1$: the second and third fields in Table 6. When $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, $M$ is a compositum of two octic dihedral CM-fields $M_{8,1}$ and $M_{8,2}$. According to [LO1, Theorem 2], there is only one such $M$ with $h^{-1}_M = 1$: the first field in Table 6.

From now on we assume that $M$ is one of these four fields and we will prove that $h_N > 1$. We classify the Galois group $\text{Gal}(N/\mathbb{Q})$. Let $K_3$ be any cubic subfield of $N$. If $K_3$ is not normal, then its normal closure $K_6$ is a dihedral real sextic field and we let $k_2$ denote the (real) quadratic subfield of $K_6$. If $K_3$ is normal, then $\text{Gal}(N/\mathbb{Q}) = D_8 \times C_6$. If $K_3$ is not normal over $Q$, then $\text{Gal}(N/\mathbb{Q})$ is isomorphic to either $D_8 \times S_3$, $D_8 \times C_2$, or $(C_3 \times D_8) \times C_2$. Let $K_4 = M_{8,1}^+ = M_{8,2}^+$. If $K_4 \cap K_2 = \mathbb{Q}$, then $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, and $\text{Gal}(N/\mathbb{Q}) = D_8 \times S_3$. If $K_4 \cap K_2 \nsubseteq \mathbb{Q}$, then $M_{8,1,2}$ is either cyclic or biquadratic bicyclic. If $M_{8,1,2}$ is cyclic, then $\text{Gal}(N/\mathbb{Q}) = D_{24} \times C_2$. If $M_{8,1,2}$ is biquadratic bicyclic, then $\text{Gal}(K_3 M_{8,1}/\mathbb{Q}) = C_3 \times D_8 = \langle a, b, c | a^3 = b^4 = c^2 = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = b^{-1} \rangle$, and $\text{Gal}(N/\mathbb{Q}) = (C_3 \times D_8) \times C_2$.

6.1. $\text{Gal}(N/\mathbb{Q}) = D_8 \times C_6$. If $\text{Gal}(M^+/\mathbb{Q}) = D_8$, then the compositum $N_1 = M_{8,1} K_3$ and $N_2 = M_{8,2} K_3$ are normal CM-subfields of $N$ with the same maximal real subfields $K_3$ and $K_4$. By [P1, Theorem 1], the compositum of the dihedral octic CM-field $\mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-9 + \sqrt{13}}/2)$ and of the cyclic cubic field of conductor 13 is the only normal CM-field of relative class number one with Galois group isomorphic to $D_8 \times C_3$. In Table 6 there is no field $M$ containing $\mathbb{Q}(\sqrt{13}, \sqrt{17})$. It follows then that $h_{N_1} > 1$. According to [CK1], there are exactly two imaginary abelian number fields with Galois group isomorphic to $C_2 \times C_2 \times C_2 \times C_2 \times C_2$ of relative class number one: $F_7(\sqrt{-1}, \sqrt{-3}, \sqrt{-7})$ and $F_7(\sqrt{-3}, \sqrt{-7}, \sqrt{-13})$, where $F_7$ denotes the cyclic
cubic field of conductor 7. In Table 6 there is no field $M$ containing $\mathbb{Q}(\sqrt{3}, \sqrt{7})$, or $\mathbb{Q}(\sqrt{5}, \sqrt{21})$, whence $h^+_{N_2} > 1$. It follows that if $\text{Gal}(M^+/\mathbb{Q}) = D_8$, then $h^+_{N_2} > 1$. If $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, then $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$ are normal CM-fields with Galois group isomorphic to $D_8 \times C_3$. Note that $M_{8,1}^+ = M_{8,2}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{17})$. According to [P] Theorem 1, $h^+_{N_1} > 1$ and $h^+_{N_2} > 1$, which implies $h^+_{N} > 1$.

6.2. $\text{Gal}(N/\mathbb{Q}) = D_8 \times S_3$. In this case $N$ has three non-normal real cubic subfields. Let $K_3$, $K_6$, $K_2$ and $K_4$ be as above. We have that $K_4 \cap K_2 = \mathbb{Q}$, $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, $M$ is the first field in Table 6. In addition, we have $K_4 = \mathbb{Q}(\sqrt{2}, \sqrt{17})$, and $K_2 = \mathbb{Q}(\sqrt{m})$ with $m \in \{5, 2 \cdot 5, 5 \cdot 17, 2 \cdot 5 \cdot 17\}$. Let $(f)$ be the conductor of the extension $K_6/k_2$ with $f$ a positive integer.

**Lemma 3.**  
1. We have $\zeta_N(s) \leq 0$ for $0 < s < 1$.
2. For each given $k_2$ in the above we can compute a bound of $f \leq C$ on the conductor $(f)$ for $N$’s such that $h^+_{N} = 1$. These bounds and the possible $f$’s are compiled in Table 7.
3. The quotient $h^+_{N}/h^+_{M}$ is the perfect fourth power of some rational integer.

**Proof.** (1) Let $\chi_{N/M^+}$ be any one of two characters associated with the cyclic sextic extension $N/M^+$. We have

$$
\frac{\zeta_N(s)}{\zeta_{M^+}(s)} = \frac{\zeta_M(s)}{\zeta_{M^+}(s)} L(s, \chi_{N/M^+}) L(s, \chi_{N/M^+}^2)^2
$$

and

$$
\frac{\zeta_M(s)}{\zeta_{M^+}(s)} = \frac{\zeta_{M_1}(s)}{\zeta_{K_1}(s)} \frac{\zeta_{M_2}(s)}{\zeta_{K_2}(s)} = L(s, \psi_1)^2 L(s, \psi_2)^2,
$$

where $\psi_i$ is the unique irreducible character of degree 2 of $\text{Gal}(M_{8,i}/\mathbb{Q})$ the dihedral group of order 8, and $L(s, \psi_i)$ denotes the Artin $L$-function associated with $\psi_i$ for $i = 1, 2$. Since $\psi_i$ is real valued, $L(s, \psi_i)$ is on the real axis and $L(s, \psi_i)^2 \geq 0$. For $M^+ = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17})$ we have verified that $\zeta_{M^+}(s) \leq 0$ for $s \in [0, 1]$, whence $\zeta_N(s) \leq 0$.

(2) Since $M/M^+$ is unramified at all finite primes, $N/N^+$ is unramified at all finite primes and $d_N/d_{N^+} = d_{N^+} = d_{K_6}^2 f^{16} N_{K_6/\mathbb{Q}}(\mathfrak{o}_{N^+/K_6})$. Using Proposition 2, we get an upper bound on $f$. Since the prime ideals lying above 2 and those above 17 split in $M/M^+$, if $(f, 2) > 1$ or $(f, 17) > 1$, then $3^2$ divides $h^+_{N}$ by [LOO] Proposition 8. Note that the prime ideals lying above 13 split in $M/M^+$, whence the relative class number of the fourth field $N$ in Table 7 is divisible by $3^4$.

**Table 7.**

<table>
<thead>
<tr>
<th>$k_2$</th>
<th>$\mu_{k_2}$</th>
<th>$\text{Res}<em>{s=1}(\zeta</em>{k_2}) \leq f \leq f$</th>
<th>$h^+_{N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}(\sqrt{5})$</td>
<td>0.0436324</td>
<td>10</td>
<td>NONE</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2} \cdot 5)$</td>
<td>0.4276490</td>
<td>41</td>
<td>37</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{5} \cdot 17)$</td>
<td>0.5861712</td>
<td>31</td>
<td>9</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2} \cdot 5 \cdot 17)$</td>
<td>1.4062136</td>
<td>38</td>
<td>13</td>
</tr>
</tbody>
</table>
(3) Note that $M_{8,1}$ and $M_{8,2}$ are cyclic over $\mathbb{Q}(\sqrt{34})$. Let $K$ be the compositum of $\mathbb{Q}(\sqrt{34})$ and $k_2$. Then Gal($N/K$) = $C_{12}$. Let $\chi$ be any one of the four characters of order 12 associated with the cyclic extension $N/K$. Similarly to point (3) of Lemma 2, we verify that $L(0, \chi) \in \mathbb{Q}$ and $h_N/h_M = (L(0, \chi)/2)^4 \sqrt{34}$.

In conclusion, we have proved that every normal CM-field with Galois group isomorphic to $D_8 \times S_3$ has class number greater than one. Our computational results are given in Table 7.

6.3. Gal($N/\mathbb{Q}$) = $D_{24} \times C_2$. In this case $N$ has three non-normal cubic fields and $M_{8,1}/k_2$ is cyclic. Let $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$. Then we have $N = N_1N_2$ with $N_1^2 = N_2^2 = K_3K_4$. If Gal($M^+/\mathbb{Q}$) = $D_8$, then Gal($N_1/\mathbb{Q}$) = $D_{24}$, and Gal($N_2/\mathbb{Q}$) = $S_3 \times C_2 \times C_2$. If Gal($M^+/\mathbb{Q}$) = $C_2 \times C_2 \times C_2$, then Gal($N_1/\mathbb{Q}$) = $D_{24} = $ Gal($N_2/\mathbb{Q}$). Using [Lef, Theorem 4.1] and [P] Theorem 1], we verify that in both cases $h_{N_1} > 1$ and $h_{N_2} > 1$. It follows that the class number of a normal CM-field with Galois group isomorphic to $D_{24} \times C_2$ is greater than one.

6.4. Gal($N/\mathbb{Q}$) = $(C_3 \times D_8) \times C_2$. In this case $N$ has three non-normal cubic fields and $M_{8,1}/k_2$ is biquadratic bicyclic. Then the Galois group of the compositum $N_1 = M_{8,1}K_3$ over $\mathbb{Q}$ is isomorphic to $C_3 \times D_8$, whence $h_{N_1} > 1$ ([P] Theorem 13]). If Gal($M^+/\mathbb{Q}$) = $D_8$, then Gal($N_2/\mathbb{Q}$) = $S_3 \times C_2 \times C_2$. If Gal($M^+/\mathbb{Q}$) = $C_2 \times C_2 \times C_2$, then Gal($N_2/\mathbb{Q}$) = $C_3 \times D_8$. By [P] Theorems 1 and 13] $h_{N_2} > 1$. Consequently, if Gal($N/\mathbb{Q}$) = $(C_3 \times D_8) \times C_2$, then $h_N > 1$.

To conclude, Theorem 1 is now proved with completion.

All computations were carried out using Pari-Gp ([Pa]) and KASH ([K]).

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