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FINITE ELEMENT APPROXIMATION OF H-SURFACES

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Abstract. In this paper a piecewise linear finite element approximation of H-surfaces, or surfaces with constant mean curvature, spanned by a given Jordan curve in $\mathbf{R}^3$ is considered. It is proved that the finite element H-surfaces converge to the exact H-surfaces under the condition that the Jordan curve is rectifiable. Several numerical examples are given.

1. Introduction

The purpose of the present paper is to study numerical methods for surfaces with constant mean curvature, sometimes called *H-surfaces*. In a series of papers [12], [13], [14] (also see [15]), the third author proposed the use of a finite element method to realize minimal surfaces parametrically. Here we show that the method is effective for H-surfaces as well.

Let $\Gamma \subset \mathbf{R}^3$ be a rectifiable Jordan curve, $B = \{(u,v) \mid u^2+v^2<1\}$, and $H>0$ a constant. The surface $\mathcal{M}$ is parametrized by $x : B \to \mathbf{R}^3$, $\mathcal{M} = \{x(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v)) \bigm| (u,v) \in B\}$, satisfying the following conditions:

$$
\begin{array}{ll}
\Delta x = 2H x_u \times x_v & \text{in } B, \\
|x_u|^2 = |x_v|^2,\ x_u \cdot x_v = 0 & \text{in } B, \\
x|_{\partial B} : \partial B \to \Gamma & : \text{ topologically onto},
\end{array}
\tag{1.1}
$$

where $\Delta x = (\Delta x^1, \Delta x^2, \Delta x^3)$ and x_u, x_v are componentwise partial derivatives with respect to u, v, respectively. Here and henceforth, $\times$ and $\cdot$ denote the wedge and the inner products in $\mathbf{R}^3$, respectively. The second condition indicates that x is an isothermal coordinate on $\mathcal{M}$, and therefore, the first equality says that the mean curvature of $\mathcal{M}$ is H everywhere. Finally, the third requirement means that $x|_{\partial B}(\partial B) = \Gamma$ and $(x|_{\partial B})^{-1}(p)$ is connected for any $p \in \Gamma$. Therefore, $\partial\mathcal{M} = \Gamma$.

Letting $R := \operatorname{diam}(\Gamma)/2$, Hildebrandt [9] and Brezis and Coron [1] proved the existence of the first and the second solutions for (1.1) in the cases of $HR \le 1$ and $HR < 1$, respectively. Here, we construct finite element approximations of the first solution, and show their H^1 convergence in the case of $HR < 1$.

Original surfaces are obtained by the method of variation. We suppose that Γ is parametrized by a continuous bijective map $\alpha \in H^{1/2} \cap C(\partial B, \mathbf{R}^3)$. We may suppose without loss of generality that $R = \max_{\partial B} |\alpha|$. Then, x solves (1.1) if and

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only if it is a critical point of E on $H^1(B, \mathbf{R}^3)$, where

$$E(x) := \int_B \left\{ |\nabla x|^2 + \frac{4}{3} H x \cdot (x_u \times x_v) \right\}.$$

We say that a continuous map $\eta : \partial B \to \partial B$ is *nondecreasing* if there is a continuous nondecreasing function $f : [0, 2\pi] \to \mathbf{R}$ such that

$$f(2\pi) - f(0) = 2\pi \text{ and } \eta(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1} f(\theta)} \text{ for all } \theta \in [0, 2\pi].$$

Let

$$\mathcal{E} := \left\{ \gamma \;\middle|\; \begin{array}{l} \gamma \in H^{1/2} \cap C(\partial B, \mathbf{R}^3), \quad \gamma(\partial B) = \Gamma, \\ \alpha^{-1} \circ \gamma : \text{nondecreasing}, \\ e^{\sqrt{-1}\theta} = (\alpha^{-1} \circ \gamma)(e^{\sqrt{-1}\theta}) \text{ with } \theta = 0, \ \pm(2\pi)/3 \end{array} \right\}.$$

The last requirement on $\alpha^{-1} \circ \gamma$ is called the *three point condition*. Any solution x of (1.1) can assume it by a conformal transformation on B. Combined with the topological onto-ness, on the other hand, it enables us to apply the well-known Courant-Lebesgue lemma [1, Lemma 9]. (For the details of this lemma and its proof, see [4] and [5, Section 4.4].)

Lemma 1.1 (Courant-Lebesgue Lemma)**.** *Let $\{\gamma_n\}$ be a sequence in $\mathcal{E}$ such that $\|\gamma_n\|_{H^{1/2}}$ remains bounded. Then, there exist a subsequence γ_{n_i} and some $\gamma \in \mathcal{E}$ such that*

$$\|\gamma_{n_i} - \gamma\|_{L^\infty} \to 0.$$

Obviously, $\alpha \in \mathcal{E} \neq \emptyset$. We take $R' > R$ with $HR' < 1$ and set

$$X := \left\{ x \in H^1(B, \mathbf{R}^3) \;\middle|\; \|x\|_{L^\infty} \leq R', \ x|_{\partial B} \in \mathcal{E} \right\}.$$

Then, we can show (see [9] and [1]) that $\inf_X E$ is attained by some $x \in X$, and that $\|x\|_{L^\infty} \leq R$ follows from the maximum principle. Hence x becomes a critical point of E. One can also show that x is analytic and continuous in B and on $\overline{B}$, respectively, and $x : B \to \mathbf{R}^3$ is regarded as a parameterization of an H-surface $\mathcal{M}$ satisfying $\partial \mathcal{M} = \Gamma$. Henceforth, we call a minimizer $x \in X$ of $\inf_X E$ the *Hildebrandt solution*.

We recall the following argument of [9] and [1]. Let $\{x_n \in X\}$ be a minimizing sequence of E, that is, $E(x_n) \to \inf_X E$ as $n \to \infty$. We replace $\{x_n\}$ by the solutions of Dirichlet problems to obtain a Hildebrandt solution. From [9, Theorem 1], Lemma 1, Lemma 2, and Remark 4 of [1] we know the following lemma holds:

Lemma 1.2. *Let $\gamma \in H^{1/2} \cap L^\infty(\partial B, \mathbf{R}^3)$ and $R := \|\gamma\|_{L^\infty(\partial B)}$. Fix $R' > R$ such that $HR' < 1$ and set*

$$K_\gamma := \left\{ y \in H^1(B, \mathbf{R}^3) \;\middle|\; y = \gamma \text{ on } \partial B \text{ and } \|y\|_{L^\infty} \leq R' \right\}. \tag{1.2}$$

Then, there exists a unique $\tilde{x} \in K_\gamma$ such that $E(\tilde{x}) = \inf_{y \in K_\gamma} E(y)$. Moreover, we have $\|\tilde{x}\|_{L^\infty(\partial B)} \leq R$.

Therefore, letting $\gamma_n := x_n|_{\partial B}$ and defining K_{γ_n} with $\gamma = \gamma_n$ in (1.2), we see that $\inf_{K_{\gamma_n}} E$ is attained by a unique $\tilde{x}_n \in K_{\gamma_n}$. Passing to a subsequence, to $\{\tilde{x}_n\}$ converges weakly to some x in $H^1(B, \mathbf{R}^3)$. We have the following lower semicontinuity on $\{x_n\}$ (see [9, Lemma 1]).

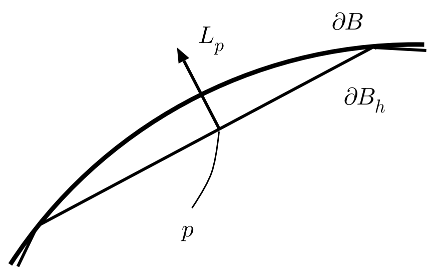

FIGURE 1.1. Extending x_h to $B\backslash B_h$

Lemma 1.3. *Let* $Y_R := \{ y \in H^1(B, \mathbf{R}^3) \mid \|y\|_{L^\infty} \le R \}$ *with a positive constant* R. *Let* $\{x_n \in Y_R\}$ *be such that* $x_n \rightharpoonup x$ *weakly in* $H^1(B, \mathbf{R}^3)$ *with some* $x \in H^1(B, \mathbf{R}^3)$. *Then, provided that* $HR < 3/2$*, we have*

$$E(x) \le \liminf_{n\to\infty} E(x_n).$$

Hence $E(x) \le \inf_X E$ follows from Lemma 1.3. We also have the uniform convergence of $\{\gamma_n = \tilde{x}_n|_{\partial B}\}$ to $x|_{\partial B}$ by the Courant-Lebesgue lemma, passing to a subsequence again. The condition $x|_{\partial B} \in \mathcal{E}$ follows from the argument of [8], and we also have $\|x\|_{L^\infty} \le R$. Therefore, x is a Hildebrandt solution.

Here, from [1, Lemma 8] and its proof, the sequence $\{\tilde{x}_n\}$ actually converges to x strongly in $H^1(B, \mathbf{R}^3)$. We shall make use of this argument later.

Now we proceed to the finite element approximation. We refer to Ciarlet [2], [3] for the basic notions of that method.

Take a family $\{\tau_h\}$ of regular and quasi-uniform triangulations of B with the size parameter $h > 0$. We impose that the three points $e^{\sqrt{-1}\theta}$ on ∂B with $\theta = 0, \pm(2\pi)/3$ are always nodal points of τ_h. Let $B_h := \left(\bigcup_{T\in\tau_h} \overline{T}\right)^\circ$. Obviously $B_h \subset B$. Let x_h be a piecewise linear continuous function defined on B_h. We extend the value of x_h to $B\backslash B_h$ in the following way. Let $p \in \partial B_h$ not be a nodal point of τ_h, and L_p the exterior normal segment on ∂B_h. Then, $x_h(q)$ for $q \in \overline{B} \cap L_p$ is defined by $x_h(q) := x_h(p)$ (see Figure 1.1).

We say $x_h|_{\partial B} \in \mathcal{E}_h$ if it is monotone, maps each nodal point of ∂B_h into Γ, and maps each of the three points $e^{\sqrt{-1}\theta}$ to $\alpha(e^{\sqrt{-1}\theta})$, where $\theta = 0, \pm(2\pi)/3$. Although $\Gamma_h = x_h(\partial B_h)$ may be regarded as a piecewise linear approximation of Γ, the function $x_h|_{\partial B}$ itself does not belong to $\mathcal{E}$. Thus, the Courant-Lebesgue lemma does not apply to $\{x_h\}$.

The following lemma is proved in [14] under the assumption that any $T \in \tau_h$ is an acute or right triangle. However, quasi-uniformity of $\{\tau_h\}$ can hold under that assumption, if one makes use of the general maximum principle of Schatz [11] in the proof.

Lemma 1.4. *Let* $\{x_h\}_{h>0} \subset H^1(B, \mathbf{R}^3)$ *be a bounded family of piecewise linear functions satisfying* $x_h|_{\partial B} \in \mathcal{E}_h$. *Then we have* $\{x_{h'}\} \subset \{x_h\}$ *with* $\gamma_{h'} = x_{h'}|_{\partial B}$ *converging uniformly to a topologically onto mapping* $\gamma : \partial B \to \Gamma$.

Let X_h be the set of piecewise linear functions $\{x_h\}$ satisfying $\|x_h\|_{L^\infty} \le R'$ and $x_h|_{\partial B} \in \mathcal{E}_h$. The finite element solution $x_h \in X_h$ of the equation (1.1), which we are now studying, is the stationary point of the functional E_h defined by

$$E_h(x) := \int_{B_h} \left\{ |\nabla x|^2 + \frac{4}{3} H x \cdot (x_u \times x_v) \right\}.$$

In particular, a minimizer $x_h \in X_h$ of $\inf_{X_h} E_h$ is a finite element solution which is called the *finite element Hildebrandt solution.* An elementary calculation shows that

$$E_h(x_h) = E(x_h) + O(h) \tag{1.3}$$

as $h \downarrow 0$ uniformly for piecewise linear continuous functions x_h satisfying

$$\int_{B_h} |\nabla x_h|^2 = O(1) \qquad \text{and} \qquad \|x_h\|_{L^\infty} \le C.$$

Because of this fact, the result obtained in this paper continues to hold if x_h is replaced by a minimizer of $\inf_{X_h} E$.

To see (1.3), let us note first that

$$\left| \int_{B \setminus B_h} \left\{ |\nabla x_h|^2 + \frac{4}{3} H x_h \cdot (x_{hu} \times x_{hv}) \right\} \right| \le \left(1 + \frac{2}{3} HC \right) \int_{B \setminus B_h} |\nabla x_h|^2$$

because $\|x_h\|_{L^\infty} \le C$. Let S be a connected component of $(B \setminus B_h)^\circ$ and take $T \in \tau_h$, a common edge of ∂B with S. Then we have

$$\int_S |\nabla x_h|^2 \le \frac{|S|}{|T|} \int_T |\nabla x_h|^2 ,$$

where $|\cdot|$ denotes the two-dimensional Lebesgue measure. Noting that $|S| = O(h^3)$ and $|T| \sim h^2$, we get

$$\int_{B \setminus B_h} |\nabla x_h|^2 \le C' h \int_{B_h} |\nabla x_h|^2 , \tag{1.4}$$

and hence the conclusion follows.

The following is the main theorem of this paper.

Theorem 1.5. *Let $\Gamma \subset \mathbf{R}^3$ be a given rectifiable Jordan curve with $R := \mathrm{diam}(\Gamma)/2$. Let $H > 0$ be such that $HR < 1$. Suppose that Γ is smooth enough so that Hildebrandt's solutions $x \in X$ belong to $W^{1,p}(B, \mathbf{R}^3)$ with $p > 2$. Let $\{\tau_h\}$ be a family of regular and quasi-uniform triangulations of B with mesh size h. Let $\{x_h \in X_h\}$ be a sequence of the finite element Hildebrandt solutions. Then, there exists a subsequence $\{x_{h'}\} \subset \{x_h\}$ converging to a Hildebrandt solution x of (1.1) strongly in $H^1(B, \mathbf{R}^3)$, and furthermore, $\{x_{h'}|_{\partial B}\}$ converges uniformly to $x|_{\partial B}$. If the Hildebrandt solution is unique, the original sequence $\{x_h \in X_h\}$ converges to x in the above sense.*

Concerning numerical methods for H-surfaces, we have Hewgill [7] and Grosse-Brauckmann and Polthier [6]. The former breaks the surface into small patches, where equations describing nonparametric H-surfaces are solved. The latter is concerned with closed H-surfaces of multiple genuses. There, area minimizers in the sphere S^3 are constructed first, and then they are conjugated to H-surfaces. In both papers the problem of convergence is not discussed.

2. Proof of Theorem 1.5

If $x \in H^1(B, \mathbf{R}^3)$ satisfies $\|x\|_{L^\infty} \leq R'$, then the inequality

$$E(x) \geq \frac{1}{3} \int_B |\nabla x|^2$$

follows. From the same reasoning, the family of approximate solutions $\{x_h\}$ defined above satisfies

$$\int_{B_h} |\nabla x_h|^2 = O(1).$$

This implies the boundedness of $\{x_h\}$ in $H^1(B, \mathbf{R}^3)$ by (1.4). Therefore, by Lemma 1.4, there exists $\{x_{h'}\} \subset \{x_h\}$ with $\gamma_{h'} = x_{h'}|_{\partial B}$ converging uniformly to a topologically onto mapping $\gamma : \partial B \to \Gamma$. We have $\gamma \in \mathcal{E}$.

By Lemma 1.2 there exists a unique minimizer $x \in H^1(B, \mathbf{R}^3)$ of

$$\inf \left\{ E(y) \;\middle|\; y \in H^1(B, \mathbf{R}^3),\ \|y\|_{L^\infty} \leq R',\ y|_{\partial B} = \gamma \right\}. \tag{2.1}$$

Let $\hat{x} \in X$ be a Hildebrandt solution satisfying $\hat{x} \in W^{1,p}(B, \mathbf{R}^3)$ for $p > 2$ and $E(\hat{x}) = \inf_X E$ (the existence of $\hat{x}$ is proven by [9]). It is obvious that $x \in X$, and hence $E(\hat{x}) \leq E(x)$.

If $\pi_{h'}$ denotes the interpolation operator and $\hat{x}_{h'} := \pi_{h'}\hat{x}$, then

$$\|\hat{x}_{h'} - \hat{x}\|_{L^\infty} \to 0 \qquad \text{and} \qquad \|\hat{x}_{h'} - \hat{x}\|_{H^1} \to 0. \tag{2.2}$$

We also have $\|\hat{x}_{h'}\|_{L^\infty} \leq R'$ and $\hat{x}_{h'}|_{\partial B} \in \mathcal{E}_{h'}$. This means $\hat{x}_{h'} \in X_{h'}$, and hence $E_{h'}(x_{h'}) \leq E_{h'}(\hat{x}_{h'})$. By (1.3), we obtain $E(x_{h'}) \leq E(\hat{x}_{h'}) + o(1)$.

Let $\tilde{x}_{h'} \in H^1(B, \mathbf{R}^3)$ be the minimizer of

$$\inf \left\{ E(y) \;\middle|\; y \in H^1(B, \mathbf{R}^3),\ \|y\|_{L^\infty} \leq R',\ y|_{\partial B} = \gamma_{h'} \right\}. \tag{2.3}$$

It is obvious that $E(\tilde{x}_{h'}) \leq E(x_{h'})$. We get

$$E(\tilde{x}_{h'}) \leq E(x_{h'}) \leq E(\hat{x}_{h'}) + o(1),$$

where $E(\hat{x}_{h'}) = E(\hat{x}) + o(1)$ by (2.2).

Now, we make use of [1, Lemma 8] concerning the convergence of the solutions of Dirichlet problems. In fact, $\tilde{x}_{h'}$ and x are the minimizers of (2.3) and (2.1), respectively, and we have $\|\gamma_{h'} - \gamma\|_{L^\infty(\partial B, \mathbf{R}^3)} \to 0$ and $\|\gamma_{h'}\|_{H^{1/2}(\partial B, \mathbf{R}^3)} = O(1)$. Under such a situation, that lemma assures $E(x) \leq \liminf E(\tilde{x}_{h'})$ in particular. We obtain

$$\begin{aligned} E(x) &\leq \liminf E(\tilde{x}_{h'}) \leq \liminf E(x_{h'}) \\ &\leq \limsup E(x_{h'}) \leq \lim E(\hat{x}_{h'}) = E(\hat{x}) = \inf_X E \end{aligned}$$

and therefore, $x \in X$ attains $\inf_X E$. It is a Hildebrandt solution, and

$$E(x_{h'}) = \inf_X E + o(1) = E(x) + o(1). \tag{2.4}$$

Although $\{x_{h'}\}$ is not a minimizing sequence of $\inf_X E$ because $x_{h'} \notin X$ by $x_{h'}(\partial B) \neq \Gamma$, we can apply the argument described in the previous section. Recall that $\gamma_{h'} = x_{h'}|_{\partial B}$, that x and $\tilde{x}_{h'}$ are minimizers of (2.1) and (2.3), respectively, that $\gamma_{h'}$ converges uniformly to γ, and that $\left\{ \|\gamma_{h'}\|_{H^{1/2}(B, \mathbf{R}^3)} \right\}$ remains bounded. Furthermore, this time $E(\tilde{x}_{h'}) \to E(x)$ follows similarly to (2.4). Under such a situation, [1, Lemma 8] and its proof give that

$$\tilde{x}_{h'} \to x \qquad \text{strongly in } H^1(B, \mathbf{R}^3). \tag{2.5}$$

Now, we turn to the convergence of the family $\{x_{h'}\}$. As we have already said, it is uniformly bounded in $H^1(B, \mathbf{R}^3)$ with $\|x_{h'}\|_{L^\infty} \le R'$. Passing to a subsequence, we have $x' \in H^1(B, \mathbf{R}^3)$ such that $x_{h'} \rightharpoonup x'$ weakly in $H^1(B, \mathbf{R}^3)$, $*$-weakly in $L^\infty(B, \mathbf{R}^3)$, and almost everywhere in B and on ∂B. In particular, $\|x'\|_{L^\infty} \le R'$. We also have the uniform convergence of $\{x_{h'}|_{\partial B}\}$ to γ, and hence $x'|_{\partial B} = \gamma \in \mathcal{E}$.

Then, Lemma 1.3 ensures

$$E(x') \le \liminf E(x_{h'}) = \inf_X E.$$

In particular, x' attains the minimum of (2.1), and therefore $x' = x$ follows from the uniqueness of the minimizer in Lemma 1.2. We have proven the weak convergence $x_{h'} \rightharpoonup x$ in $H^1(B, \mathbf{R}^3)$ and the uniform convergence $x_{h'}|_{\partial B} \to x|_{\partial B}$. Now we shall show that the former convergence is actually strong in $H^1(B, \mathbf{R}^3)$ and complete the proof of Theorem 1.5.

For this purpose, we take $\varphi_{h'}$ satisfying

$$\Delta\varphi_{h'} = 0 \quad \text{in } B \qquad \text{and} \qquad \varphi_{h'} = \gamma_{h'} - \gamma \quad \text{on } \partial B.$$

Then, the maximum principle implies $\|\varphi_{h'}\|_{L^\infty(B)} \le \|\gamma_{h'} - \gamma\|_{L^\infty(\partial B)} \to 0$. Also, the Dirichlet principle gives

$$\int_B |\nabla\varphi_{h'}|^2 \le \int_B |\nabla(\tilde{x}_{h'} - x)|^2 \to 0$$

because $\tilde{x}_{h'} - x|_{\partial B} = \gamma_{h'} - \gamma$ holds with (2.5). We obtain

$$\varphi_{h'} \to 0 \qquad \text{in both } H^1(B, \mathbf{R}^3) \text{ and } C(\overline{B}, \mathbf{R}^3), \tag{2.6}$$

which implies $E(x_{h'}) = E(x + \varphi_{h'}) + o(1)$ by (2.4).

We can now follow the argument of the proof of [1, Lemma 8]. (Namely, (56) implies (57) under the assumptions given there.) Again by (2.6), we get

$$E(x_{h'} - \varphi_{h'}) = E(x) + o(1).$$

Besides $x_{h'} - \varphi_{h'}|_{\partial B} = \gamma$, we have $\|x_{h'} - \varphi_{h'}\|_{L^\infty} \le R''$ for $h' > 0$ sufficiently small, where $R'' > R'$ and $R''H < 1$. By Lemma 1.2, on the other hand, any minimizer $\tilde{x}$ of $\inf_{\tilde{X}} E$ satisfies $\|\tilde{x}\|_{L^\infty} \le R$ for

$$\tilde{X} = \left\{ x \in H^1(B, \mathbf{R}^3) \mid \|x\|_{L^\infty} \le R'',\ x|_{\partial B} \in \mathcal{E} \right\}.$$

This fact implies that $x \in X$ also attains $\inf_{\tilde{X}} E$, and in particular, it attains (2.1) with R' replaced by R''. The family $\{x_{h'} - \varphi_{h'}\}$ is a minimizing sequence for that problem, and then the strong convergence $x_{h'} - \varphi_{h'} \to x$ in $H^1(B, \mathbf{R}^3)$ follows from Lemma 1.2 and [1, Lemma 8].

The strong convergence $x_{h'} \to x$ in $H^1(B, \mathbf{R}^3)$ is now a consequence of (2.6), and the proof is complete.

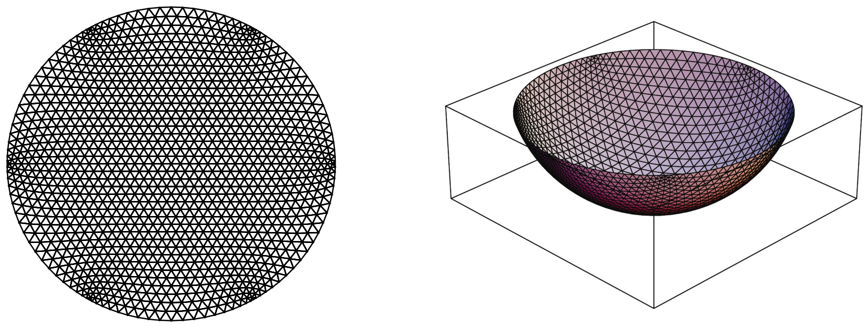

FIGURE 3.1. The triangulation and FE H-surface with $H = 0.95$

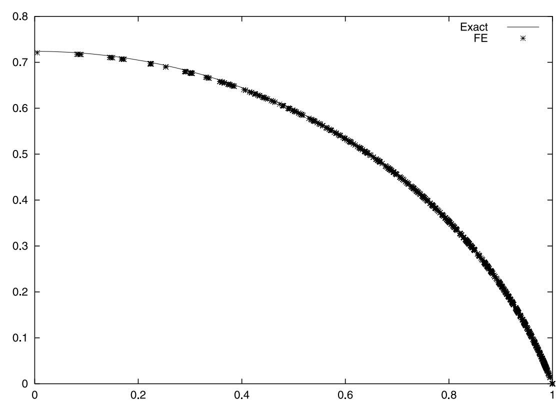

FIGURE 3.2. The comparison betwee the exact and FE H-surfaces

3. Numerical examples

In this section we give several numerical examples. The first example is the simplest one: the contour γ is the circle $(\cos t, \sin t, 0)$ $(-\pi \leq t \leq \pi)$. Figure 3.1 shows the triangulation of the unit disk B and the finite element H-surface with $H = 0.95$.

We surely know that the image of the exact H-surface is a part of the sphere with center $z_0 := (0, 0, \sqrt{1/H^2 - 1})$ and radius $1/H$. Figure 3.2 shows the comparison between the exact and finite element H-surfaces. The solid line is the graph of the function

$$f(r) := (1/H^2 - r)^{1/2} - (1/H^2 - 1)^{1/2} \quad (0 \leq r \leq 1)$$

and for the finite element H-surface (x_h^1, x_h^2, x_h^3), the point $(\sqrt{(x_h^1(p_i))^2 + (x_h^2(p_i))^2}, -x_h^3(p_i))$ is dotted at each nodal point p_i.

To see how finite element H-surfaces converge, we compute the above example with several triangulations. Table 3.3 and the graph shown in Figure 3.4 are the result. In the table and the graph "h" stands for the mesh size of the triangulation and "error" stands for $\max_{p_i} \left| \, 1/H - |x_h(p_i) - z_0| \, \right|$ for nodal points p_i. The convergence rate seems quadratic.

TABLE 3.3.

h	error
9.336×10^{-2}	5.709×10^{-3}
6.996×10^{-2}	3.251×10^{-3}
4.676×10^{-2}	1.460×10^{-3}
3.514×10^{-2}	8.252×10^{-4}

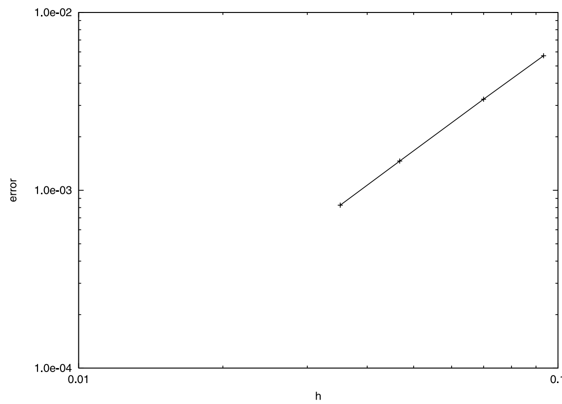

FIGURE 3.4.

The next example is finite element H-surfaces with the contour γ defined by $(x(t), y(t), z(t))$, where

$$x(t) := (1 + q\cos t)\cos 2t,$$
$$y(t) := (1 + q\cos t)\sin 2t,$$
$$z(t) := p\sin t$$

$(-\pi \le t \le \pi)$ with $p := 0.6$ and $q := -0.2$. Figures 3.5 and 3.6 show the finite element H-surfaces with $H = 0.0$, $H = 0.9$, respectively. In the figures *ViewPoint* is the coordinate of the viewpoint.

The authors have observed that, when H approaches 1.0, the numerical scheme becomes unstable and computation is finally aborted. Developing a numerical scheme for computing finite element H-surfaces around and beyond the *turning point* $H = R^{-1}$ (that is, computing *large solutions*) is an interesting problem.

For the next example we define the contour γ by $(x(t), y(t), z(t))$, where

$$x(t) := (1 + q\cos 3t)\cos 2t,$$
$$y(t) := (1 + q\cos 3t)\sin 2t,$$
$$z(t) := p\sin 3t$$

$(-\pi \le t \le \pi)$ with $p := 0.25$ and $q := 0.25$. Figures 3.7 and 3.8 show the finite element H-surfaces with $H = 0.0$ and $H = 0.9$, respectively.

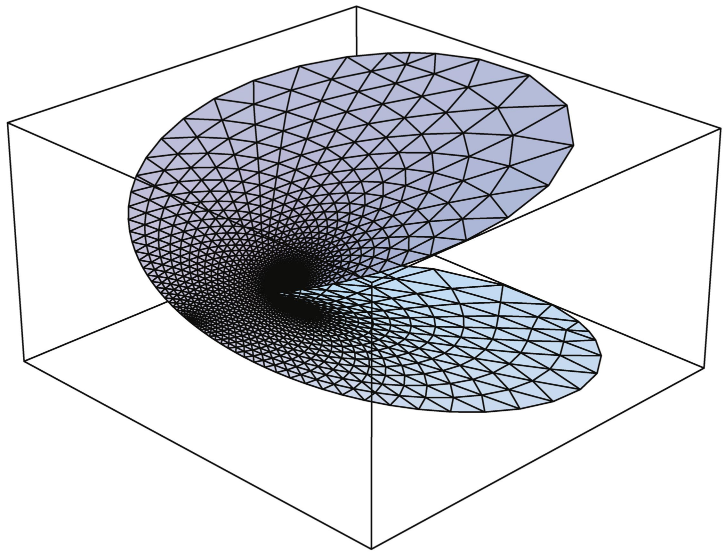

FIGURE 3.5. The FE H-surface with $H = 0.0$, *View Point* $= (3, 3, 2)$

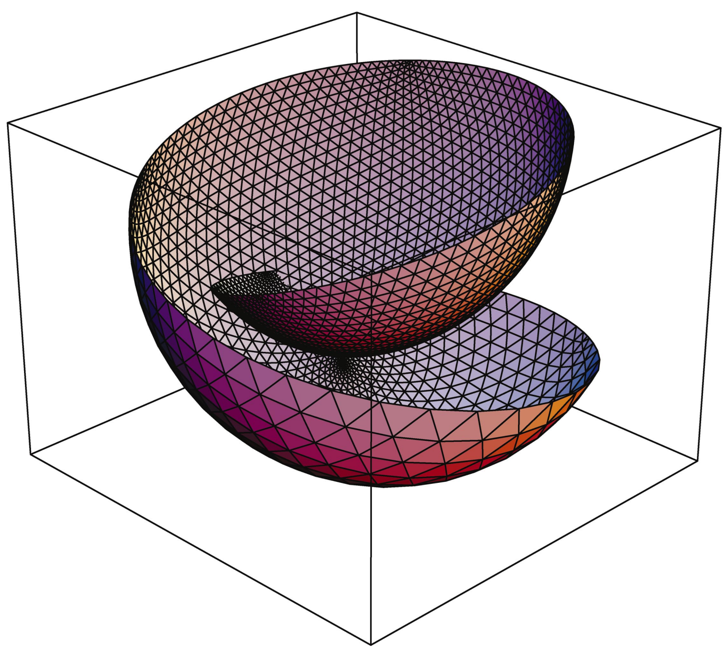

FIGURE 3.6. The FE H-surface with $H = 0.9$, *ViewPoint* $= (3, 3, 2)$

It is well-known that, if $H = 0.0$, then the Hildebrandt solution (or the Douglas-Radó solution to the Plateau problem) does not have any branch points [10]. Hence, we have to notice that these FE H-surfaces do not approximate the Hildebrandt solution because the FE H-surface with $H = 0.0$ has a branch point. Showing the

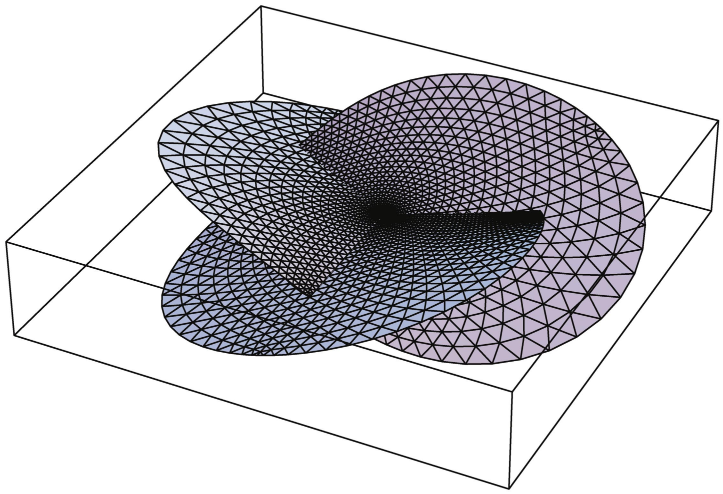

FIGURE 3.7. The FE H-surface with $H = 0.0$, *ViewPoint* $= (4, 2, 3)$

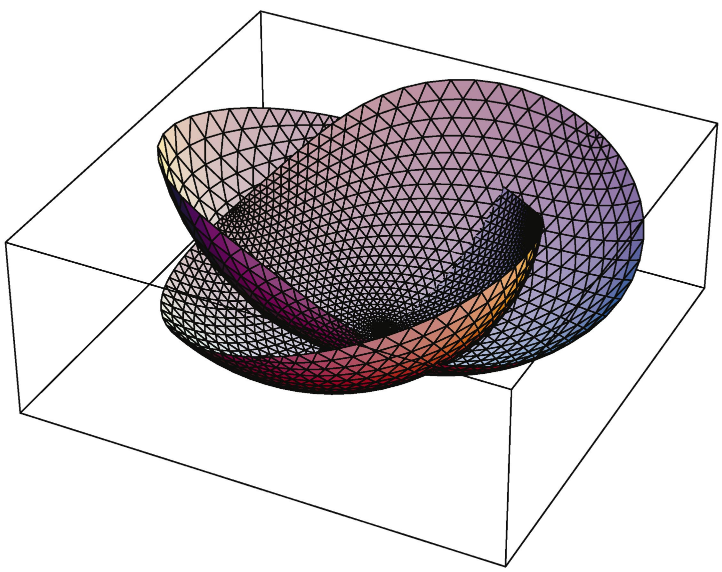

FIGURE 3.8. The FE H-surface with $H = 0.9$, *ViewPoint* $= (4, 2, 3)$

existence of exact H-surface branches associated with such FE H-surfaces other than *small/large solutions* is an interesting problem. The authors are planning to develop finite element analysis for FE H-surfaces which are not associated with Hildebrandt solutions.

All computations were carried out on a PC with Celeron 300A, Linux Kernel 2.2.14, and Fujitsu's Fortran 90 compiler. Each example took a few minutes to compute. Mathematica and gnuplot were used to draw the figures.

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