MATHEMATICS OF COMPUTATION
Volume 72, Number 242, Pages 839–863
S 0025-5718(02)01455-2
Article electronically published on June 25, 2002

A MASS FORMULA FOR UNIMODULAR LATTICES WITH NO ROOTS

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Abstract. We derive a mass formula for n-dimensional unimodular lattices having any prescribed root system. We use Katsurada's formula for the Fourier coefficients of Siegel Eisenstein series to compute these masses for all root systems of even unimodular 32-dimensional lattices and odd unimodular lattices of dimension $n \leq 30$. In particular, we find the mass of even unimodular 32-dimensional lattices with no roots, and the mass of odd unimodular lattices with no roots in dimension $n \leq 30$, verifying Bacher and Venkov's enumerations in dimensions 27 and 28. We also compute better lower bounds on the number of inequivalent unimodular lattices in dimensions 26 to 30 than those afforded by the Minkowski-Siegel mass constants.

1. Introduction

First we review some definitions. For more information, see [9] and [21].

An n-dimensional *lattice* Λ is the set $\mathrm{Z}v_1 + \cdots + \mathrm{Z}v_n$ of all integer linear combinations of a basis $\{v_1, \ldots, v_n\}$ for R^n. We associate with Λ the *Gram matrix* A with i,j-th entry the inner product (v_i, v_j). (The matrix A is a positive definite quadratic form, and much of what follows may be reformulated in the language of quadratic forms.) The *determinant* of Λ is defined to be $\det(A)$, and Λ is called *unimodular* if its determinant is 1. We say that Λ is *integral* if (v, v') is an integer for all $v, v' \in \Lambda$. Such a lattice is called *even* (or *Type II*) if (v, v) is always even, and *odd* (or *Type I*) otherwise. The *dual* of Λ is $\Lambda' = \{v \in \mathrm{R}^n : (v, x) \in \mathrm{Z} \text{ for all } x \in \Lambda\}$, and the determinant of Λ is the order of the finite abelian group Λ'/Λ. Since we shall (with the exception of the occasional dual lattice) primarily be concerned with integral lattices, we generally omit the adjective integral in what follows. The *norm* of a vector $v \in \Lambda$ is defined to be (v, v), the square of its length. We say that a lattice is *decomposable* (or *reducible*) if it can be written as an orthogonal direct sum of two nonzero sublattices, and is *indecomposable* (or *irreducible*) otherwise.

Each n-dimensional unimodular lattice Λ has a vector u such that $(u, x) \equiv (x, x) \pmod 2$ for all $x \in \Lambda$. Such a vector is called a *parity vector* [9, Preface to 3rd edition, p. xxxiv], or a *characteristic vector* [4], or a *canonical element* [32]. The set of parity vectors forms a coset $u + 2\Lambda$ in $\Lambda/2\Lambda$, and each parity vector satisfies $(u, u) \equiv n \pmod 8$.

Let N be an n-dimensional lattice with Gram matrix A, and let M be an m-dimensional lattice with Gram matrix B. We say that N is *represented* by M if

Received by the editor March 29, 2001 and, in revised form, May 8, 2001.
2000 *Mathematics Subject Classification.* Primary 11H55; Secondary 11E41.
This work was partially supported by grants from the NSF and the Royal Society.

there exists an $m \times n$ integral matrix X for which $X^t B X = A$. We let $r(M, N)$ denote the number of representations of N by M. When $m = n$, we say that M and N are (integrally) *equivalent* if there exists an integral matrix X with determinant ± 1 for which $X^t B X = A$. M and N are in the same *genus* if they are equivalent over the p-adic integers Z_p for each prime p (including $p = \infty$, for which $\mathrm{Z}_p = \mathrm{Q}$). We define $\mathrm{Aut}(N)$ to be the group of $n \times n$ integral matrices X for which $X^t A X = A$. (Note that the definitions of $\det(N)$, $r(M, N)$ and equivalence are independent of the integral bases chosen for the lattices, and so is $\mathrm{Aut}(N)$, up to isomorphism.) The *theta series* of a lattice Λ is defined by

$$\Theta_\Lambda(q) = \sum_{v \in \Lambda} q^{(v,v)} = \sum_{k=0}^{\infty} r(\Lambda, k) q^k,$$

where $q = e^{\pi i z}$.

For our purposes, a *root* is a vector of norm 1 or 2, and the *root system* of Λ is the set of roots in Λ. The lattice generated by a root system is called a *root lattice*, and we define the *rank* of a root system to be the dimension of the corresponding root lattice. Root lattices are completely classified: they are direct sums of the irreducible root lattices Z, $A_n\,(n \geq 1), D_n\,(n \geq 4), E_6, E_7$, and E_8. We use the same notation to refer to root systems, and for brevity we sometimes write the root system

$$n_1 A_1 \oplus \cdots \oplus n_j E_8 \quad \text{as } A_1^{n_1} \cdots E_8^{n_j}.$$

In this paper we are concerned with the problem of classifying unimodular lattices, and also the subproblem of classifying unimodular lattices without roots (which correspond to denser sphere-packings), up to equivalence.

The Minkowski-Siegel mass formula (see [7]) gives the sum of the reciprocals of the orders of the automorphism groups of all inequivalent lattices in a given genus. The mass constants can be used to verify that an enumeration of inequivalent lattices in a given genus is complete. They also give a lower bound for the number of inequivalent lattices in a given genus (sometimes called the *class number*).

In dimensions divisible by 8 there are two genera of unimodular lattices, Type I and Type II (except in dimension 0, in which there is just one lattice, of Type II). In all other dimensions there is only one genus of unimodular lattices, Type I. The lower bounds provided by the mass constants show that the number of unimodular lattices increases super-exponentially as a function of the dimension. Unimodular lattices have been completely enumerated in dimensions $n \leq 25$, but number more than 900 million in dimension 30. Unimodular lattices without roots have been completely enumerated in dimensions $n \leq 28$, but number more than 8×10^{20} in dimension 33. In both cases, somewhere between dimension 26 and dimension 32 there is a transition from being completely classified to being numerous enough to make classification unappealing, so information about what is going on in these dimensions is of interest.

Our approach is based on a suggestion by Borcherds that the Fourier coefficients of Siegel Eisenstein series of degree $4k$ could be used to derive something analogous to the Minkowski-Siegel mass formula, but which gives the mass of all the even unimodular lattices of dimension $8k$ having any given root system.

We wrote a computer program that uses Katsurada's formula (see [15]) for the Fourier coefficients to calculate the masses for all possible root systems of even unimodular 32-dimensional lattices. From these masses, we used the methods of

[9, Chapter 16] to compute the masses, for each root system, of unimodular lattices in dimensions $n \leq 30$.

In particular, we have a mass formula for those lattices which have the empty root system (that is, which have no roots). This formula verifies the known masses in dimensions $n \leq 28$, and provides new lower bounds for the number of unimodular lattices without roots in dimensions 29 to 32.

We also used our program to compute better lower bounds on the total number of odd unimodular lattices in dimensions 26 to 30, and even unimodular lattices in dimension 32, than those gotten from the Minkowski-Siegel mass constants.

Our results may be viewed as a coarse classification of lattices, as inequivalent lattices may have the same root system. For dimensions $n \leq 23$, in which it happens to be the case that a unimodular lattice in a given genus is completely determined by its root system, our results coincide with the previously known enumerations. The same is true for the even unimodular 24-dimensional lattices (which are known as the *Niemeier lattices*).

2. A mass formula for even unimodular lattices having any given root system

Let Ω be the set of inequivalent even unimodular lattices of dimension $8k$. We define the *mass* m of Ω by

$$m = \sum_{\Lambda \in \Omega} \frac{1}{|\mathrm{Aut}(\Lambda)|}.$$

By the Minkowski-Siegel mass formula, for $k > 0$ we have

$$m = \frac{|B_{4k}|}{8k} \prod_{j=1}^{4k-1} \frac{|B_{2j}|}{4j},$$

where B_i is the ith Bernoulli number. (See [9] and [7] for this, and for the mass formulae for other genera of lattices.)

For an n-dimensional lattice N, we define a weighted average number of representations of N by the lattices $\Lambda \in \Omega$ by

$$a(N) = \frac{1}{m} \sum_{\Lambda \in \Omega} \frac{r(\Lambda, N)}{|\mathrm{Aut}(\Lambda)|}.$$

Let $\{R_1, ..., R_s\}$ be the set of all the root lattices of dimension n or less with no vectors of norm 1. Each R_i is the direct sum of lattices A_j $(j \geq 1), D_j$ $(j \geq 4), E_6, E_7,$ and E_8. Let Ω_i be the set of lattices in Ω having root system R_i, and let $m(R_i) = \sum_{\Lambda \in \Omega_i} |\mathrm{Aut}(\Lambda)|^{-1}$ be the mass of those lattices in Ω which have root system R_i, so that $m = m(R_1) + \cdots + m(R_s)$.

Proposition 1. *Let U be the $s \times s$ matrix with i,j-th entry $r(R_j, R_i)$, let v be the vector $(m(R_1), \ldots, m(R_s))^t$, and let w be the vector $(a(R_1), \ldots, a(R_s))^t$. Then $\frac{1}{m} Uv = w$. Furthermore, U is invertible, so $m(R_i) = (mU^{-1}w)_i$ gives the mass of the lattices in Ω having root system R_i.*

Proof. Observe that if R is any root lattice and if S is the root system of a lattice Λ, then $r(\Lambda, R) = r(S, R)$, so for $j = 1, \ldots, s$ we have

$$\begin{aligned} a(R_j) &= \frac{1}{m} \sum_{\Lambda \in \Omega} \frac{r(\Lambda, R_j)}{|\mathrm{Aut}(\Lambda)|} = \frac{1}{m} \sum_{i=1}^{s} \sum_{\Lambda \in \Omega_i} \frac{r(\Lambda, R_j)}{|\mathrm{Aut}(\Lambda)|} \\ &= \frac{1}{m} \sum_{i=1}^{s} r(R_i, R_j) \sum_{\Lambda \in \Omega_i} \frac{1}{|\mathrm{Aut}(\Lambda)|} \\ &= \frac{1}{m} \sum_{i=1}^{s} r(R_i, R_j)\, m(R_i). \end{aligned}$$

Thus $\frac{1}{m} Uv = w$. We may assume the R_i's are ordered so that their dimensions are non-decreasing, and so that among those with the same dimension the determinants are non-increasing. With this ordering, $r(R_j, R_i) = 0$ whenever $i > j$, so U is upper triangular. Since each diagonal element $r(R_i, R_i) = |\mathrm{Aut}(R_i)|$ is positive, we have $\det(U) \neq 0$, so $v = mU^{-1}w$. □

Remark 2. The values $a(R_j)$ are the Fourier coefficients of Siegel Eisenstein series. Borcherds, Freitag and Weissauer [6] used a relation similar to $\frac{1}{m} Uv = w$ to compute the coefficients of a cusp form from the known values of $m(R_i)$ in dimension 24. We shall do the inverse in dimension 32: use the values of $a(R_j)$ to derive the values $m(R_i)$. We discuss how to compute $a(R_j)$ in Section 7. The problem of computing $r(R_i, R_j)$ is largely combinatorial; we discuss it in Section 8.

3. Masses of 32-dimensional even unimodular lattices with any given root system

Let $m_n^{\mathrm{II}}(R)$ and $m_n^{\mathrm{I}}(R)$ denote the masses of the n-dimensional even and odd unimodular lattices having root system R, and let $m_n(R)$ denote their sum. Let $w(R)$ denote the order of the Weyl group of R; $w(R)$ is the product of the orders of the Weyl groups of the irreducible components of R, where $w(A_n) = (n+1)!, w(D_n) = 2^{n-1}n!, w(E_6) = 2^7 3^4 5 \cdot 7, w(E_7) = 2^{10} 3^4 5 \cdot 7$, and $w(E_8) = 2^{14} 3^5 5^2 7$. It is sometimes more convenient to list the values of $m_n^{\mathrm{II}}(R) \cdot w(R)$ than it is to list the values of $m_n^{\mathrm{II}}(R)$ alone, but the latter can easily be recovered from the former.

We used a computer to calculate $m_{32}^{\mathrm{II}}(R)$ for each root system R with $\mathrm{rank}(R) \leq 32$. The computation took about two weeks on a Sun Ultra 60, running a program written in Common Lisp and compiled with Franz Inc.'s Allegro CL. (We discuss several issues related to the implementation in Sections 7 to 9.)

Theorem 3. *The nonzero values of $m_{32}^{II}(R) \cdot w(R)$ with $\mathrm{rank}(R) \leq 9$ are as listed in Table 1. (The list of all* 13218 *nonzero values of $m_{32}^{II}(R) \cdot w(R)$ is available electronically at* [18]*.)*

The root system R of a lattice Λ is called *complete* if the sublattice of Λ generated by R has finite index in Λ, or equivalently if $\mathrm{rank}(R) = \dim(\Lambda)$. The classification of even unimodular lattices with complete root systems is closely related to the classification of certain self-dual codes (see [33]).

Corollary 4 (Kervaire [17])**.** *There are* 119 *complete root systems R that occur as root systems of indecomposable* 32*-dimensional even unimodular lattices.*

TABLE 1. Masses of 32-dimensional even unimodular lattices

dim	root system R	$m_{32}^{\mathrm{II}}(R)\cdot w(R)$	(as decimal)	comments
0	$\emptyset$	$\frac{131003733128202332665891 7}{238863431761920000}$	5484461.50	no roots
1	A_1	$\frac{111536168182433}{5677056}$	19646832.00	
2	A_1^2	$\frac{72024731351193941}{1857945600}$	38765792.00	
2	A_2	$\frac{1327104974887}{2939328}$	451499.44	
3	A_1^3	$\frac{6904800898075}{124416}$	55497696.00	
3	A_1A_2	$\frac{977951251237}{445440}$	2195472.50	
3	A_3	$\frac{329127961}{74240}$	4433.30	
4	A_1^4	$\frac{30223371257980501}{471859200}$	64051672.00	
4	$A_1^2A_2$	$\frac{19867101805}{3456}$	5748582.50	
4	A_2^2	$\frac{1772535692573}{42598400}$	41610.38	
4	A_1A_3	$\frac{21073837}{768}$	27439.89	
4	A_4	$\frac{8397751}{384000}$	21.87	
4	D_4	$\frac{35841940559}{157212057600}$	0.23	see [2]
5	A_1^5	$\frac{14457125482723}{230400}$	62747940.00	
5	$A_1^3A_2$	$\frac{10626384230783}{995328}$	10676264.00	
5	$A_1A_2^2$	$\frac{673556587}{2560}$	263108.06	
5	$A_1^2A_3$	$\frac{200386803709}{2211840}$	90597.34	
5	A_2A_3	$\frac{8085187}{5760}$	1403.68	
5	A_1A_4	$\frac{46917823}{269568}$	174.05	
5	A_1D_4	$\frac{473}{240}$	1.97	
5	A_5	$\frac{73}{960}$	0.08	
5	D_5	$\frac{433}{3317760}$	0.00	see [2]
6	A_1^6	$\frac{355695555290333}{6635520}$	53604776.00	
6	$A_1^4A_2$	$\frac{22484458507}{1440}$	15614208.00	
6	$A_1^2A_2^2$	$\frac{16820220686833}{19169280}$	877457.06	
6	$A_1^3A_3$	$\frac{5452147363}{25920}$	210345.19	
6	A_2^3	$\frac{9785018477}{1866240}$	5243.17	
6	$A_1A_2A_3$	$\frac{680633479}{61440}$	11078.02	
6	$A_1^2A_4$	$\frac{11264777}{15360}$	733.38	
6	A_3^2	$\frac{323400013}{20447232}$	15.82	
6	$A_1^2D_4$	$\frac{622763}{69120}$	9.01	
6	A_2A_4	$\frac{6059}{512}$	11.83	
6	A_2D_4	$\frac{344077}{2446080}$	0.14	
6	A_1A_5	$\frac{123853}{159744}$	0.78	
6	A_1D_5	$\frac{1}{648}$	0.00	
6	A_6	$\frac{1}{4608}$	0.00	

TABLE 1. Masses of 32-dimensional even unimodular lattices (continued)

dim	root system R	$m_{32}^{\mathrm{II}}(R) \cdot w(R)$	(as decimal)	comments
6	D_6	$\frac{1}{18720000}$	0.00	unique
6	E_6	$\frac{1}{1268047872}$	0.00	unique
7	A_1^7	$\frac{23307759701}{576}$	40464860.00	
7	$A_1^5A_2$	$\frac{290429642677}{15360}$	18908180.00	
7	$A_1^3A_2^2$	$\frac{32712111919}{16128}$	2028280.70	
7	$A_1^4A_3$	$\frac{491983248817}{1290240}$	381311.40	
7	$A_1A_2^3$	$\frac{641243179}{15360}$	41747.60	
7	$A_1^2A_2A_3$	$\frac{528426689}{11520}$	45870.37	
7	$A_1^3A_4$	$\frac{828763}{384}$	2158.24	
7	$A_2^2A_3$	$\frac{238819303}{552960}$	431.89	
7	$A_1^3D_4$	$\frac{2242333}{77760}$	28.84	
7	$A_1A_3^2$	$\frac{2750969}{17920}$	153.51	
7	$A_1A_2A_4$	$\frac{57178267}{483840}$	118.18	
7	$A_1^2A_5$	$\frac{24071}{5760}$	4.18	
7	$A_1A_2D_4$	$\frac{959}{640}$	1.50	
7	A_3A_4	$\frac{297043}{860160}$	0.35	
7	A_2A_5	$\frac{51383}{774144}$	0.07	
7	A_3D_4	$\frac{133}{30720}$	0.00	
7	$A_1^2D_5$	$\frac{11}{1152}$	0.01	
7	A_1A_6	$\frac{521}{188160}$	0.00	
7	A_2D_5	$\frac{1}{6912}$	0.00	
7	A_7	$\frac{1}{1376256}$	0.00	
8	A_1^8	$\frac{2005621383142854931}{73987522560}$	27107562.00	
8	$A_1^6A_2$	$\frac{2015000372681}{103680}$	19434804.00	
8	$A_1^4A_2^2$	$\frac{1327413084613}{368640}$	3600838.50	
8	$A_1^5A_3$	$\frac{980176289}{1728}$	567231.70	
8	$A_1^2A_2^3$	$\frac{13761649541}{80640}$	170655.37	
8	$A_1^3A_2A_3$	$\frac{42164571593}{322560}$	130718.54	
8	A_2^4	$\frac{176324027322323}{180592312320}$	976.37	
8	$A_1^4A_4$	$\frac{9542951321}{1935360}$	4930.84	
8	$A_1A_2^2A_3$	$\frac{691208023}{161280}$	4285.76	
8	$A_1^4D_4$	$\frac{371231029}{5160960}$	71.93	
8	$A_1^2A_3^2$	$\frac{6071138573}{7741440}$	784.24	
8	$A_1^2A_2A_4$	$\frac{1655917}{2688}$	616.04	
8	$A_2A_3^2$	$\frac{1794539}{120960}$	14.84	
8	$A_1^3A_5$	$\frac{2245489}{143360}$	15.66	

TABLE 1. Masses of 32-dimensional even unimodular lattices (continued)

dim	root system R	$m_{32}^{\mathrm{II}}(R)\cdot w(R)$	(as decimal)	comments
8	$A_1^2A_2D_4$	$\frac{6113}{720}$	8.49	
8	$A_2^2A_4$	$\frac{237}{40}$	5.93	
8	$A_1A_3A_4$	$\frac{200995}{48384}$	4.15	
8	$A_2^2D_4$	$\frac{18917}{266112}$	0.07	
8	$A_1A_2A_5$	$\frac{7739}{8960}$	0.86	
8	$A_1^3D_5$	$\frac{829}{20160}$	0.04	
8	$A_1A_3D_4$	$\frac{353}{6720}$	0.05	
8	$A_1^2A_6$	$\frac{2287}{120960}$	0.02	
8	A_4^2	$\frac{432673039}{170311680000}$	0.00	
8	A_3A_5	$\frac{1049}{483840}$	0.00	
8	$A_1A_2D_5$	$\frac{43}{20160}$	0.00	
8	A_2A_6	$\frac{1}{3780}$	0.00	
8	A_4D_4	$\frac{1}{15360}$	0.00	
8	D_4^2	$\frac{1867}{1937768448}$	0.00	
8	A_3D_5	$\frac{19}{1935360}$	0.00	
8	$A_1^2D_6$	$\frac{607}{85155840}$	0.00	
8	A_1A_7	$\frac{5}{1064448}$	0.00	
8	A_8	$\frac{1}{185794560}$	0.00	
8	A_2E_6	$\frac{1}{277136640}$	0.00	unique
8	D_8	$\frac{1}{1002795171840}$	0.00	odd Leech
8	E_8	$\frac{1}{83155536130867200000}$	0.00	Leech
9	A_1^9	$\frac{4668288705497}{290304}$	16080690.00	
9	$A_1^7A_2$	$\frac{5526347655971}{322560}$	17132774.00	
9	$A_1^5A_2^2$	$\frac{740728763}{144}$	5143950.00	
9	$A_1^6A_3$	$\frac{1568559349553}{2211840}$	709164.94	
9	$A_1^3A_2^3$	$\frac{1368567316381}{2903040}$	471425.60	
9	$A_1^4A_2A_3$	$\frac{1271443287}{4480}$	283804.30	
9	$A_1A_2^4$	$\frac{23738023}{2560}$	9272.67	
9	$A_1^5A_4$	$\frac{2142275851}{233280}$	9183.28	
9	$A_1^2A_2^2A_3$	$\frac{927969995}{43008}$	21576.68	
9	$A_1^5D_4$	$\frac{158461}{1080}$	146.72	
9	$A_1^3A_3^2$	$\frac{36859871}{13440}$	2742.55	
9	$A_1^3A_2A_4$	$\frac{237492443}{107520}$	2208.82	
9	$A_2^3A_3$	$\frac{3078967}{17280}$	178.18	
9	$A_1^4A_5$	$\frac{457601}{10080}$	45.40	
9	$A_1^3A_2D_4$	$\frac{21454849}{645120}$	33.26	

TABLE 1. Masses of 32-dimensional even unimodular lattices (continued)

dim	root system R	$m_{32}^{\mathrm{II}}(R) \cdot w(R)$	(as decimal)	comments
9	$A_1A_2A_3^2$	$\frac{6031423}{32256}$	186.99	
9	$A_1A_2^2A_4$	$\frac{989231}{13440}$	73.60	
9	$A_1^2A_3A_4$	$\frac{1223851}{46080}$	26.56	
9	$A_1^2A_2A_5$	$\frac{6696269}{1182720}$	5.66	
9	$A_1A_2^2D_4$	$\frac{2927}{2880}$	1.02	
9	A_3^3	$\frac{31211779}{163296000}$	0.19	
9	$A_1^4D_5$	$\frac{1029287}{7464960}$	0.14	
9	$A_1^2A_3D_4$	$\frac{66941}{184320}$	0.36	
9	$A_2A_3A_4$	$\frac{3827}{8064}$	0.47	
9	$A_1^3A_6$	$\frac{34339}{384000}$	0.09	
9	$A_2^2A_5$	$\frac{783383}{13063680}$	0.06	
9	$A_1A_4^2$	$\frac{1021}{32256}$	0.03	
9	$A_2A_3D_4$	$\frac{1}{252}$	0.00	
9	$A_1^2A_2D_5$	$\frac{37}{2520}$	0.01	
9	$A_1A_3A_5$	$\frac{4813}{129024}$	0.04	
9	$A_1A_2A_6$	$\frac{547}{126720}$	0.00	
9	$A_1A_4D_4$	$\frac{1}{1260}$	0.00	
9	$A_2^2D_5$	$\frac{1}{8000}$	0.00	
9	$A_1^2A_7$	$\frac{43}{430080}$	0.00	
9	$A_1A_3D_5$	$\frac{1}{15840}$	0.00	
9	A_4A_5	$\frac{13}{295680}$	0.00	
9	A_3A_6	$\frac{1}{466560}$	0.00	
9	A_5D_4	$\frac{1}{1512000}$	0.00	
9	A_2A_7	$\frac{1}{504000}$	0.00	
9	$A_1^3E_6$	$\frac{1}{1512000}$	0.00	unique
9	$A_1A_2D_6$	$\frac{1}{645120}$	0.00	
9	A_4D_5	$\frac{1}{1774080}$	0.00	
9	A_1A_8	$\frac{1}{20401920}$	0.00	
9	D_4D_5	$\frac{1}{41287680}$	0.00	
9	A_3D_6	$\frac{1}{52254720}$	0.00	
9	$A_1^2D_7$	$\frac{1}{177408000}$	0.00	unique
9	A_3E_6	$\frac{1}{489646080}$	0.00	unique
9	A_9	$\frac{1}{3592512000}$	0.00	
9	A_2E_7	$\frac{1}{991533312000}$	0.00	unique
9	D_9	$\frac{1}{84610842624000}$	0.00	shorter Leech

Proof. There are 143 root systems R for which $\operatorname{rank}(R) = 32$ and $m_{32}^{\mathrm{II}}(R) > 0$. Of these, the root system D_{16}^2 and the 23 root systems containing E_8 correspond to decomposable lattices. The remaining 119 root systems correspond to indecomposable lattices. □

Kervaire [17], extending the work of Koch and Venkov [24], has proven a stronger result: there are exactly 132 indecomposable even unimodular 32-dimensional lattices with complete root systems, with 119 different root systems occurring.

Corollary 5. *There are at least ten million* 32*-dimensional even unimodular lattices without roots.*

Proof. Each such lattice has at least two automorphisms, so the number of such lattices is at least $2 \times m_{32}^{\mathrm{II}}(\emptyset) = 1.096 \times 10^7$. We will have more to say about lattices without roots is Section 5. □

Remark 6. If there is a single even unimodular 32-dimensional lattice with root system R, then the corresponding value $m_{32}^{\mathrm{II}}(R) \cdot w(R)$ is of the form $1/q$ for some $q \in \mathrm{N}$; the converse often holds (but not always: for $R = A_1^7 A_2^3 A_3 D_7$, $m_{32}^{\mathrm{II}}(R) \cdot w(R)$ splits as $1/4 = 1/12 + 1/6$; see [4]). We give one example below, and shall see more examples in Section 4 (Niemeier lattices) and Section 5 (unimodular lattices with no roots). The last column of Table 1 labels as "unique" those lattices whose uniqueness follows from [4], and in some cases gives the name of the lower dimensional lattice that when glued to R produces the unique even unimodular 32-dimensional lattice with root system R.

Example 7. In [12], Elkies and Gross construct a 26-dimensional even lattice L_0 of determinant 3 with no roots, for which $|\mathrm{Aut}(L_0)| = 2^{13}3^5 7^2 13 = 1268047872$, and sketch a proof of its uniqueness that uses Euclidean lattices. (Borcherds [4, Chapter 5.7] had previously proved its existence and uniqueness using Lorentzian lattices.) By [4], even 26-dimensional lattices of determinant 3 with no roots are in one-to-one correspondence with even unimodular 32-dimensional lattices having root system E_6, where the order of the automorphism group of the 32-dimensional lattice is $w(E_6)$ times the order the automorphism group of the corresponding 26-dimensional lattice. Since $m_{32}^{\mathrm{II}}(E_6) \cdot w(E_6) = |\mathrm{Aut}(L_0)|^{-1}$, L_0 must be the unique even 26-dimensional lattice of determinant 3 with no roots, and there must also be a unique 32-dimensional even unimodular lattice with root system E_6. (This also follows from the classification of 27-dimensional unimodular lattices with no roots in [2], as the one-to-one correspondence mentioned above also extends to 27-dimensional unimodular lattices with no roots and with a parity vector of norm 3 [4].)

Remark 8. In an earlier draft we pointed out the large mass of lattices with root systems A_1^k for small k, and remarked that roots seem to have a propensity for being orthogonal to one another. Peters has since sent us a preprint [31] explaining this: he observes that the sum of the masses $m_{32}^{\mathrm{II}}(\emptyset) + \sum_{k=1}^{32} m_{32}^{\mathrm{II}}(A_1^k)$ in [18] is 97.25% of the total mass of the genus of even unimodular 32-dimensional lattices, and shows that a lower bound of 97.11% of the total mass can be derived from the small size of the Fourier coefficient corresponding to A_2 in the Siegel Eisenstein series of degree 2 and weight 16. The idea is that an even unimodular lattice represents A_2 if and only if it has non-orthogonal roots u and v (with $u \neq \pm v$).

4. Masses of unimodular lattices of dimension $n \leq 30$ with any given root system

In [9, Chapter 16], Conway and Sloane describe a correspondence between unimodular lattices of dimension $n \leq 23$ and orbits of norm 4 vectors in even unimodular 24-dimensional lattices. They use this correspondence to produce, from the list of Niemeier lattices, a list of the root systems and automorphism group orders of all unimodular lattices of dimension $n \leq 23$. Unimodular lattices of dimension $n \leq 31$ likewise correspond to orbits of norm 4 vectors in even unimodular 32-dimensional lattices, and we can use this correspondence to compute $m_n(R)$ for all root systems R and all $n \leq 30$.

Let $v = 2e$ be a vector of norm 4 in an even unimodular lattice Λ of dimension $n = 32$. Then $L_{31} = \{x \in \Lambda^\perp : x + ne \in \Lambda \text{ for some } n \in \mathrm{Z}\}$ is an odd 31-dimensional unimodular lattice, and $|\mathrm{Aut}(L_{31})| = 2|\mathrm{Aut}(\Lambda)|/c(v)$, where $c(v)$ is the number of images of v under $\mathrm{Aut}(\Lambda)$. If the lattice L_{31} has exactly $2k$ vectors of norm 1, we can write $L_{31} = L_{31-k} \oplus \mathrm{Z}^k$, where L_{31-k} has minimal norm 2 and $|\mathrm{Aut}(L_{31-k})| = |\mathrm{Aut}(L_{31})|/(2^k k!)$. (We shall call L_{31-k} the *reduced* lattice corresponding to Λ and v.) This construction gives a one-to-one correspondence between orbits of norm 4 vectors in 32-dimensional even unimodular lattices and unimodular lattices with no vectors of norm 1 in dimensions less than 32. (See [9] or [4]).

We shall mainly be concerned with norm 4 vectors v that are the sum of two orthogonal roots r and s. For such vectors, knowing just the root system of Λ allows us to compute the dimension and the root system of the reduced lattice L_{31-k}. Some useful information, distilled from [9, Table 16.8], is provided in the table below.

R	$\#r$	$\hat{R}$	shape of v	$\#v$	$\tilde{R}$	$\dim(L_{31-k})$
A_n	$4\binom{n+1}{2}$	A_{n-2}	any	$6\binom{n+1}{4}$	A_{n-4}	29
D_4	24	A_1^3	any	24	$\emptyset$	28
$D_n\,(n \geq 5)$	$4\binom{n}{2}$	D_{n-2}	$(\pm 1^4, 0^{n-4})$	$16\binom{n}{4}$	D_{n-4}	28
			$(\pm 2, 0^{n-1})$	$2n$	$\emptyset$	$32-n$
E_6	72	A_5	any	270	$\emptyset$	27
E_7	126	D_6	any	756	A_1	26
E_8	240	E_7	any	2160	$\emptyset$	24

The interpretation of this table is as follows:

Case 1: r and s are from different components R and S of the root system of Λ. Then the reduced lattice L_{31-k} has dimension 30, and has the same root system as Λ, except for the components R and S from which r and s are taken, which are transformed to $\hat{R}$ and $\hat{S}$ as given in the third column. The second column gives the number of roots $\#r$ in each component R, from which we can compute the number of vectors $v = r + s$ with $r \in R$ and $s \in S$.

Case 2: r and s are from the same component R of the root system of Λ. Then the dimension of the reduced lattice is given in the last column, and the root system of L_{32-k} is the same as that of Λ except the component R is replaced by $\tilde{R}$. Note that for $R = D_n$ with $n \geq 5$, $\tilde{R}$ and $\dim(L_{32-k})$ depend on the shape of v. The column headed $\#v$ gives the number of norm 4 vectors of that shape in the component R.

Remark 9. Norm 4 vectors that are not the sum of two orthogonal roots correspond to reduced lattices of dimension 31. For this reason we have not computed the

mass of unimodular 31-dimensional lattices with root system R. (That mass could be computed from the mass of even unimodular 40-dimensional lattices with root system $R \oplus D_9$.)

Computing the number $c(v)$ (and hence the order of $\mathrm{Aut}(L_{31-k})$ in terms of the order of $\mathrm{Aut}(\Lambda)$) sometimes requires additional information about Λ, such as whether there are automorphisms of Λ that permute the components of its root system with multiplicity greater than one. But for our purposes this is not an impediment, as we demonstrate in the following example.

Example 10. Let Λ be an even unimodular 32-dimensional lattice with root system $A_1^4 D_5$ and let $v = s + t$, where s is a root from one of the four components A_1 and t is a root from the component D_5. Then the corresponding reduced lattice L_{31-k} is 30-dimensional, has root system $A_1^4 A_3$ (since $D_3 = A_3$), and has automorphism group order $|\mathrm{Aut}(L_{30})| = (2|\mathrm{Aut}(\Lambda)|/c(v))/(2^1 \cdot 1!) = |\mathrm{Aut}(\Lambda)|/c(v)$. Since each component A_1 has 2 roots and the component D_5 has 40 roots, there are $4 \cdot 2 \cdot 40 = 320$ such vectors v, which form anywhere from one to four orbits under $\mathrm{Aut}(\Lambda)$, depending on whether there are any automorphisms permuting the components A_1. Because of this ambiguity, we do not know exactly what $c(v)$ is. But suppose the 320 vectors break into m orbits, $V_1, \ldots, V_m$, with representatives $v_1, \ldots, v_m$. Then

$$\sum_{i=1}^{m} c(v_i) = \sum_{i=1}^{m} |V_i| = 320,$$

so the total mass of the lattices that correspond to the lattice Λ and any of these vectors v is

$$\sum_{i=1}^{m} \frac{c(v_i)}{|\mathrm{Aut}(\Lambda)|} = \frac{320}{|\mathrm{Aut}(\Lambda)|}.$$

Since from Table 1 the mass of all 32-dimensional lattices Λ_j with root system $A_1^4 D_5$ is $\sum_j |\mathrm{Aut}(\Lambda_j)|^{-1} = 1029287/7464960$, these lattices contribute $\sum_j 320|\mathrm{Aut}(\Lambda_j)|^{-1}$ $= 320 \cdot 1029287/7464960$ towards the total mass of 30-dimensional unimodular lattices with root system $A_1^4 A_3$. By similarly accounting for contributions from orbits of norm 4 vectors in even unimodular 32-dimensional lattices with other root systems, we can compute the exact mass of the 30-dimensional unimodular lattices with root system $A_1^4 A_3$.

We have, in this manner, computed the mass of n-dimensional unimodular lattices having any given root system for all $n \leq 30$. We have not provided a table of these masses (as they can easily be derived from the table of masses of 32-dimensional even unimodular lattices with any given root system in [18]), but we will use these masses in Section 6 to find lower bounds on the class numbers of unimodular lattices in dimensions up to 30.

Example 11. Even unimodular 24-dimensional lattices with root system R correspond to even unimodular 32-dimensional lattices with root system $R \oplus E_8$, and $m_{24}^{\mathrm{II}}(R) = m_{32}^{\mathrm{II}}(R \oplus E_8) \cdot w(E_8) \cdot c$, where c is the multiplicity of E_8 in $R \oplus E_8$. We see from the table in [18] that there are 24 root systems of the form $R \oplus E_8$ for which $m_{32}^{\mathrm{II}}(R \oplus E_8) > 0$. Since even unimodular 24-dimensional lattices happen to be uniquely determined by their root systems, the corresponding values of $m_{24}^{\mathrm{II}}(R)$

are precisely of the form $|\mathrm{Aut}(\Lambda)|^{-1}$, where Λ is the Niemeier lattice having root system R.

5. Mass formula for unimodular lattices with no roots

Recall that $m_n(\emptyset)$ denotes the mass of the n-dimensional unimodular lattices with no roots. By considering all the orbits of norm 4 vectors $v = r + s$ in even unimodular 32-dimensional lattices that correspond to reduced lattices with no roots, we have

$$\begin{aligned}
m_n(\emptyset) &= m_{32}^{\mathrm{II}}(D_{32-n})\, w(D_{32-n}) \quad \text{for } n \le 26, n \ne 24,\\
m_{24}(\emptyset) &= m_{32}^{\mathrm{II}}(D_8)\, w(D_8) + m_{32}^{\mathrm{II}}(E_8)\, w(E_8),\\
m_{27}(\emptyset) &= m_{32}^{\mathrm{II}}(D_5)\, w(D_5) + m_{32}^{\mathrm{II}}(E_6)\, w(E_6),\\
m_{28}(\emptyset) &= m_{32}^{\mathrm{II}}(D_4)\, 3w(D_4) + m_{32}^{\mathrm{II}}(D_5)\, w(D_5),\\
m_{29}(\emptyset) &= m_{32}^{\mathrm{II}}(A_3)\, w(A_3) + m_{32}^{\mathrm{II}}(A_4)\, w(A_4),\\
m_{30}(\emptyset) &= m_{32}^{\mathrm{II}}(A_1^2)\, w(A_1^2) + m_{32}^{\mathrm{II}}(A_1A_2)\, w(A_1A_2) + m_{32}^{\mathrm{II}}(A_2^2)\, w(A_2^2).
\end{aligned}$$

These masses $m_n(\emptyset)$ may be computed by looking up the values of $m_{32}^{\mathrm{II}}(R)$ in Table 1 and [18]; the results are listed in Table 2, split into $m_n^{\mathrm{I}}(\emptyset)$ and $m_n^{\mathrm{II}}(\emptyset)$ for n divisible by 8. Table 2 also includes lower bounds on $m_n^{\mathrm{I}}(\emptyset)$ for $n = 31$ and 32, which we explain below.

Suppose Λ is an even unimodular 32-dimensional lattice with no roots. It follows from a theta function argument as in [9, Theorem 7.17] that Λ has 146880 vectors v of norm 4. For each orbit of these vectors, the corresponding reduced lattice is 31-dimensional and has no roots. Thus these lattices Λ contribute

$$\frac{146880}{2}\, m_{32}^{\mathrm{II}}(\emptyset) = \frac{22270634631794396553201589}{55292461056000} \approx 4.03 \times 10^{11}$$

towards the mass of 31-dimensional lattices with no roots. There are almost certainly additional contributions from norm 4 glue vectors v in even unimodular 32-dimensional lattices with root systems A_1^k for some k, but we will not attempt to account for these. We can restate this in terms of parity vectors:

Proposition 12. *The mass of* 31*-dimensional unimodular lattices with no roots and with no parity vectors of norm* 7 *is* $(146880/2)\, m_{32}^{II}(\emptyset)$.

Proof. Let L be a 31-dimensional unimodular lattice with no roots, and Λ the corresponding 32-dimensional even unimodular lattice. As in [9, p. 414], Λ is equal to $\{x + y \,|\, x \in L^i, y \in \frac{i}{2} + 2\mathbb{Z}, \text{for } i = 0, 1, 2, 3\} \subset (L^0)' \oplus \frac{1}{2}\mathbb{Z}$, where L^0 is the sublattice of L consisting of vectors of even norm; L^0, L^1, L^2 and L^3 are the cosets of L^0 in its dual $(L^0)'$; and $L^0 \cup L^2 = L$. The only ways the vector $x + y$ in Λ can have norm 2 is if $y = 0$ and $(x, x) = 2$ with $x \in L^0$, if $y = 1$ and $(x, x) = 1$ with $x \in L^2$, or if $y = \pm 1/2$ and $(x, x) = 7/4$ with $x \in L^1$ or $x \in L^3$. In the first two cases x would be in L, which would contradict L having no roots. Thus Λ has a root if and only if there is a vector x of norm $7/4$ in $L^1 \cup L^3$. But in this case x is in $(L^0)' \setminus L$ — the so-called *shadow* of L — and $2x$ has norm 7. Since the parity vectors of L are precisely twice the shadow vectors [9, Preface to 3rd edition, p. xxxiv], L has a parity vector of norm 7 if and only if Λ has a root. (The 31-dimensional unimodular lattices with no roots and no parity vectors of norm 7 must have parity vectors of norm 15 by [11].) □

TABLE 2. Masses of n-dimensional unimodular lattices without roots. Odd lattices appear above, even lattices below.

n	$m_n^{\mathrm{I}}(\emptyset)$	(as decimal)	number of lattices
0-22	0	0	0
23	$\frac{1}{84610842624000}$	1.18×10^{-14}	1 (shorter Leech lattice)
24	$\frac{1}{1002795171840}$	9.97×10^{-13}	1 (odd Leech lattice)
25	0	0	0
26	$\frac{1}{18720000}$	5.34×10^{-8}	1 (classified in [4])
27	$\frac{206867}{1585059840}$	1.31×10^{-4}	3 (classified in [2])
28	$\frac{17924389897}{26202009600}$	6.84×10^{-1}	38 (classified in [2])
29	$\frac{49612728929}{11136000}$	4.46×10^{3}	more than 8900
30	$\frac{7180069576834562839}{175111372800}$	4.10×10^{7}	more than 82000000
31	?	$> 4 \times 10^{11}$	more than 8×10^{11}
32	?	$> 5 \times 10^{15}$	more than 1×10^{16}
33	?	$> 4 \times 10^{20}$	more than 8×10^{20} [8]

n	$m_n^{\mathrm{II}}(\emptyset)$	(as decimal)	number of lattices
0	1	1	1 (empty lattice)
8	0	0	0
16	0	0	0
24	$\frac{1}{8315553613086720000}$	1.20×10^{-19}	1 (Leech lattice)
32	$\frac{131003733128202332665891 7}{238863431761920000}$	5.48×10^{6}	more than 10000000

We can construct 32-dimensional odd unimodular lattices with no roots from 32-dimensional even unimodular lattices with no roots as follows. Let Λ be a 32-dimensional even unimodular lattice with no roots, and let $b \in \Lambda/2\Lambda$ be a nonzero element with norm divisible by 4. Then there is a unique 32-dimensional odd lattice L containing $\Lambda_b := \{v \in \Lambda \mid (v, b) \in 2\mathbb{Z}\}$ (see [4, Chapter 0.2]). (Λ and L are *neighbors*, meaning their intersection has index two in each of them.) If b is not represented by a vector of norm 0 or 4, then L has no roots. Of the 2^{32} elements in $\Lambda/2\Lambda$, $2^{31} + 2^{15}$ have norm congruent to 0 mod 4, by Milgram's formula [27, Appendix 4] applied to the lattice $\sqrt{2}\,\Lambda$. (Milgram's formula says that if M is an even lattice and M' its dual, then

$$\sum_{u \in M'/M} \exp(2\pi i (u,u)/2) = (\det M)^{1/2} \exp(2\pi i \sigma/8),$$

where σ is the signature of M, which is equal to its dimension when M is positive definite.) Note that $(b, b) \bmod 4$ depends only on $b \bmod 2\Lambda$. Vectors v and w of norm 4 in Λ are equivalent mod 2Λ if and only if $v = \pm w$, so 146880/2

elements of $\Lambda/2\Lambda$ are represented by vectors of norm 4. Thus there are $2^{31} + 2^{15} - 146880/2 - 1 = 2147442975$ elements of $\Lambda/2\Lambda$ represented by vectors of norm divisible by 4 but not represented by vectors of norm 0 or 4. Since each odd 32-dimensional unimodular lattice has two even neighbors, this implies that $m_{32}^{\mathrm{I}}(\emptyset) \geq (2147442975/2) \cdot m_{32}^{\mathrm{II}}(\emptyset) = 5.89 \times 10^{15}$. (This is almost certainly an underestimate, since an even neighbor of an odd lattice with no roots can have roots.)

As an immediate consequence of the values we computed for $m_n(\emptyset)$ in Table 2, we have:

Corollary 13 ([9, 4, 2])**.** *There exist odd unimodular n-dimensional lattices without roots for $n = 23, 24,$ and $26, \ldots, 32$, but not for $n = 1, \ldots, 22$ or $n = 25$.*

Unimodular lattices with no roots are known to exist in all dimensions $n \geq 26$ (see [8], [28]). In dimensions $n \leq 28$ they have already been completely enumerated, and so the masses $m_n(\emptyset)$ may also be computed by summing the reciprocals of the automorphism group orders of these lattices. (Happily, this agrees with our mass formula in each case.) For the even lattices, there is the empty lattice in dimension 0, and the Leech lattice Λ_{24} in dimension 24. The Leech lattice Λ_{24} was discovered by Leech in 1965 [25] and was shown to be the unique even unimodular 24-dimensional lattice without roots by Niemeier [29] and by Conway [9, Chapter 12] around 1969. For the odd lattices, there is the shorter Leech lattice in dimension 23, the odd Leech lattice O_{24} in dimension 24 [30], the lattice S_{26} in dimension 26, 3 lattices in dimension 27, and 38 lattices in dimension 28. The lattice S_{26} was constructed by Conway in the 1970's and was shown to be the unique unimodular 26-dimensional lattice without roots by Borcherds in 1984 [4]. Borcherds also found one of the 27-dimensional lattices. The full enumerations in dimensions 27 and 28 are due to Bacher and Venkov [2].

Remark 14. In the cases of the Leech lattice and S_{26}, with our mass formula the uniqueness follows immediately from the constructions, simply by verifying that $|\mathrm{Aut}(\Lambda_{24})|^{-1} = m_{24}^{\mathrm{II}}(\emptyset)$ and $|\mathrm{Aut}(S_{26})|^{-1} = m_{26}^{\mathrm{I}}(\emptyset)$.

Examples of unimodular lattices with no roots have been constructed for dimensions 29 to 32 (including the 15 exceptional even unimodular 32-dimensional lattices classified in [24]), and a nonconstructive analytic argument shows that they exist in all dimensions $n \geq 33$ (see the Conway-Thompson Theorem [27, p. 46] for $n \geq 37$, and [7] for $33 \leq n \leq 36$.) In fact for $n \geq 33$ this argument gives a lower bound for the mass of lattices without roots that is close to the total mass of the genus, so there are a great many lattices without roots (see Remark 16). The idea is that the coefficients a_1 and a_2 of the average theta series

$$\frac{1}{m} \sum_{\Lambda \in \Omega} \frac{\Theta_\Lambda(q)}{|\mathrm{Aut}(\Lambda)|} = 1 + a_1 q^1 + a_2 q^2 + a_3 q^3 + \cdots$$

give the average number of vectors of norm 1 and norm 2, taken over all the lattices in the genus Ω. If $a_1 + a_2$ is less than 2, then there must be some $\Lambda \in \Omega$ with no vectors of norm 1 or 2 — this is the case when Ω is the genus of n-dimensional unimodular lattices for $n \geq 33$. For $n = 33$, $a_1 + a_2$ is approximately 1.42, and this implies that $m_{33}^{\mathrm{I}}(\emptyset) \geq 4.04 \times 10^{20}$, so there are more than 8×10^{20} 33-dimensional unimodular lattices without roots [8]. For $n \leq 32$, the average number of roots is greater than 2, so this argument does not apply. Notice that the coefficient a_i is

the same as the average number of representations $a(N)$ we defined in Section 2, in the special case where N is the 1×1 matrix (i).

Remark 15. In dimensions $n \leq 33$, odd unimodular lattices with no roots have minimal norm 3, except in dimension 32, in which they can have minimal norm 3 or 4. In dimensions 24 and 32, even unimodular lattices with no roots have minimal norm 4. See [8].

Remark 16. Let m'_n be the mass of odd n-dimensional unimodular lattices with only trivial automorphisms. There are no such lattices for $n \leq 28$ [2], but Bacher has found one for $n = 29$ [1]. Bannai [3] showed that $m'_n/m_n \to 1$ as $n \to \infty$. For $n > 1$ any lattice with roots has nontrivial automorphisms, so $m'_n/m_n \leq m_n(\emptyset)/m_n$. Below we list $m_n(\emptyset)/m_n$ for $26 \leq n \leq 30$, and lower bounds on $m_n(\emptyset)/m_n$ for $31 \leq n \leq 33$.

n	$m_n(\emptyset)/m_n$	ref.
26	0.000116	[4]
27	0.000856	[2]
28	0.00658	[2]
29	0.0300	
30	0.0908	
31	>0.135	
32	>0.136	
33	>0.287	[8]

6. Lower bounds on class numbers

Let Ω be the set of inequivalent lattices in a genus of dimension $n > 0$, m the mass of that genus, and $m(R)$ the mass of those lattices having root system R. Each lattice $\Lambda \in \Omega$ has at least two automorphisms, $1 : x \mapsto x$ and $-1 : x \mapsto -x$; from this we get the well-known lower bound $|\Omega| \geq \lceil 2m \rceil$. For each root r of Λ, the reflection

$$x \mapsto x - 2\frac{(x,r)}{(r,r)}r$$

is also in $\mathrm{Aut}(\Lambda)$. Define $w'(R)$ to be the order of the subgroup of $\mathrm{Aut}(\Lambda)$ generated by reflections and by -1. Then $w'(R) = w(R)$ if the map -1 is already in the Weyl group of R, and $w'(R) = 2w(R)$ otherwise. (The Weyl group of R contains -1 if and only if $\mathrm{rank}(R) = \dim(\Lambda)$ and each component of R is A_1, E_7, E_8 or D_k for even k.) Then there are at least $\lceil m(R)\, w'(R) \rceil$ lattices with root system R, so we get an improved lower bound,

$$|\Omega| \geq \sum_R \lceil m(R)\, w'(R) \rceil.$$

We can do slightly better still, as follows: Write $m(R)\, w'(R) = q + a/b$ with $q, a, b \in \mathrm{Z}, a < b$, and $\gcd(a, b) = 1$, and define a modified ceiling function by $\langle q + a/b \rangle = q$ if $a = 0$, $q + 1$ if $a = 1$, and $q + 2$ if $a > 1$. It can easily be shown that there are at least $\langle m(R)\, w'(R) \rangle$ lattices with root system R, so that

$$|\Omega| \geq \sum_R \langle m(R)\, w'(R) \rangle. \tag{1}$$

Evaluating this sum with the value of $m_{32}^{\mathrm{II}}(R)$ we computed for each R gives $|\Omega| \geq 1162109024$.

Corollary 17. *There are at least one billion even unimodular* 32*-dimensional lattices.*

TABLE 3. Comparison of the actual number α_n of unimodular lattices of dimension n, the lower bound β_n computed from (1), the number r_n of distinct root systems occuring (including those with components Z), and the Minkowski-Siegel mass constant m_n. Odd lattices are listed above, even lattices below. The last column is included only when $2m_n$ provides a nontrivial lower bound for α_n.

dim n	actual α_n^{I}	$\geq$	bound β_n^{I}	r_n^{I}	mass m_n^{I}	$\beta_n^{\mathrm{I}}/2m_n^{\mathrm{I}}$
0	0	$\geq$	0	0	0	
1	1	$\geq$	1	1	0.5	
2	1	$\geq$	1	1	0.125	
3	1	$\geq$	1	1	2.083×10^{-2}	
4	1	$\geq$	1	1	2.604×10^{-3}	
5	1	$\geq$	1	1	2.604×10^{-4}	
6	1	$\geq$	1	1	2.170×10^{-5}	
7	1	$\geq$	1	1	1.551×10^{-6}	
8	1	$\geq$	1	1	9.688×10^{-8}	
9	2	$\geq$	2	2	6.100×10^{-9}	
10	2	$\geq$	2	2	4.485×10^{-10}	
11	2	$\geq$	2	2	4.213×10^{-11}	
12	3	$\geq$	3	3	5.267×10^{-12}	
13	3	$\geq$	3	3	9.031×10^{-13}	
14	4	$\geq$	4	4	2.186×10^{-13}	
15	5	$\geq$	5	5	7.705×10^{-14}	
16	6	$\geq$	6	6	4.093×10^{-14}	
17	9	$\geq$	9	9	3.402×10^{-14}	
18	13	$\geq$	13	13	4.583×10^{-14}	
19	16	$\geq$	16	16	1.033×10^{-13}	
20	28	$\geq$	28	28	4.002×10^{-13}	
21	40	$\geq$	40	40	2.735×10^{-12}	
22	68	$\geq$	68	68	3.377×10^{-11}	
23	117	$\geq$	117	117	7.710×10^{-10}	
24	273	$\geq$	273	266	3.330×10^{-8}	
25	665	$\geq$	657	609	2.781×10^{-6}	
26	?	$\geq$	2307	1695	4.586×10^{-4}	
27	?	$\geq$	14179	4492	1.524×10^{-1}	
28	?	$\geq$	327972	9213	1.040×10^{2}	1569.2
29	?	$\geq$	37938009	20298	1.486×10^{5}	127.7
30	?	$\geq$	20169641025	67848	4.520×10^{8}	22.3
31	?	$\geq$	?	?	2.980×10^{12}	
32	?	$\geq$	?	?	4.328×10^{16}	

dim n	actual α_n^{II}	$\geq$	bound β_n^{II}	r_n^{II}	mass m_n^{II}	$\beta_n^{\mathrm{II}}/2m_n^{\mathrm{II}}$
0	1	$\geq$	1	1	1	
8	1	$\geq$	1	1	1.435×10^{-9}	
16	2	$\geq$	2	2	2.489×10^{-18}	
24	24	$\geq$	24	24	7.937×10^{-15}	
32	?	$\geq$	1162109024	13218	4.031×10^{7}	14.4

For comparison, $\lceil 2m_{32}^{\mathrm{II}} \rceil = 80618466$, so our lower bound is 14.4 times larger than the lower bound obtained by doubling the Minkowski-Siegel mass constant.

We also computed $m_n(R)$ for each $n \le 30$ and each root system R, and used (1) to find lower bounds on the number of odd unimodular lattices. (In the cases where a single $m_n(R)$ is expressed as a sum of masses corresponding to different root systems of 32-dimensional even unimodular lattices, such as $m_{28}(\emptyset) = m_{32}^{\mathrm{II}}(D_4)\,3w(D_4) + m_{32}^{\mathrm{II}}(D_5)\,w(D_5)$, we bounded the number for each summand individually, which gives a better overall bound.)

Unimodular lattices in dimensions $n \le 25$ have been completely enumerated: even unimodular lattices of dimension 8 by Mordell, of dimension 16 by Witt, and of dimension 24 by Niemeier [29] (see also Venkov [33]); odd unimodular lattices of dimension $n \le 16$ by Kneser [23], of dimension $n \le 23$ by Conway and Sloane [9, Chapter 16], and of dimension 24 and 25 by Borcherds [4].

Table 3 gives our lower bound β_n on the number of unimodular lattices in dimension $n \le 30$, and our computation of the number of distinct root systems r_n that occur in these lattices (including root systems with components Z, which do not occur for even lattices). The table includes for comparison the actual number α_n of unimodular lattices in dimension $n \le 25$, taken from [9, Table 2.2], and the Minkowski-Siegel mass constants m_n, taken from [9, Tables 16.3 and 16.5]. (The counts of odd unimodular lattices include those with vectors of norm 1 to facilitate comparison with m_n; since any integral lattice with a vector of norm 1 is of the form $\mathrm{Z}^k \oplus \Lambda$ where Λ has minimal (nonzero) norm 2, the counts of lattices with no vectors of norm 1 can be recovered from β_n; similarly, the number of distinct root systems with no components Z can be recovered from r_n, with the caution that 8 of the 24 root systems of even unimodular 24-dimensional lattices also occur as root systems of odd unimodular 24-dimensional lattices.) Our lower bounds agree exactly with the actual numbers for $n \le 24$. This is to be expected for $n \le 23$ since in those dimensions an odd or even n-dimensional unimodular lattice is uniquely determined by its root system R, and $\langle m(R)\,w'(R)\rangle$ is exactly equal to the number of lattices with root system R when there are 0 or 1 such lattices. This is also true for 24-dimensional even unimodular lattices. Our lower bound in dimension 25 is within two percent of the actual number, and our lower bounds in dimensions 26 to 30 (for which the actual numbers are not known) are the best we are aware of.

7. Computing the numbers $a(R)$

For a half-integral $n \times n$ matrix B, define

$$c_{n,k}(B) = (-1)^{nk/2} 2^{n(k-(n-1)/2)} (\det B)^{(2k-n-1)/2}\, b(B,k) \prod_{i=2k-n+1}^{2k} \frac{\pi^{i/2}}{\Gamma(i/2)},$$

where

$$b(B,k) = \sum_{R \in S_n(\mathrm{Q})/S_n(\mathrm{Z})} \exp(2\pi i \operatorname{tr}(BR)) \mu(R)^{-k}$$

is the *Siegel series* (with $\mu(R)$ equal to the product of denominators of elementary divisors of R). Put $\epsilon_{n,k} = 1/2$ if $n = k-1$ or $n = k > 1$, and 1 otherwise.

Theorem 18 (Siegel)**.** *Let N be a lattice with $\dim(N) = n \leq 8k$, and let*

$$a(N) = \frac{1}{m_{8k}^{II}} \sum_{\Lambda \in \Omega} \frac{r(\Lambda, N)}{|\mathrm{Aut}(\Lambda)|}$$

with the sum taken over even unimodular lattices of dimension $8k$. Then $a(N) = \epsilon_{n,8k} c_{n,4k}(B)$, where B is $1/2$ times the Gram matrix of N.

Proof. See [21, Theorem 6.8.1], and note that the product of local densities

$$\prod_p \alpha_p(M_p, N_p)$$

is equal to $b(M, k)$ when N is an even unimodular lattice of dimension $2k$ [20]. □

Remark 19. The *Siegel Eisenstein series* of degree n and weight k is defined to be

$$E_{n,k}(Z) = \sum_{\{C,D\}} |CZ + D|^{-k},$$

where $\{C, D\}$ runs over all representatives of the equivalence classes of coprime pairs of $n \times n$ matrices. For $k > n$, $c_{n,k}(C)$ is the coefficient in the Fourier expansion

$$E_{n,k}(Z) = \sum_C c_{n,k}(C) \exp(2\pi i \operatorname{tr}(CZ)),$$

where C runs over all positive semi-definite half-integral $n \times n$ matrices.

The first explicit formula for $b(B, k)$ for arbitrary n is due to Katsurada [15], and was published in 1999. Prior to this, an explicit formula for the coefficients of $E_{n,k}(Z)$ was known only for $n \leq 3$: The case $n = 1$ is well-known (see for example [32, Chapter VII]), Maaß [26] gives an explicit formula for $n = 2$ (see also Kaufhold [16]), and Katsurada [14] gives an explicit formula for $n = 3$ (extending partial results by Kitaoka [19]).

Let B be a nondegenerate symmetric half-integral $n \times n$ matrix over Z_p. It follows from [22] that

$$b(B,s) = \left(\zeta(s) \prod_{i=1}^{\lfloor r/2 \rfloor} \zeta(2s-2i)\right)^{-1} \prod_{p \mid D(B)} F_p(B; p^{-s}) \times \begin{cases} L(s - r/2; \chi_B) & \text{if } r \text{ is even,} \\ 1 & \text{if } r \text{ is odd,} \end{cases}$$

for certain polynomials $F_p(B; X)$. Here ζ is the Riemann zeta function, and $L(s, \chi_B)$ is a Dirichlet L-series whose values may be computed using the method in [7].

The explicit formula for $F_p(B; X)$ in [15, Theorem 4.3] is not itself well-suited for calculation, since the outer index of summation takes 2^n values, but we can use Katsurada's recursion relations [15, Theorems 4.1 and 4.2] as part of a practical algorithm for computing $F_p(B; X)$. We will state these recursion relations below (without proof); first we will need to introduce some of the notation from [15].

For $a = p^r c$ with $r \in \mathrm{Z}$ and $c \in \mathrm{Z}_p^*$, define $\chi_p(a) = (\frac{c}{p})$ for r even and 0 for r odd (where $(\frac{-}{p})$ is the Legendre symbol mod p), and define

$$\chi_2(a) = \begin{cases} +1 & \text{if } r \equiv 0 \bmod 2,\ c \equiv 1 \bmod 8, \\ -1 & \text{if } r \equiv 0 \bmod 2,\ c \equiv 5 \bmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathrm{ord}_p(a)$ to be the exact power of p dividing a, and define $i_p(B)$ to be the least integer t for which $p^t B^{-1}$ is half-integral. Let $(\ ,\ )_p$ denote the Hilbert symbol

over $\mathbb{Q}_p$ (see [32]), and let h_p denote the Hasse invariant (see [21]). For odd n, define

$$\eta_p(B) = h_p(B)(\det B, (-1)^{(n-1)/2}\det B)_p(-1,-1)_p^{(n^2-1)/8}.$$

For even n, define

$$\xi_p(B) = \chi_p((-1)^{n/2}\det B)$$

and $\xi'_p(B) = 1 + \xi_p(B) - \xi_p(B)^2$. By convention, $\xi_p(B) = \xi'_p(B) = 1$ for B the empty matrix. Define $D(B) = 2^{2\lfloor n/2\rfloor}\det B$, $d_p(B) = \mathrm{ord}_p(D(B))$, and

$$\delta_p(B) = \begin{cases} 2\lfloor (d_p(B)+1-\delta_{2p})/2\rfloor & \text{if } n \text{ is even,} \\ d_p(B) & \text{if } n \text{ is odd,} \end{cases}$$

where δ_{2p} is the Kronecker delta.

We shall suppress most of the subscripts p in what follows. Let p be any prime and suppose B and B_2 are nondegenerate half-integral matrices of rank n and $n-1$ respectively over $\mathbb{Z}_p$. Put $\delta = \delta(B)$ and $\tilde{\delta} = \delta(B_2)$. If n is even, put $\xi = \xi(B)$, $\xi' = \xi'(B)$, and $\tilde{\eta} = \eta(B_2)$; if n is odd, put $\tilde{\xi} = \xi(B_2)$, $\tilde{\xi}' = \xi'(B_2)$, and $\eta = \eta(B)$. (By convention $\tilde{\eta} = \tilde{\eta}' = 1$ if $B_2 = \emptyset$, and $\tilde{\delta} = 0$ if $n = 1$.) Then define rational functions $C(B, B_2; X)^{(1)}$ and $C(B, B_2; X)^{(0)}$ in X by

$$C(B,B_2;X)^{(1)} = \begin{cases} \frac{1-p^{n/2}\xi X}{1-p^{n+1}X^2} & \text{if } n \text{ is even,} \\ \frac{1}{1-p^{(n+1)/2}\tilde{\xi}X} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$C(B,B_2;X)^{(0)} = \begin{cases} \frac{(-1)^{\xi+1}\xi'\tilde{\eta}(1-p^{n/2+1}X\xi)(p^{n/2}X)^{\delta-\tilde{\delta}+\xi^2}p^{\delta/2})}{1-p^{n+1}X^2} & \text{if } n \text{ is even,} \\ \frac{(-1)^{\tilde{\xi}}\tilde{\xi}'\eta(p^{(n-1)/2}X)^{\delta-\tilde{\delta}+2-\tilde{\xi}^2}p^{(2\delta-\tilde{\delta}+2)/2}}{1-p^{(n+1)/2}\tilde{\xi}X} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 20 (Katsurada [15])**.** *Let $B_1 = (b_1)$ and B_2 be nondegenerate half-integral matrices of degree 1 and $n-1$, respectively, over $\mathbb{Z}_p$, and put $B = B_1 \perp B_2$. Assume that $ord(b_1) \geq i(B_2) - 1 + 2\delta_{2p}$. Then we have*

$$F_p(B;X) = C(B,B_2;X)^{(1)}F(B_2;pX) + C(B,B_2;X)^{(0)}F_p(B_2;X).$$

Let B_2 be a nondegenerate half-integral matrix of degree $n-2$ over $\mathbb{Z}_2$, let $H = \left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right)$ and let $Y = \left(\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1 \end{smallmatrix}\right)$. Let $B_1 = 2^m K$ with $K = H$ or Y, or $B_1 = 2^m u_1 \perp 2^m u_2$ with $u_1, u_2 \in \mathbb{Z}_2^*$, and put $B = B_1 \perp B_2$, $\delta = \delta(B)$, $\tilde{\delta} = \delta(2^m \perp B_2)$, $\hat{\delta} = \delta(B_2)$, and

$$\sigma = \begin{cases} (2\tilde{\delta} - \delta - \hat{\delta} + 2)/2 & \text{if } n \text{ is even, } B_1 = 2^m u_1 \perp 2^m u_2 \text{ and } d(B) \text{ is odd} \\ & \text{or if } n \text{ is even, } B_1 = 2^m K, \text{ and } \xi(B_2) = 0, \\ 2 & \text{if } n \text{ is odd, } B_1 = 2^m K, \text{ and } d(2^m \perp B_2) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If n is even, put $\xi = \xi(B)$, $\xi' = \xi'(B)$, $\hat{\xi} = \xi(B_2)$, $\hat{\xi}' = \xi'(B_2)$, and

$$\tilde{\eta} = \begin{cases} \eta(2^m u_2 \perp B_2) & \text{if } B_1 = 2^m u_1 \perp 2^m u_2, \\ & \text{and } d(B_2) \text{ is even,} \\ (-1)^{((n-1)^2-1)/8}h(B_2)(2^m, (-1)^{(n-1)/2}\det B_2)_2 & \text{if } B_1 = 2^m K, \\ & \text{and } \xi(B_2) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

If n is odd, put $\eta = \eta(B)$, $\hat{\eta} = \eta(B_2)$, $\tilde{\xi}' = 1$, and $\tilde{\xi} = 1$ if $B_1 = 2^m K$ and $d(2^m \perp B_2)$ is even, and 0 otherwise. Define four rational functions in X by

$$C(B, B_2; X)^{(11)} = \begin{cases} \frac{1-2^{n/2}\xi X}{1-2^{n+1}X^2} & \text{if } n \text{ is even,} \\ \frac{1}{1-2^{(n+1)/2}\tilde{\xi}X} & \text{if } n \text{ is odd,} \end{cases}$$

$$C(B, B_2; X)^{(10)} = \begin{cases} \frac{(-1)^{\xi+1}\xi'\tilde{\eta}(1-2^{n/2+1}X\xi)(2^{n/2}X)^{\delta-\tilde{\delta}+\xi^2+\sigma}2^{\delta/2})}{1-2^{n+1}X^2} & \text{if } n \text{ is even,} \\ \frac{(-1)^{\tilde{\xi}}\eta(2^{(n-1)/2}X)^{\delta-\tilde{\delta}+2-\tilde{\xi}^2+\sigma}2^{(2\delta-\tilde{\delta}+2+\sigma)/2}}{1-2^{(n+1)/2}\tilde{\xi}X} & \text{if } n \text{ is odd,} \end{cases}$$

$$C(B, B_2; X)^{(21)} = \begin{cases} \frac{1}{1-2^{n/2}\hat{\xi}X} & \text{if } n \text{ is even,} \\ \frac{1-2^{(n-1)/2}\tilde{\xi}X}{1-2^n X^2} & \text{if } n \text{ is odd,} \end{cases}$$

$$C(B, B_2; X)^{(20)} = \begin{cases} \frac{(-1)^{\hat{\xi}}\hat{\xi}'\tilde{\eta}(2^{(n-2)/2}X)^{\tilde{\delta}-\hat{\delta}+2-\hat{\xi}^2-\sigma}2^{(2\tilde{\delta}-\hat{\delta}+2-2\sigma)/2}}{1-2^{n/2}\hat{\xi}X} & \text{if } n \text{ is even,} \\ \frac{(-1)^{\tilde{\xi}+1}\hat{\eta}(1-2^{(n+1)/2}X\tilde{\xi})(2^{(n-1)/2}X)^{\tilde{\delta}-\hat{\delta}+\tilde{\xi}^2-\sigma}2^{(\tilde{\delta}-\sigma)/2})}{1-2^n X^2} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 21 (Katsurada [15])**.** *Let $B_1 = 2^m u_1 \perp 2^m u_2$ with $u_1, u_2 \in \mathrm{Z}_2^*$ or $B = 2^m K$ with $K = H$ or Y. Let B_2 be a half-integral matrix of degree $n-2$ over Z_2 which is also in $GL_{n-2}(\mathrm{Q}_2)$, and put $B = B_1 \perp B_2$. Assume that $m \geq i(B_2) + 1$. Then we have*

$$\begin{aligned} F_2(B; X) &= C(B, B_2; X)^{(11)} C(B, B_2; 2X)^{(21)} F_2(B_2; 4X) \\ &+ C(B, B_2; X)^{(11)} C(B, B_2; 2X)^{(20)} F_2(B_2; 2X) \\ &+ C(B, B_2; X)^{(10)} C(B, B_2; X)^{(21)} F_2(B_2; 2X) \\ &+ C(B, B_2; X)^{(10)} C(B, B_2; X)^{(20)} F_2(B_2; X). \end{aligned}$$

For $p \neq 2$, any nondegenerate symmetric half-integral $n \times n$ matrix B can be diagonalized over the p-adic integers Z_p, $B \cong p^{e_1} u_1 \perp \ldots \perp p^{e_n} u_n$ with $e_1 \geq \cdots \geq e_n \geq 0$ and with $u_i \in \{1, \epsilon\}$ for all i, where ϵ is any quadratic nonresidue (see [34]). Then $F_p(B; X)$ can be computed by repeated applications of Theorem 20 above.

Note that in the course of recursively computing $F_p(B; X)$ for $X = p^{-k}$ in this manner, one occasionally encounters zeros of the denominators of the functions $C^{(1)}$ and $C^{(0)}$. (This is only a problem when X is a negative power of p.) Rather than attempting to simplify the expressions symbolically, we instead computed $F_p(B; 1), F_p(B; p), \ldots, F_p(B; p^n)$, and then used Lagrangian interpolation to compute $F_p(B; p^{-k})$. The reason for choosing $X = 1, p, \ldots, p^n$ rather than $n+1$ other numbers is that for these numbers the recursive subproblems overlap: we need only evaluate $F_p(p^{e_i} u_i \perp \ldots \perp p^{e_n} u_n; X)$ for $X = 1, p, \ldots, p^{2n-i}$, starting with $i = n$, and working down to $i = 1$, which gives $F_p(B; 1), \ldots, F_p(B; p^n)$.

For $p = 2$, any nondegenerate symmetric half-integral matrix B is equivalent over Z_2 to a matrix of the form $2^{e_1}(U_1 \perp V_1) \perp \ldots \perp 2^{e_m}(U_m \perp V_m)$ with $e_1 > \cdots > e_m \geq 0$, $U_i = \emptyset$ or u_1 or $u_1 \perp u_2$ for $u_1, u_2 \in \{\pm 1, \pm 3\}$, and $V_i = \emptyset$ or $H \perp \cdots \perp H$ or $H \perp \cdots \perp H \perp Y$ (see [34]). Then $F_2(B; X)$ can be computed by repeated applications of Theorems 20 and 21 above. (We again used an interpolation scheme, similar to the one described for $p \neq 2$.)

8. Computing the numbers $r(R_i, R_j)$

The method described in this section is essentially the one used for some of the computations in [6] (although the algorithm itself is not described in that paper). In this section we write $r(R, R')$ as $\mathrm{emb}(R', R)$, since a representation of R' by R is the same as a linear map from R' into R that preserves inner products. For any irreducible root systems S and T it is routine to compute the number of embeddings $\mathrm{emb}(S, T)$ of S into T, and to determine the root system of the orthogonal complement of S in T for each of these embeddings. There are at most two orbits of embeddings of S into T, so we write $\mathrm{emb}(S, T) = \mathrm{emb}_1(S, T) + \mathrm{emb}_2(S, T)$, where there are $\mathrm{emb}_1(S, T)$ embeddings of S into T for which the orthogonal complement is $\mathrm{comp}_1(S, T)$ and there are $\mathrm{emb}_2(S, T)$ embeddings of S into T for which the orthogonal complement is $\mathrm{comp}_2(S, T)$.

Example 22. We demonstrate how to compute the two orbits of embeddings of A_3 into D_n. Recall from [9] that $A_n = \{(x_1, \ldots, x_{n+1}) \in \mathrm{Z}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}$ and $D_n = \{(x_1, \ldots, x_n) \in \mathrm{Z}^n : x_1 + \cdots + x_n \in 2\mathrm{Z}\}$. A_3 is generated by three roots $v_1 = (1, -1, 0, 0)$, $v_2 = (0, 1, -1, 0)$, and $v_3 = (0, 0, 1, -1)$ with $(v_1, v_1) = (v_2, v_2) = (v_3, v_3) = 2$, $(v_1, v_2) = (v_2, v_3) = -1$ and $(v_1, v_3) = 0$. We can map v_1 to any of the $4C(n, 2)$ roots r_1 in D_n (all permutations of $(\pm 1, \pm 1, 0, \ldots, 0)$), where $C(n, k)$ denotes the binomial coefficient.

We can then map v_2 to any of the $4(n-2)$ roots r_2 of D_n that have inner product -1 with r_1. Let us say r_1 is supported in coordinates i and j and r_2 is supported in coordinates j and k. (Clearly $k \neq i$.) Then there are two cases:

(a) We can map v_3 to the root r_3 supported in coordinates i and j that has inner product 0 with r_1 and inner product -1 with r_2. In this case the roots of D_n orthogonal to r_1, r_2, and r_3 form the system D_{n-3}.

(b) We can map v_3 to $2(n-3)$ roots r_3 supported in coordinates k and l ($l \neq i, j$) that have inner product 0 with r_1 and inner product -1 with r_2. In this case the roots of D_n orthogonal to r_1, r_2, and r_3 form the system D_{n-4}.

Hence there are $4C(n, 2) \cdot 4(n-2) \cdot 1 = C(n, 3) \cdot |\mathrm{Aut}(A_3)|$ ways to embed A_3 into D_n with complement D_{n-3}, and there are $4C(n, 2) \cdot 4(n-2) \cdot 2(n-3) = 2^3 C(n, 4) \cdot |\mathrm{Aut}(A_3)|$ ways to embed A_3 into D_n with complement D_{n-4}.

The computations for other irreducible root systems are similar. All nonzero values of emb_1, emb_2, comp_1, and comp_2 are given in Table 4.

If R and R' are root lattices, then we can write $R = S_1 \oplus \cdots \oplus S_k$ and $R' = T_1 \oplus \cdots \oplus T_m$, where each S_i and T_i is an irreducible root lattice, and we can compute $r(R', R)$ recursively via the formula

$$\begin{aligned} r(R', R) &= \mathrm{emb}(R, R') \\ &= \mathrm{emb}\Big(\bigoplus_{i \le k} S_i, \bigoplus_{i \le m} T_i\Big) \\ &= \sum_{j=1}^{m} \mathrm{emb}_1(S_k, T_j)\, \mathrm{emb}\Big(\bigoplus_{i \le k-1} S_i, \mathrm{comp}_1(S_k, T_j) \oplus \bigoplus_{i \le m, i \neq j} T_i\Big) \\ &+ \sum_{j=1}^{m} \mathrm{emb}_2(S_k, T_j)\, \mathrm{emb}\Big(\bigoplus_{i \le k-1} S_i, \mathrm{comp}_2(S_k, T_j) \oplus \bigoplus_{i \le m, i \neq j} T_i\Big). \end{aligned}$$

TABLE 4. Embeddings of irreducible root systems into one another. (In the fourth and last columns, A_0, D_0, D_1, D_2 and D_3 should be interpreted as $\emptyset, \emptyset, \emptyset, A_1A_1$ and A_3, respectively.)

S	T	$\frac{\mathrm{emb}_1(S,T)}{\lvert\mathrm{Aut}(S)\rvert}$	$\mathrm{comp}_1(S,T)$	$\frac{\mathrm{emb}_2(S,T)}{\lvert\mathrm{Aut}(S)\rvert}$	$\mathrm{comp}_2(S,T)$
$\emptyset$	T	1	T	$-$	$-$
A_i	$A_j\ (j \geq i)$	$C(j+1, i+1)$	A_{j-i-1}	$-$	$-$
A_1	D_j	$C(j,2) \cdot 2$	A_1D_{j-2}	$-$	$-$
A_3	D_j	$C(j,4) \cdot 2^3$	D_{j-4}	$C(j,3)$	D_{j-3}
$A_i\ (i \neq 1,3)$	$D_j\ (j > i)$	$C(j, i+1) \cdot 2^i$	D_{j-i-1}	$-$	$-$
D_i	$D_j\ (j \geq i)$	$C(j,i)$	D_{j-i}	$-$	$-$
A_1	E_6	$2^2 3^2$	A_5	$-$	$-$
A_2	E_6	$2^3 3 \cdot 5$	A_2A_2	$-$	$-$
A_3	E_6	$2 \cdot 3^3 5$	A_1A_1	$-$	$-$
A_4	E_6	$2^3 3^3$	A_1	$-$	$-$
A_5	E_6	$2^2 3^2$	A_1	$-$	$-$
D_4	E_6	$3^2 5$	$\emptyset$	$-$	$-$
D_5	E_6	3^3	$\emptyset$	$-$	$-$
E_6	E_6	1	$\emptyset$	$-$	$-$
A_1	E_7	$3^2 7$	D_6	$-$	$-$
A_2	E_7	$2^4 3 \cdot 7$	A_5	$-$	$-$
A_3	E_7	$2^2 3^2 5 \cdot 7$	A_3A_1	$-$	$-$
A_4	E_7	$2^5 3^2 7$	A_2	$-$	$-$
A_5	E_7	$2^4 3 \cdot 7$	A_2	$2^4 3^2 7$	A_1
A_6	E_7	$2^5 3^2$	$\emptyset$	$-$	$-$
A_7	E_7	$2^2 3^2$	$\emptyset$	$-$	$-$
D_4	E_7	$3^2 5 \cdot 7$	$A_1A_1A_1$	$-$	$-$
D_5	E_7	$2 \cdot 3^3 7$	A_1	$-$	$-$
D_6	E_7	$3^2 7$	A_1	$-$	$-$
E_6	E_7	$2^2 7$	$\emptyset$	$-$	$-$
E_7	E_7	1	$\emptyset$	$-$	$-$
A_1	E_8	$2^3 3 \cdot 5$	E_7	$-$	$-$
A_2	E_8	$2^5 5 \cdot 7$	E_6	$-$	$-$
A_3	E_8	$2^3 3^3 5 \cdot 7$	D_5	$-$	$-$
A_4	E_8	$2^7 3^3 7$	A_4	$-$	$-$
A_5	E_8	$2^7 3^2 5 \cdot 7$	A_2A_1	$-$	$-$
A_6	E_8	$2^8 3^3 5$	A_1	$-$	$-$
A_7	E_8	$2^5 3^3 5$	A_1	$2^6 3^3 5$	$\emptyset$
A_8	E_8	$2^6 3 \cdot 5$	$\emptyset$	$-$	$-$
D_4	E_8	$2 \cdot 3^2 5^2 7$	D_4	$-$	$-$
D_5	E_8	$2^3 3^3 5 \cdot 7$	A_3	$-$	$-$
D_6	E_8	$2^2 3^3 5 \cdot 7$	A_1A_1	$-$	$-$
D_7	E_8	$2^3 3^3 5$	$\emptyset$	$-$	$-$
D_8	E_8	$3^3 5$	$\emptyset$	$-$	$-$
E_6	E_8	$2^5 5 \cdot 7$	A_2	$-$	$-$
E_7	E_8	$2^3 3 \cdot 5$	A_1	$-$	$-$
E_8	E_8	1	$\emptyset$	$-$	$-$

A direct implementation of this algorithm does a lot of redundant computation on certain inputs, some of which we can circumvent with dynamic programming or *memoization* (see [10, Chapter 16]). Our implementation also does several things to reduce the amount of computation when there are direct summands in R or R' with multiplicity greater than one. But computing $r(R', R)$ is an NP-hard problem, since it is an NP-complete problem to determine if $r(R', R) > 0$, as we show below.

Proposition 23. *The problem of determining whether a root system R embeds into a root system R' is NP-complete.*

Proof. We shall reduce 3-Partition (see [13]) to this problem. Let $S = \{s_1, \ldots, s_{3k}\}$ be an instance of 3-Partition, with the s_i positive integers summing to kt. Then S can be partitioned into k sets each consisting of 3 elements with sum t if and only if the root system $D_{4s_1} \oplus \cdots \oplus D_{4s_{3k}}$ embeds into the root system $\bigoplus_{i=1}^{k} D_{4t}$. □

Remark 24. Since 3-Partition is strongly NP-complete, the problem of whether one root system embeds into another remains NP-complete if the Gram matrices of root lattices are used as input (rather that the list of components A_i, D_i, and E_i as above).

Since we need to compute $r(R, R')$ for all pairs of root systems rather than just one pair, the amortized computational cost would be reduced considerably by using dynamic programming. But the dynamic programming table becomes unmanageably large in dimension 32, so we instead use a hash table, which is purged periodically, for memoization.

9. Eliminating root systems *a priori*

There are 405844 root systems of rank $n \leq 32$ with no vectors of norm 1, corresponding to all direct sums of $A_i, D_i\,(i \geq 4), E_6, E_7$ and E_8, where the order of the summands does not matter and the sum of the subscripts is at most 32. Since computing the number of embeddings $r(R_i, R_j)$ can be time-consuming, we used the following congruences, due to Borcherds [4], to eliminate some root systems from consideration:

Let roots(R) denote the number of roots of R. If R is the root system of a 32-dimensional even unimodular lattice, then:

if R contains E_8, then $\text{roots}(R) \equiv 0 \pmod{24}$,
if R contains E_7, then $\text{roots}(R) \equiv 0 \pmod{12}$,
if R contains E_6, then $\text{roots}(R) \equiv 0 \pmod{6}$,
if R contains D_6, then $\text{roots}(R) \equiv 0 \pmod{4}$,
if R contains D_7, then $\text{roots}(R) \equiv 0 \pmod{8}$,
if R contains D_8, then $\text{roots}(R) \equiv 0 \pmod{8}$,
if R contains $D_n, n > 8$, then $\text{roots}(R) \equiv 0 \pmod{16}$.

Also note that if $a(R) = 0$, then $m(R) = 0$. If $\text{rank}(R) = 32$ and $\det(R)$ is not a perfect square, then $a(R)$ must be 0, so we eliminated those root systems as well. This left 135443 root systems. We ordered them so that $\dim(R_i) \leq \dim(R_j)$ if $i < j$, and so that $\det(R_i) \geq \det(R_j)$ if $i < j$ and $\dim(R_i) = \dim(R_j)$.

As the matrix U with $U_{i,j} = r(R_i, R_j)$ would still contain around 10 billion elements, we did not explicitly construct and invert it. Rather, we computed each element in the matrix when it was required for solving $\frac{1}{m}Uv = w$ by back-substitution, with

$$m(R_i) = \frac{1}{r(R_i, R_i)} \left\{ m \cdot a(R_i) - \sum_{j>i} r(R_j, R_i)\, m(R_j) \right\}.$$

If $m(R_j)$ has already been computed to be 0, then the values $r(R_j, R_i)$ need not be computed, since they make no contribution to this sum.

Acknowledgments

The author would like to thank Richard Borcherds for many helpful suggestions, Richard Fateman for advice on Lisp, and Xerox PARC for the use of its computers.

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