MATHEMATICS OF COMPUTATION
Volume 72, Number 242, Pages 865–889
S 0025-5718(02)01470-9
Article electronically published on October 18, 2002

RELATIVE INVARIANTS OF SOME 2-SIMPLE PREHOMOGENEOUS VECTOR SPACES

TAKEYOSHI KOGISO, GO MIYABE, MIYUKI KOBAYASHI, AND TATSUO KIMURA

ABSTRACT. In this paper, we shall construct explicitly irreducible relative invariants of two 2-simple prehomogeneous vector spaces. Together with a preprint by the same authors, this completes the list of all relative invariants of regular 2-simple prehomogeneous vector spaces of type I.

1. INTRODUCTION

Let G be a connected reductive algebraic group defined over the complex number field $\mathbb{C}$, V a finite dimensional vector space, and $\rho : G \longrightarrow GL(V)$ a rational representation of G. Such a triplet (G, ρ, V) is called a prehomogeneous vector space (abbreviated P.V.) if V has an open G-orbit, and a triplet is called irreducible if ρ is an irreducible representation. Furthermore, such a triplet (G, ρ, V) is called simple (resp. n-simple) if the derived subgroup $[G, G]$ is a simple algebraic group (resp. the product of n-simple algebraic groups). A nonzero rational function $F(x)$ is called a relative invariant corresponding to a character $\chi : G \to GL_1$ if it satisfies the relation $F(\rho(g)x) = \chi(g)F(x)$ as a rational function for all $g \in G$.

A complete list of irreducible prehomogeneous vector spaces is given by M. Sato and T. Kimura in [1]. At the same time, the relative invariants are constructed for almost all of these spaces. However, for some complicated prehomogeneous vector spaces, such as classification numbers (6), (7), (10), (20), (21) and (24) in [1], the construction of relative invariants had not been settled. In 1971, an irreducible relative invariant of (20) was constructed in [6] and, in 1981, that of (6), (7) was constructed in [3]. In 1990, relative invariants for (10), (21), (24) were constructed in [7] by some complicated calculations. In 1995, that of (10) and (21) was constructed in [8] by using the notion of the quotient space. For the case of nonirreducible prehomogeneous vector spaces, in 1983, T. Kimura studied the case of nonirreducible simple prehomogeneous vector spaces. In 1988, T. Kimura, S. Kasai, M. Inuzuka and O. Yasukura [9] completed the classifications of nonirreducible reduced 2-simple prehomogeneous vector spaces of type I. See [9] for the definition of type I and type II.

Recently, the relative invariants are constructed in [10] and [11] for almost all of these spaces of type I except for the following two cases:

(regular 8) $(GL_1^2 \times SL_5 \times SL_8, \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, \mathrm{Alt}_5^{\oplus 8} \oplus V(8)^*)$,

(regular 40) $(GL_1^2 \times \mathrm{Spin}_{10} \times SL_{14}, (\text{a half-spin rep.}) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, V(16)^{\oplus 14} \oplus V(14)^*)$.

Received by the editor April 20, 2000 and, in revised form, May 29, 2001.
2000 *Mathematics Subject Classification.* Primary 11S05, 11S90.

These cases are the most complicated and difficult cases in [9]. The purpose of this paper is to construct explicitly irreducible relative invariants of the above 2 cases and to complete the construction of irreducible relative invariants for all 2-simple prehomogeneous vector spaces of type I. In this paper, we have reduced this construction problem to determine some polynomials. Although we used the computer software *Mathematica* [12] to decide the explicit form of polynomials, once we obtain them, it is not necessary to use the computer to check them.

2. Notations and preliminaries

We denote by Alt_n (resp. Sym_n) the totality of $n\times n$ alternating matrices (resp. $n\times n$ symmetric matrices). For $X\in\mathrm{Alt}_{2n}$, let $Pf(X)$ be the Pfaffian of X so that we have $Pf(X)^2=\det X$ and $Pf(AX^tA)=\det A\cdot Pf(X)$ for $A\in GL_{2n}$. We denote Λ' (resp. χ) the even half-spin representation (resp. the vector representation) of Spin_{10}. Note that the dual representation Λ'^* of Λ' is the odd half-spin representation of Spin_{10}. For the infinitesimal representation of Λ', see (5.38) in [1].

For $X\in M_n$ (=the totality of $n\times n$ matrices), let $\Delta(X)$ be the cofactor matrix of X so that we have $X\cdot\Delta(X)=\Delta(X)\cdot X=\det X\cdot I_n$.

When we prove the irreducibility of relative invariants, we often use the following facts.

Lemma 2.1 (cf. [1]). *Let (G,ρ,V) be a P.V. with a generic point v_0.*

(1) *For a character $\chi:G\to GL_1$, there exists a relative invariant corresponding to χ if and only if $\chi|_{G_{v_0}}=1$, where $G_{v_0}=\{g\in G;\rho(g)v_0=v_0\}$.*
(2) *Any irreducible component of a relative invariant is also a relative invariant.*

Lemma 2.2 (cf. [1], §4 Proposition 18). *There is a one-to-one correspondence between relative invariants $f(x)$ of $(G\times SL_n,\rho\otimes\Lambda_1,V(m)\otimes V(n))$ $(m>n\geq 1)$ and relative invariants $\tilde f(\tilde x)$ of its castling transform $(G\times SL_{m-n},\rho^*\otimes\Lambda_1,V(m)^*\otimes V(m-n))$. Moreover, there exists a positive integer d for each $f(x)$ such that $\deg f(x)=nd$ and $\deg\tilde f(\tilde x)=(m-n)d$. If f is irreducible, then $\tilde f$ is also irreducible.*

Moreover, we prove the following lemma to construct some G-equivariant mapping in the first example.

Lemma 2.3. *For $X=(x_{ij})=(x_1\mid\cdots\mid x_n)\in M(n+2,n)$, let $X^{(i,j)}\in M(n)$ be the matrix obtained from X by subtracting i-th and j-th rows. For $i<j$, put $\tilde{x_{ij}}=(-1)^{i+j+1}\det X^{(i,j)}$ and define the alternating matrix $\varphi(X)=(\tilde{x_{ij}})\in\mathrm{Alt}_{n+2}$. Then the map $\varphi:M(n+2,n)\to\mathrm{Alt}_{n+2}$ satisfies $\varphi(AX)=\det A\cdot{}^tA^{-1}\varphi(X)A^{-1}$ for $A\in GL_{n+2}$.*

Proof. Put $Y=AX=(y_1\mid\cdots\mid y_n)$. For $i<j$, we have

$$\begin{aligned}
&e_i\wedge e_j\wedge x_1\wedge\cdots\wedge x_n\\
&\quad=\det X^{(i,j)}e_i\wedge e_j\wedge e_1\wedge\cdots\wedge e_{i-1}\wedge e_{i+1}\wedge\cdots\wedge e_{j-1}\wedge e_{j+1}\wedge\cdots\wedge e_{n+2}\\
&\quad=\tilde{x_{ij}}e_1\wedge\cdots\wedge e_{n+2}.
\end{aligned}$$

Then, by the action of $A = (a_{ij}) \in GL_{n+2}$, we have

$$\begin{aligned}
\tilde{x}_{ij} \det A \cdot e_1 \wedge \cdots \wedge e_{n+2} &= \tilde{x_{ij}}(Ae_1) \wedge \cdots \wedge (Ae_{n+2}) \\
&= \left(\sum_{l=1}^{n+2} e_l a_{li}\right) \wedge \left(\sum_{k=1}^{n+2} e_k a_{ki}\right) \wedge y_1 \wedge \cdots \wedge y_n \\
&= \sum_{l<k} a_{li} a_{kj} e_l \wedge e_k \wedge y_1 \wedge \cdots \wedge y_n - \sum_{k<l} a_{li} a_{kj} e_k \wedge e_l \wedge y_1 \wedge \cdots \wedge y_n \\
&= \sum_{l<k} (a_{li} a_{kj} - a_{ki} a_{lj}) e_l \wedge e_k \wedge y_1 \wedge \cdots \wedge y_n \\
&= \sum_{l<k} (a_{li} a_{kj} - a_{ki} a_{lj}) \tilde{y_{lk}} \wedge e_1 \wedge \cdots \wedge e_{n+2}.
\end{aligned}$$

Hence, we have $\sum_{l<k}(a_{li}a_{kj} - a_{ki}a_{lj})\tilde{y_{lk}} = \det A \cdot \tilde{x_{ij}}$, which is equivalent to ${}^tA\varphi(Y)A = \det A \cdot \varphi(X)$. This implies that $\varphi(AX) = \det A \cdot {}^tA^{-1}\varphi(X)A^{-1}$. □

3. Explicit construction of irreducible relative invariants

In this section, H denotes the generic isotropy subgroup of a P.V. (G, ρ, V), and $H_1 \sim H_2$ implies that H_1 and H_2 are locally isomorphic, namely their Lie algebras are isomorphic. N denotes the number of the basic irreducible relative invariants.

3.1. **Explicit construction of irreducible relative invariants of** $(GL_1^2 \times SL_5 \times SL_8, \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$ **with** $H \sim SO_2,\ N = 2$**.** To investigate relative invariants, we may assume that $G = GL_5 \times GL_8$ acts on $V = \mathrm{Alt}_5^{\oplus 8} \oplus M(8,1)$ by

$$x = ((X_1, X_2, \cdots, X_8), Y) \mapsto ((AX_1{}^tA, \cdots, AX_8{}^tA){}^tB, {}^tB^{-1}Y)$$

for $x = ((X_1, \cdots, X_8), Y) \in V$ and $g = (A, B) \in G$. We shall construct the irreducible relative invariants of this prehomogeneous vector space by the following steps.

Step 1. For $Y = {}^t(y_1, \cdots, y_8) \in M(8,1)$, we put $X \cdot Y = \sum_{i=1}^{8} y_i X_i \in \mathrm{Alt}_5$. Then we obtain that

$$X \cdot Y \mapsto AX \cdot Y{}^tA \tag{3.1}$$

and hence we have

$$\Delta(X \cdot Y) \mapsto (\det A)^{2\,t}A^{-1}\Delta(X \cdot Y)A^{-1}, \tag{3.2}$$

where $\Delta(X \cdot Y)$ is the cofactor matrix of the odd size alternating matrix the $X \cdot Y \in \mathrm{Alt}_5$. Note that $\Delta(X \cdot Y) \in \mathrm{Sym}_5$.

Step 2. For

$$X_i = \begin{pmatrix} 0 & x_{12}^{(i)} & x_{13}^{(i)} & x_{14}^{(i)} & x_{15}^{(i)} \\ -x_{12}^{(i)} & 0 & x_{23}^{(i)} & x_{24}^{(i)} & x_{25}^{(i)} \\ -x_{13}^{(i)} & -x_{23}^{(i)} & 0 & x_{34}^{(i)} & x_{35}^{(i)} \\ -x_{14}^{(i)} & -x_{24}^{(i)} & -x_{34}^{(i)} & 0 & x_{45}^{(i)} \\ -x_{15}^{(i)} & -x_{25}^{(i)} & -x_{35}^{(i)} & -x_{45}^{(i)} & 0 \end{pmatrix} \in \mathrm{Alt}_5 \ (i = 1, 2, \cdots, 8),$$

we put $\tilde{X}_i := {}^t(x_{12}^{(i)}, x_{13}^{(i)}, x_{14}^{(i)}, x_{15}^{(i)}, x_{23}^{(i)}, x_{24}^{(i)}, x_{25}^{(i)}, x_{34}^{(i)}, x_{35}^{(i)}, x_{45}^{(i)}) \in M(10,1)$.

The action $X_i \mapsto AX_i{}^tA$ induces $\tilde{X}_i \mapsto \Lambda_2(A)\tilde{X}_i$ ($\Lambda_2(A) \in GL_{10}$). Then we put $Z = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_8) \in M(10, 8)$ and define the 10×10-alternating matrix $\tilde{Z} = (z_{ij})_{1 \le i < j \le 10}$ with $z_{ij} = (-1)^{i+j} \det Z^{(i,j)}$, where $Z^{(i,j)}$ is the 8×8-matrix obtained from Z by substracting the i-th and the j-th rows. Then, by Lemma 2.3, we have $\tilde{Z} \mapsto (\det \Lambda_2(A))(\det B)^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}$. Note that $\det \Lambda_2(A) = (\det A)^4$.

Step 3. If we put $\Phi(\tilde{Z}) := (\varphi(\tilde{Z})_{i,j})_{1 \le i \le j \le 5} \in \mathrm{Sym}_5$ for $\tilde{Z} \in \mathrm{Alt}_{10}$ with the following entries, then we have Lemma 3.1.

$$\varphi_{11} = z_{510}^2 + z_{69}^2 + z_{78}^2 - 2z_{59}z_{610} - 2z_{68}z_{79} + 2z_{58}z_{710} - 2z_{67}z_{89} + 2z_{57}z_{810} - 2z_{56}z_{910},$$
$$\varphi_{22} = z_{210}^2 + z_{39}^2 + z_{48}^2 - 2z_{29}z_{310} + 2z_{28}z_{410} - 2z_{38}z_{49} + 2z_{24}z_{810} - 2z_{34}z_{89} - 2z_{23}z_{910},$$
$$\varphi_{33} = z_{110}^2 + z_{37}^2 + z_{46}^2 - 2z_{17}z_{310} - 2z_{36}z_{47} + 2z_{16}z_{410} - 2z_{34}z_{67} + 2z_{14}z_{610} - 2z_{13}z_{710},$$
$$\varphi_{44} = z_{19}^2 + z_{27}^2 + z_{45}^2 - 2z_{17}z_{29} - 2z_{25}z_{47} + 2z_{15}z_{49} - 2z_{24}z_{57} + 2z_{14}z_{59} - 2z_{12}z_{79},$$
$$\varphi_{55} = z_{18}^2 + z_{26}^2 + z_{35}^2 - 2z_{16}z_{28} - 2z_{25}z_{36} + 2z_{15}z_{38} - 2z_{23}z_{56} + 2z_{13}z_{58} - 2z_{12}z_{68},$$
$$\begin{aligned}\varphi_{12} = &-z_{210}z_{510} - z_{410}z_{58} + z_{310}z_{59} + z_{29}z_{610} + z_{49}z_{68} - z_{39}z_{69}\\ &-z_{28}z_{710} - z_{48}z_{78} + z_{38}z_{79} - z_{27}z_{810} + z_{45}z_{810} + z_{37}z_{89} - z_{46}z_{89} + z_{26}z_{910} - z_{35}z_{910},\end{aligned}$$
$$\begin{aligned}\varphi_{13} = &\ z_{110}z_{510} + z_{410}z_{56} - z_{310}z_{57} - z_{19}z_{610} - z_{45}z_{610} + z_{39}z_{67}\\ &-z_{48}z_{67} - z_{47}z_{68} + z_{46}z_{69} + z_{18}z_{710} + z_{35}z_{710} + z_{37}z_{78} - z_{36}z_{79} + z_{17}z_{810} - z_{16}z_{910},\end{aligned}$$
$$\begin{aligned}\varphi_{14} = &\ z_{45}z_{510} - z_{49}z_{56} + z_{210}z_{57} + z_{48}z_{57} + z_{47}z_{58} - z_{110}z_{59}\\ &-z_{46}z_{59} - z_{29}z_{67} + z_{19}z_{69} - z_{25}z_{710} - z_{27}z_{78} - z_{18}z_{79} + z_{26}z_{79} - z_{17}z_{89} + z_{15}z_{910},\end{aligned}$$
$$\begin{aligned}\varphi_{15} = &-z_{35}z_{510} - z_{210}z_{56} + z_{39}z_{56} - z_{38}z_{57} + z_{110}z_{58} - z_{37}z_{58}\\ &+z_{36}z_{59} + z_{25}z_{610} + z_{28}z_{67} - z_{19}z_{68} + z_{27}z_{68} - z_{26}z_{69} + z_{18}z_{78} - z_{15}z_{810} + z_{16}z_{89},\end{aligned}$$
$$\begin{aligned}\varphi_{23} = &-z_{110}z_{210} + z_{19}z_{310} + z_{27}z_{310} - z_{37}z_{39} - z_{18}z_{410} - z_{26}z_{410}\\ &+z_{38}z_{47} - z_{46}z_{48} + z_{36}z_{49} - z_{24}z_{610} + z_{34}z_{69} + z_{23}z_{710} - z_{34}z_{78} - z_{14}z_{810} + z_{13}z_{910},\end{aligned}$$
$$\begin{aligned}\varphi_{24} = &-z_{210}z_{27} + z_{110}z_{29} + z_{29}z_{37} - z_{19}z_{39} + z_{25}z_{410} - z_{28}z_{47}\\ &+z_{45}z_{48} + z_{18}z_{49} - z_{35}z_{49} + z_{24}z_{510} - z_{34}z_{59} + z_{24}z_{78} - z_{23}z_{79} + z_{14}z_{89} - z_{12}z_{910},\end{aligned}$$
$$\begin{aligned}\varphi_{25} = &\ z_{210}z_{26} - z_{110}z_{28} - z_{25}z_{310} - z_{29}z_{36} + z_{19}z_{38} + z_{35}z_{39}\\ &-z_{38}z_{45} + z_{28}z_{46} - z_{18}z_{48} - z_{23}z_{510} + z_{34}z_{58} - z_{24}z_{68} + z_{23}z_{69} + z_{12}z_{810} - z_{13}z_{89},\end{aligned}$$
$$\begin{aligned}\varphi_{34} = &-z_{110}z_{19} + z_{17}z_{210} - z_{27}z_{37} + z_{17}z_{39} - z_{15}z_{410} - z_{45}z_{46}\\ &+z_{26}z_{47} + z_{35}z_{47} - z_{16}z_{49} - z_{14}z_{510} + z_{34}z_{57} + z_{24}z_{67} - z_{14}z_{69} + z_{12}z_{710} + z_{13}z_{79},\end{aligned}$$
$$\begin{aligned}\varphi_{35} = &\ z_{110}z_{18} - z_{16}z_{210} + z_{15}z_{310} + z_{27}z_{36} - z_{35}z_{37} - z_{17}z_{38}\\ &+z_{36}z_{45} - z_{26}z_{46} + z_{16}z_{48} + z_{13}z_{510} - z_{34}z_{56} - z_{12}z_{610} - z_{23}z_{67} + z_{14}z_{68} - z_{13}z_{78},\end{aligned}$$
$$\begin{aligned}\varphi_{45} = &-z_{18}z_{19} - z_{26}z_{27} + z_{17}z_{28} + z_{16}z_{29} + z_{25}z_{37} - z_{15}z_{39}\\ &-z_{35}z_{45} + z_{25}z_{46} - z_{15}z_{48} + z_{24}z_{56} + z_{23}z_{57} - z_{14}z_{58} - z_{13}z_{59} + z_{12}z_{69} + z_{12}z_{78}.\end{aligned}$$

Lemma 3.1. *For every $A \in GL_5, B \in GL_8$, $\tilde{Z} \in \mathrm{Alt}_{10}$, we have*

$$\tilde{Z} \mapsto (\det A)^4(\det B)^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}$$

and

$$\Phi((\det A)^4(\det B)\ {}^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}) = (\det A)^6(\det B)^2 \cdot A\Phi(\tilde{Z})^tA. \tag{3.3}$$

Proof. It is enough to prove the equivariance (3.3) in the case when A is one of the fundamental matrices

$$A_u = \begin{pmatrix} 1 & \varepsilon & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad A_d = \mathrm{diag}(a, 1, 1, 1, 1),$$

or permutation matrices.

Checking (3.3) for diagonal or permutation matrices is easy. Note that $\det A_d = a$ and $\Phi(a^{4t}\Lambda_2(A_d)^{-1}\tilde{Z}\Lambda_2(A_d)^{-1}) = a^6 A_d \Phi(\tilde{Z})^t A_d$. For A_u, we consider the action of A_u. Since $\det A_u = 1$,

$$\Phi(\tilde{Z}) \mapsto \Phi({}^t\Lambda_2(A_u)^{-1}\tilde{Z}\Lambda_2(A_u)^{-1}).$$

Then we have $\varphi_{11} \mapsto \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^2\varphi_{22}$, $\varphi_{1j} \mapsto \varphi_{1j} + \varepsilon\varphi_{2j}$ $(2 \le j \le 5)$, $\varphi_{lk} \mapsto \varphi_{lk}$ $(2 \le l \le k \le 5)$. Hence, we have $\Phi({}^t\Lambda_2(A_u)^{-1}\tilde{Z}\Lambda_2(A_u)^{-1}) = A_u\Phi(\tilde{Z})^t A_u$. □

Remark A. By using the computer software *Mathematica* [12], we calculate the above polynomials φ_{ij} $(1 \le i \le j \le 5)$ along the following program.

(i) First, we construct the polynomial φ_{11}. For the action of the diagonal matrix $\mathrm{diag}(a_1, a_2, a_3, a_4, a_5)$, denote by T_1 the polynomial corresponding to the weight $a_1^2(\det A)^{10}$:

$$\begin{aligned} T_1 = {} & k_1 z_{56}z_{910} + k_2 z_{57}z_{810} + k_3 z_{58}z_{710} + k_4 z_{59}z_{610} + k_5 z_{510}^2 + k_6 z_{510}z_{69} \\ & + k_7 z_{510}z_{78} + k_8 z_{69}^2 + k_9 z_{69}z_{78} + k_{10} z_{78}^2 + k_{11} z_{67}z_{89} + k_{12} z_{68}z_{79}. \end{aligned}$$

(ii) Next, we calculate the invariant polynomial TT_1 under the action of permutation matrices (23), (24), (25), (35), (45) on T_1:

$$\begin{aligned} TT_1 = {} & l_1(z_{56}z_{910} + z_{67}z_{89} - z_{57}z_{810} + z_{68}z_{79} - z_{58}z_{710} + z_{59}z_{610}) \\ & + l_2(z_{510}^2 + z_{69}^2 + z_{78}^2). \end{aligned}$$

(iii) Next, we calculate the invariant polynomial TTT_1 under the action of unipotent matrices:

$$\begin{pmatrix} 1 & & & & \\ & 1 & \varepsilon & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & \varepsilon & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \varepsilon \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & \varepsilon & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \varepsilon \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \varepsilon \\ & & & & 1 \end{pmatrix},$$

and we can construct the polynomial φ_{11} uniquely up to the constant

$$\begin{aligned} \varphi_{11} = {} & z_{510}^2 + z_{69}^2 + z_{78}^2 - 2z_{59}z_{610} - 2z_{68}z_{79} \\ & - 2z_{58}z_{710} - 2z_{67}z_{89} - 2z_{57}z_{810} - 2z_{56}z_{910}. \end{aligned}$$

(iv) From the explicit form of φ_{11}, we can construct the other φ_{ij} by the action of the generators of GL_5 on φ_{11}.

Step 4. If we put $\Psi(\tilde{Z}) := (\psi(\tilde{z})_{i,j})_{1\le i<j\le 5} \in \mathrm{Alt}_5$ for $\tilde{Z} \in \mathrm{Alt}_{10}$ with the entries below, then we have Lemma 3.2.

$$
\begin{aligned}
\psi_{12} = & -z_{110}z_{15}z_{210} + z_{110}^2 z_{25} - z_{110}z_{19}z_{26} + 2z_{17}z_{210}z_{26} + z_{110}z_{18}z_{27} \\
& - 2z_{16}z_{210}z_{27} - z_{110}z_{17}z_{28} + z_{110}z_{16}z_{29} + z_{15}z_{19}z_{310} - 3z_{17}z_{25}z_{310} \\
& + 2z_{15}z_{27}z_{310} - z_{110}z_{19}z_{35} + z_{17}z_{210}z_{35} + z_{19}^2 z_{36} + 2z_{27}^2 z_{36} \\
& - 3z_{17}z_{29}z_{36} - z_{18}z_{19}z_{37} - 2z_{26}z_{27}z_{37} + z_{17}z_{28}z_{37} + 2z_{16}z_{29}z_{37} \\
& - 2z_{27}z_{35}z_{37} + 2z_{25}z_{37}^2 + z_{17}z_{19}z_{38} - z_{17}z_{27}z_{38} - z_{16}z_{19}z_{39} + z_{17}z_{26}z_{39} \\
& + 2z_{17}z_{35}z_{39} - 2z_{15}z_{37}z_{39} - z_{15}z_{18}z_{410} + 3z_{16}z_{25}z_{410} \\
& - 2z_{15}z_{26}z_{410} - z_{15}z_{35}z_{410} + z_{110}z_{18}z_{45} - z_{16}z_{210}z_{45} + z_{15}z_{310}z_{45} \\
& + 2z_{27}z_{36}z_{45} - 2z_{35}z_{37}z_{45} - 2z_{17}z_{38}z_{45} + 2z_{36}z_{45}^2 - z_{18}z_{19}z_{46} - 2z_{26}z_{27}z_{46} \\
& + 2z_{17}z_{28}z_{46} + z_{16}z_{29}z_{46} + 2z_{25}z_{37}z_{46} - z_{15}z_{39}z_{46} - 2z_{26}z_{45}z_{46} - 2z_{35}z_{45}z_{46} \\
& + 2z_{25}z_{46}^2 + z_{18}^2 z_{47} + 2z_{26}^2 z_{47} - 3z_{16}z_{28}z_{47} + 2z_{26}z_{35}z_{47} + 2z_{35}^2 z_{47} - 6z_{25}z_{36}z_{47} \\
& + 3z_{15}z_{38}z_{47} - z_{17}z_{18}z_{48} + z_{16}z_{27}z_{48} - z_{15}z_{37}z_{48} + 2z_{16}z_{45}z_{48} \\
& - 2z_{15}z_{46}z_{48} + z_{16}z_{18}z_{49} - z_{16}z_{26}z_{49} - 2z_{16}z_{35}z_{49} \\
& + 3z_{15}z_{36}z_{49} + z_{110}z_{12}z_{510} - z_{17}z_{23}z_{510} + z_{16}z_{24}z_{510} - z_{14}z_{26}z_{510} \\
& + z_{13}z_{27}z_{510} - 2z_{14}z_{35}z_{510} + 2z_{13}z_{45}z_{510} - z_{14}z_{210}z_{56} - z_{27}z_{34}z_{56} \\
& + z_{24}z_{37}z_{56} + z_{14}z_{39}z_{56} + z_{12}z_{410}z_{56} - z_{34}z_{45}z_{56} + z_{24}z_{46}z_{56} \\
& - 2z_{23}z_{47}z_{56} - z_{13}z_{49}z_{56} + z_{13}z_{210}z_{57} - z_{12}z_{310}z_{57} + z_{26}z_{34}z_{57} + z_{34}z_{35}z_{57} \\
& - 2z_{24}z_{36}z_{57} + z_{23}z_{37}z_{57} - z_{14}z_{38}z_{57} + z_{23}z_{46}z_{57} + z_{13}z_{48}z_{57} + z_{110}z_{14}z_{58} \\
& + z_{17}z_{34}z_{58} - 2z_{14}z_{37}z_{58} - z_{14}z_{46}z_{58} + 3z_{13}z_{47}z_{58} - z_{110}z_{13}z_{59} - z_{16}z_{34}z_{59} \\
& + 3z_{14}z_{36}z_{59} - z_{13}z_{37}z_{59} - 2z_{13}z_{46}z_{59} - z_{12}z_{19}z_{610} - z_{15}z_{24}z_{610} \\
& + 3z_{14}z_{25}z_{610} - z_{12}z_{27}z_{610} - 2z_{12}z_{45}z_{610} + z_{24}z_{26}z_{67} - z_{23}z_{27}z_{67} + z_{14}z_{28}z_{67} \\
& - z_{13}z_{29}z_{67} - 2z_{25}z_{34}z_{67} + z_{24}z_{35}z_{67} + z_{12}z_{39}z_{67} - z_{23}z_{45}z_{67} - z_{12}z_{48}z_{67} \\
& - z_{14}z_{19}z_{68} - z_{17}z_{24}z_{68} + 2z_{14}z_{27}z_{68} + z_{14}z_{45}z_{68} - 3z_{12}z_{47}z_{68} + z_{13}z_{19}z_{69} \\
& + z_{17}z_{23}z_{69} - 2z_{14}z_{26}z_{69} + z_{15}z_{34}z_{69} - z_{14}z_{35}z_{69} + z_{12}z_{37}z_{69} \\
& + 2z_{12}z_{46}z_{69} + z_{12}z_{18}z_{710} + z_{15}z_{23}z_{710} - 3z_{13}z_{25}z_{710} + z_{12}z_{26}z_{710} \\
& + 2z_{12}z_{35}z_{710} + z_{14}z_{18}z_{78} + z_{16}z_{24}z_{78} - 2z_{13}z_{27}z_{78} - z_{15}z_{34}z_{78} \\
& + 2z_{12}z_{37}z_{78} - z_{13}z_{45}z_{78} + z_{12}z_{46}z_{78} - z_{13}z_{18}z_{79} - z_{16}z_{23}z_{79} \\
& + 2z_{13}z_{26}z_{79} + z_{13}z_{35}z_{79} - 3z_{12}z_{36}z_{79} - 2z_{14}z_{15}z_{810} + 2z_{12}z_{17}z_{810} \\
& + 2z_{14}z_{16}z_{89} - 2z_{13}z_{17}z_{89} + 2z_{13}z_{15}z_{910} - 2z_{12}z_{16}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{13} = & -z_{15}z_{210}^2 + z_{110}z_{210}z_{25} + z_{19}z_{210}z_{26} - z_{18}z_{210}z_{27} - 2z_{110}z_{19}z_{28} + z_{17}z_{210}z_{28} \\
& + 2z_{110}z_{18}z_{29} - z_{16}z_{210}z_{29} - 2z_{19}z_{25}z_{310} - z_{25}z_{27}z_{310} \\
& + 3z_{15}z_{29}z_{310} + z_{210}z_{27}z_{35} - z_{110}z_{29}z_{35} - z_{19}z_{29}z_{36} + z_{27}z_{29}z_{36} - z_{27}z_{28}z_{37} \\
& + z_{18}z_{29}z_{37} - 2z_{29}z_{35}z_{37} + 2z_{19}^2 z_{38} + z_{27}^2 z_{38} - 3z_{17}z_{29}z_{38} - 2z_{18}z_{19}z_{39} \\
& - z_{26}z_{27}z_{39} + 2z_{17}z_{28}z_{39} + z_{16}z_{29}z_{39} + 2z_{19}z_{35}z_{39} + 2z_{25}z_{37}z_{39} \\
& - 2z_{15}z_{39}^2 + 2z_{18}z_{25}z_{410} + z_{25}z_{26}z_{410} - 3z_{15}z_{28}z_{410} - z_{25}z_{35}z_{410} - z_{210}z_{26}z_{45} \\
& + z_{110}z_{28}z_{45} + z_{25}z_{310}z_{45} + 2z_{29}z_{36}z_{45} - 2z_{19}z_{38}z_{45} - 2z_{35}z_{39}z_{45} \\
& + 2z_{38}z_{45}^2 + z_{19}z_{28}z_{46} - z_{26}z_{29}z_{46} + z_{25}z_{39}z_{46} - 2z_{28}z_{45}z_{46} - z_{18}z_{28}z_{47}
\end{aligned}
$$

$$
\begin{aligned}
&+ z_{26}z_{28}z_{47} + 2z_{28}z_{35}z_{47} - 3z_{25}z_{38}z_{47} - 2z_{18}z_{19}z_{48} - z_{26}z_{27}z_{48} + z_{17}z_{28}z_{48} \\
&+ 2z_{16}z_{29}z_{48} + z_{25}z_{37}z_{48} - 2z_{15}z_{39}z_{48} + 2z_{18}z_{45}z_{48} - 2z_{35}z_{45}z_{48} \\
&+ 2z_{25}z_{46}z_{48} - 2z_{15}z_{48}^2 + 2z_{18}^2z_{49} + z_{26}^2z_{49} - 3z_{16}z_{28}z_{49} - 2z_{18}z_{35}z_{49} + 2z_{35}^2z_{49} \\
&- 3z_{25}z_{36}z_{49} + 6z_{15}z_{38}z_{49} + z_{12}z_{210}z_{510} - z_{19}z_{23}z_{510} + z_{18}z_{24}z_{510} - z_{14}z_{28}z_{510} \\
&+ z_{13}z_{29}z_{510} - 2z_{24}z_{35}z_{510} + 2z_{23}z_{45}z_{510} - z_{210}z_{24}z_{56} - z_{29}z_{34}z_{56} + 2z_{24}z_{39}z_{56} \\
&+ z_{24}z_{48}z_{56} - 3z_{23}z_{49}z_{56} + z_{210}z_{23}z_{57} + z_{28}z_{34}z_{57} - 3z_{24}z_{38}z_{57} + z_{23}z_{39}z_{57} \\
&+ 2z_{23}z_{48}z_{57} + z_{110}z_{24}z_{58} + z_{19}z_{34}z_{58} - z_{24}z_{37}z_{58} - z_{14}z_{39}z_{58} + z_{12}z_{410}z_{58} \\
&- z_{34}z_{45}z_{58} + z_{23}z_{47}z_{58} - z_{14}z_{48}z_{58} + 2z_{13}z_{49}z_{58} - z_{110}z_{23}z_{59} - z_{12}z_{310}z_{59} \\
&- z_{18}z_{34}z_{59} + z_{34}z_{35}z_{59} + z_{24}z_{36}z_{59} + 2z_{14}z_{38}z_{59} - z_{13}z_{39}z_{59} - z_{23}z_{46}z_{59} \\
&- z_{13}z_{48}z_{59} + 2z_{24}z_{25}z_{610} - 2z_{12}z_{29}z_{610} + 2z_{24}z_{28}z_{67} - 2z_{23}z_{29}z_{67} \\
&- 2z_{19}z_{24}z_{68} + z_{24}z_{27}z_{68} + z_{14}z_{29}z_{68} + z_{24}z_{45}z_{68} - 3z_{12}z_{49}z_{68} + 2z_{19}z_{23}z_{69} \\
&- z_{24}z_{26}z_{69} - z_{14}z_{28}z_{69} - z_{25}z_{34}z_{69} + 2z_{12}z_{39}z_{69} - z_{23}z_{45}z_{69} + z_{12}z_{48}z_{69} \\
&- 2z_{23}z_{25}z_{710} + 2z_{12}z_{28}z_{710} + 2z_{18}z_{24}z_{78} - z_{23}z_{27}z_{78} - z_{13}z_{29}z_{78} + z_{25}z_{34}z_{78} \\
&- z_{24}z_{35}z_{78} + z_{12}z_{39}z_{78} + 2z_{12}z_{48}z_{78} - 2z_{18}z_{23}z_{79} + z_{23}z_{26}z_{79} \\
&+ z_{13}z_{28}z_{79} + z_{23}z_{35}z_{79} - 3z_{12}z_{38}z_{79} + z_{12}z_{19}z_{810} \\
&- 3z_{15}z_{24}z_{810} + z_{14}z_{25}z_{810} + z_{12}z_{27}z_{810} - 2z_{12}z_{45}z_{810} + z_{14}z_{18}z_{89} \\
&- z_{13}z_{19}z_{89} - z_{17}z_{23}z_{89} + z_{16}z_{24}z_{89} + 2z_{15}z_{34}z_{89} \\
&- z_{14}z_{35}z_{89} - z_{12}z_{37}z_{89} + z_{13}z_{45}z_{89} + z_{12}z_{46}z_{89} - z_{12}z_{18}z_{910} \\
&+ 3z_{15}z_{23}z_{910} - z_{13}z_{25}z_{910} - z_{12}z_{26}z_{910} + 2z_{12}z_{35}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{14} = {}& 2z_{110}z_{18}z_{210} - 2z_{16}z_{210}^2 + 2z_{110}z_{210}z_{26} - 2z_{110}^2z_{28} - 2z_{18}z_{19}z_{310} \\
&+ z_{15}z_{210}z_{310} - z_{110}z_{25}z_{310} - z_{19}z_{26}z_{310} - z_{18}z_{27}z_{310} - 2z_{26}z_{27}z_{310} \\
&+ 3z_{17}z_{28}z_{310} + 3z_{16}z_{29}z_{310} + 2z_{210}z_{27}z_{36} \\
&- 2z_{110}z_{29}z_{36} + z_{25}z_{310}z_{37} - z_{210}z_{35}z_{37} - z_{29}z_{36}z_{37} - z_{28}z_{37}^2 \\
&+ 2z_{110}z_{19}z_{38} - 2z_{17}z_{210}z_{38} + z_{27}z_{37}z_{38} - z_{15}z_{310}z_{39} \\
&+ z_{110}z_{35}z_{39} + z_{19}z_{36}z_{39} + z_{18}z_{37}z_{39} + z_{26}z_{37}z_{39} \\
&- z_{17}z_{38}z_{39} - z_{16}z_{39}^2 + 2z_{18}^2z_{410} + 2z_{18}z_{26}z_{410} + 2z_{26}^2z_{410} \\
&- 6z_{16}z_{28}z_{410} + z_{35}^2z_{410} - 3z_{25}z_{36}z_{410} + 3z_{15}z_{38}z_{410} - z_{310}z_{35}z_{45} \\
&+ z_{210}z_{36}z_{45} - z_{110}z_{38}z_{45} - 2z_{210}z_{26}z_{46} + 2z_{110}z_{28}z_{46} \\
&+ 2z_{25}z_{310}z_{46} + z_{29}z_{36}z_{46} - z_{19}z_{38}z_{46} - z_{35}z_{39}z_{46} \\
&+ 2z_{38}z_{45}z_{46} - 2z_{28}z_{46}^2 + 3z_{28}z_{36}z_{47} - z_{1}z_{38}z_{47} \\
&- 2z_{26}z_{38}z_{47} - z_{35}z_{38}z_{47} - 2z_{110}z_{18}z_{48} + 2z_{16}z_{210}z_{48} \\
&- 2z_{15}z_{310}z_{48} - z_{27}z_{36}z_{48} + z_{35}z_{37}z_{48} + z_{17}z_{38}z_{48} - 2z_{36}z_{45}z_{48} + 2z_{18}z_{46}z_{48} \\
&+ 2z_{26}z_{46}z_{48} - 2z_{16}z_{48}^2 - 2z_{18}z_{36}z_{49} - z_{26}z_{36}z_{49} + z_{35}z_{36}z_{49} \\
&+ 3z_{16}z_{38}z_{49} + 2z_{13}z_{210}z_{510} - 2z_{110}z_{23}z_{510} - z_{34}z_{35}z_{510} - z_{24}z_{36}z_{510} \\
&+ z_{14}z_{38}z_{510} + z_{23}z_{46}z_{510} - z_{13}z_{48}z_{510} + z_{24}z_{310}z_{56} - 2z_{210}z_{34}z_{56} + z_{34}z_{39}z_{56} \\
&- 3z_{23}z_{410}z_{56} + z_{34}z_{48}z_{56} + 2z_{23}z_{310}z_{57} - 2z_{34}z_{38}z_{57} - z_{14}z_{310}z_{58} \\
&+ 2z_{110}z_{34}z_{58} - z_{34}z_{37}z_{58} + 3z_{13}z_{410}z_{58} - z_{34}z_{46}z_{58} \\
&- 2z_{13}z_{310}z_{59} + 2z_{34}z_{36}z_{59} - z_{12}z_{210}z_{610} + z_{19}z_{23}z_{610} + z_{18}z_{24}z_{610}
\end{aligned}
$$

$$
\begin{aligned}
&+ z_{24}z_{26}z_{610} - 2z_{14}z_{28}z_{610} - z_{13}z_{29}z_{610} + z_{25}z_{34}z_{610}\\
&+ z_{23}z_{45}z_{610} + z_{12}z_{48}z_{610} - z_{210}z_{23}z_{67}\\
&+ 3z_{28}z_{34}z_{67} - z_{24}z_{38}z_{67} - z_{23}z_{39}z_{67}\\
&+ 2z_{23}z_{48}z_{67} + z_{14}z_{210}z_{68} - z_{110}z_{24}z_{68} - z_{19}z_{34}z_{68} + z_{27}z_{34}z_{68}\\
&- 2z_{12}z_{410}z_{68} + z_{24}z_{46}z_{68} + z_{23}z_{47}z_{68} - z_{14}z_{48}z_{68} - z_{13}z_{49}z_{68}\\
&+ z_{110}z_{23}z_{69} + z_{12}z_{310}z_{69} - z_{18}z_{34}z_{69} - 2z_{26}z_{34}z_{69} + z_{14}z_{38}z_{69}\\
&+ z_{13}z_{39}z_{69} - 2z_{23}z_{46}z_{69} - 2z_{18}z_{23}z_{710} - z_{23}z_{26}z_{710}\\
&+ 3z_{13}z_{28}z_{710} - z_{23}z_{35}z_{710} - z_{12}z_{38}z_{710} - z_{13}z_{210}z_{78} + z_{12}z_{310}z_{78}\\
&+ 2z_{18}z_{34}z_{78} + z_{26}z_{34}z_{78} - z_{24}z_{36}z_{78} - z_{23}z_{37}z_{78} + 2z_{13}z_{48}z_{78}\\
&+ 2z_{23}z_{36}z_{79} - 2z_{13}z_{38}z_{79} + z_{110}z_{12}z_{810} + z_{14}z_{18}z_{810}\\
&- z_{17}z_{23}z_{810} - 2z_{16}z_{24}z_{810} + z_{14}z_{26}z_{810} + z_{13}z_{27}z_{810} - z_{15}z_{34}z_{810}\\
&- z_{13}z_{45}z_{810} - z_{12}z_{46}z_{810} - z_{110}z_{13}z_{89}\\
&+ 3z_{16}z_{34}z_{89} - z_{14}z_{36}z_{89} - z_{13}z_{37}z_{89} + 2z_{13}z_{46}z_{89} - z_{13}z_{18}z_{910}\\
&+ 3z_{16}z_{23}z_{910} - 2z_{13}z_{26}z_{910} + z_{13}z_{35}z_{910} + z_{12}z_{36}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{15} = {}& 2z_{110}z_{19}z_{210} - 2z_{17}z_{210}^2 + 2z_{110}z_{210}z_{27} - 2z_{110}^2z_{29} - 2z_{19}^2z_{310}\\
&- 2z_{19}z_{27}z_{310} - 2z_{27}^2z_{310} + 6z_{17}z_{29}z_{310} + 2z_{210}z_{27}z_{37} - 2z_{110}z_{29}z_{37} - 2z_{29}z_{37}^2\\
&+ 2z_{110}z_{19}z_{39} - 2z_{17}z_{210}z_{39} + 2z_{19}z_{37}z_{39} + 2z_{27}z_{37}z_{39} - 2z_{17}z_{39}^2\\
&+ 2z_{18}z_{19}z_{410} + z_{15}z_{210}z_{410} - z_{110}z_{25}z_{410} + z_{19}z_{26}z_{410} + z_{18}z_{27}z_{410}\\
&+ 2z_{26}z_{27}z_{410} - 3z_{17}z_{28}z_{410} - 3z_{16}z_{29}z_{410} - 2z_{25}z_{37}z_{410}\\
&+ 2z_{15}z_{39}z_{410} + z_{35}z_{410}z_{45} - z_{310}z_{45}^2\\
&- z_{25}z_{410}z_{46} + z_{210}z_{45}z_{46} + z_{39}z_{45}z_{46} - z_{29}z_{46}^2 - 2z_{210}z_{26}z_{47} + 2z_{110}z_{28}z_{47}\\
&+ 3z_{25}z_{310}z_{47} - z_{210}z_{35}z_{47} + 3z_{29}z_{36}z_{47} + z_{28}z_{37}z_{47}\\
&- 2z_{19}z_{38}z_{47} - z_{27}z_{38}z_{47} - z_{26}z_{39}z_{47}\\
&- 2z_{35}z_{39}z_{47} + z_{38}z_{45}z_{47} - z_{28}z_{46}z_{47} + z_{15}z_{410}z_{48} - z_{110}z_{45}z_{48} - z_{37}z_{45}z_{48}\\
&+ z_{19}z_{46}z_{48} + z_{27}z_{46}z_{48} + z_{18}z_{47}z_{48} - z_{17}z_{48}^2 - 2z_{110}z_{18}z_{49} + 2z_{16}z_{210}z_{49}\\
&- 3z_{15}z_{310}z_{49} + z_{110}z_{35}z_{49} - z_{19}z_{36}z_{49} - 2z_{27}z_{36}z_{49} - z_{18}z_{37}z_{49} + 2z_{35}z_{37}z_{49}\\
&+ 3z_{17}z_{38}z_{49} + z_{16}z_{39}z_{49} - z_{36}z_{45}z_{49} + z_{26}z_{46}z_{49} - z_{16}z_{48}z_{49} + 2z_{14}z_{210}z_{510}\\
&- 2z_{110}z_{24}z_{510} - z_{24}z_{37}z_{510} + z_{14}z_{39}z_{510} - z_{34}z_{45}z_{510} + z_{23}z_{47}z_{510}\\
&- z_{13}z_{49}z_{510} - 2z_{24}z_{410}z_{56} + 2z_{34}z_{49}z_{56} + 3z_{24}z_{310}z_{57} - 2z_{210}z_{34}z_{57}\\
&- z_{34}z_{39}z_{57} - z_{23}z_{410}z_{57} - z_{34}z_{48}z_{57}\\
&+ 2z_{14}z_{410}z_{58} - 2z_{34}z_{47}z_{58} - 3z_{14}z_{310}z_{59} + 2z_{110}z_{34}z_{59}\\
&+ z_{34}z_{37}z_{59} + z_{13}z_{410}z_{59} + z_{34}z_{46}z_{59}\\
&+ 2z_{19}z_{24}z_{610} + z_{24}z_{27}z_{610} - 3z_{14}z_{29}z_{610} + z_{24}z_{45}z_{610}\\
&+ z_{12}z_{49}z_{610} - z_{210}z_{24}z_{67} + 3z_{29}z_{34}z_{67}\\
&- 2z_{24}z_{39}z_{67} + z_{24}z_{48}z_{67} + z_{23}z_{49}z_{67} + 2z_{24}z_{47}z_{68} - 2z_{14}z_{49}z_{68} + z_{14}z_{210}z_{69}\\
&- 2z_{19}z_{34}z_{69} - z_{27}z_{34}z_{69} + 2z_{14}z_{39}z_{69} - z_{12}z_{410}z_{69}\\
&- z_{24}z_{46}z_{69} - z_{23}z_{47}z_{69} - z_{12}z_{210}z_{710}\\
&- z_{19}z_{23}z_{710} - z_{18}z_{24}z_{710} - z_{23}z_{27}z_{710} + z_{14}z_{28}z_{710}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2z_{13}z_{29}z_{710} + z_{25}z_{34}z_{710} - z_{24}z_{35}z_{710} \\
&- z_{12}z_{39}z_{710} - z_{110}z_{24}z_{78} + z_{19}z_{34}z_{78} \\
&+ 2z_{27}z_{34}z_{78} - 2z_{24}z_{37}z_{78} - z_{12}z_{410}z_{78} \\
&+ z_{14}z_{48}z_{78} + z_{13}z_{49}z_{78} - z_{13}z_{210}z_{79} + z_{110}z_{23}z_{79} \\
&+ 2z_{12}z_{310}z_{79} + z_{18}z_{34}z_{79} - z_{26}z_{34}z_{79} \\
&+ z_{24}z_{36}z_{79} + z_{23}z_{37}z_{79} - z_{14}z_{38}z_{79} - z_{13}z_{39}z_{79} + z_{14}z_{19}z_{810} - 3z_{17}z_{24}z_{810} \\
&+ 2z_{14}z_{27}z_{810} - z_{14}z_{45}z_{810} - z_{12}z_{47}z_{810} \\
&- z_{110}z_{14}z_{89} + 3z_{17}z_{34}z_{89} - 2z_{14}z_{37}z_{89} \\
&+ z_{14}z_{46}z_{89} + z_{13}z_{47}z_{89} + z_{110}z_{12}z_{910} \\
&- z_{13}z_{19}z_{910} + 2z_{17}z_{23}z_{910} + z_{16}z_{24}z_{910} \\
&- z_{14}z_{26}z_{910} - z_{13}z_{27}z_{910} - z_{15}z_{34}z_{910} + z_{14}z_{35}z_{910} + z_{12}z_{37}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{23} = &-z_{15}z_{210}z_{510} + z_{110}z_{25}z_{510} + z_{19}z_{35}z_{510} - z_{27}z_{35}z_{510} \\
&- z_{18}z_{45}z_{510} + z_{26}z_{45}z_{510} + z_{12}z_{510}^2 + z_{19}z_{210}z_{56} \\
&- 2z_{210}z_{27}z_{56} + z_{110}z_{29}z_{56} + z_{29}z_{37}z_{56} - 2z_{19}z_{39}z_{56} \\
&+ z_{27}z_{39}z_{56} + 2z_{25}z_{410}z_{56} - z_{28}z_{47}z_{56} + z_{45}z_{48}z_{56} \\
&+ 2z_{18}z_{49}z_{56} - z_{26}z_{49}z_{56} - z_{35}z_{49}z_{56} + z_{24}z_{510}z_{56} \\
&- z_{18}z_{210}z_{57} + 2z_{210}z_{26}z_{57} - z_{110}z_{28}z_{57} - 2z_{25}z_{310}z_{57} \\
&- z_{29}z_{36}z_{57} + 2z_{19}z_{38}z_{57} - z_{27}z_{38}z_{57} + z_{35}z_{39}z_{57} \\
&- z_{38}z_{45}z_{57} + z_{28}z_{46}z_{57} - 2z_{18}z_{48}z_{57} + z_{26}z_{48}z_{57} \\
&- z_{23}z_{510}z_{57} - 2z_{110}z_{19}z_{58} + z_{17}z_{210}z_{58} + z_{110}z_{27}z_{58} \\
&+ z_{19}z_{37}z_{58} - 2z_{27}z_{37}z_{58} + z_{17}z_{39}z_{58} - 2z_{15}z_{410}z_{58} \\
&- z_{45}z_{46}z_{58} - z_{18}z_{47}z_{58} + 2z_{26}z_{47}z_{58} + z_{35}z_{47}z_{58} \\
&- z_{16}z_{49}z_{58} - z_{14}z_{510}z_{58} + 2z_{34}z_{57}z_{58} + 2z_{110}z_{18}z_{59} \\
&- z_{16}z_{210}z_{59} - z_{110}z_{26}z_{59} + 2z_{15}z_{310}z_{59} - z_{19}z_{36}z_{59} \\
&+ 2z_{27}z_{36}z_{59} - z_{35}z_{37}z_{59} - z_{17}z_{38}z_{59} + z_{36}z_{45}z_{59} \\
&+ z_{18}z_{46}z_{59} - 2z_{26}z_{46}z_{59} + z_{16}z_{48}z_{59} + z_{13}z_{510}z_{59} \\
&- 2z_{34}z_{56}z_{59} - 2z_{19}z_{25}z_{610} + z_{25}z_{27}z_{610} + z_{15}z_{29}z_{610} \\
&- z_{25}z_{45}z_{610} - 3z_{12}z_{59}z_{610} - z_{19}z_{28}z_{67} + z_{27}z_{28}z_{67} \\
&+ z_{18}z_{29}z_{67} - z_{26}z_{29}z_{67} + z_{25}z_{39}z_{67} - z_{25}z_{48}z_{67} + z_{24}z_{58}z_{67} \\
&- z_{23}z_{59}z_{67} + 2z_{19}^2z_{68} - 2z_{19}z_{27}z_{68} + 2z_{27}^2z_{68} - 2z_{17}z_{29}z_{68} \\
&+ z_{45}^2z_{68} - 3z_{25}z_{47}z_{68} + 3z_{15}z_{49}z_{68} - 3z_{24}z_{57}z_{68} \\
&+ 3z_{14}z_{59}z_{68} - 2z_{18}z_{19}z_{69} + 2z_{19}z_{26}z_{69} - 2z_{26}z_{27}z_{69} \\
&+ z_{17}z_{28}z_{69} + z_{16}z_{29}z_{69} + z_{25}z_{37}z_{69} - 2z_{15}z_{39}z_{69} \\
&- z_{35}z_{45}z_{69} + 2z_{25}z_{46}z_{69} - z_{15}z_{48}z_{69} + z_{24}z_{56}z_{69} \\
&+ 2z_{23}z_{57}z_{69} - 2z_{14}z_{58}z_{69} - z_{13}z_{59}z_{69} + 2z_{12}z_{69}^2 \\
&+ 2z_{18}z_{25}z_{710} - z_{25}z_{26}z_{710} - z_{15}z_{28}z_{710} + z_{25}z_{35}z_{710} \\
&+ 3z_{12}z_{58}z_{710} - 2z_{18}z_{19}z_{78} + 2z_{18}z_{27}z_{78} - 2z_{26}z_{27}z_{78} \\
&+ z_{17}z_{28}z_{78} + z_{16}z_{29}z_{78} + 2z_{25}z_{37}z_{78} - z_{15}z_{39}z_{78}
\end{aligned}
$$

$$
\begin{aligned}
&- z_{35}z_{45}z_{78} + z_{25}z_{46}z_{78} - 2z_{15}z_{48}z_{78} + 2z_{24}z_{56}z_{78} \\
&+ z_{23}z_{57}z_{78} - z_{14}z_{58}z_{78} - 2z_{13}z_{59}z_{78} + 2z_{12}z_{69}z_{78} \\
&+ 2z_{12}z_{78}^2 + 2z_{18}^2z_{79} - 2z_{18}z_{26}z_{79} + 2z_{26}^2z_{79} - 2z_{16}z_{28}z_{79} \\
&+ z_{35}^2z_{79} - 3z_{25}z_{36}z_{79} + 3z_{15}z_{38}z_{79} - 3z_{23}z_{56}z_{79} \\
&+ 3z_{13}z_{58}z_{79} - 6z_{12}z_{68}z_{79} + z_{15}z_{19}z_{810} + z_{17}z_{25}z_{810} \\
&- 2z_{15}z_{27}z_{810} + z_{15}z_{45}z_{810} + 3z_{12}z_{57}z_{810} + z_{17}z_{18}z_{89} \\
&- z_{16}z_{19}z_{89} - z_{17}z_{26}z_{89} + z_{16}z_{27}z_{89} + z_{15}z_{37}z_{89} \\
&- z_{15}z_{46}z_{89} + z_{14}z_{56}z_{89} - z_{13}z_{57}z_{89} - 2z_{12}z_{67}z_{89} \\
&- z_{15}z_{18}z_{910} - z_{16}z_{25}z_{910} + 2z_{15}z_{26}z_{910} - z_{15}z_{35}z_{910} - 3z_{12}z_{56}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{24} = {}& 2z_{110}z_{18}z_{510} - 2z_{16}z_{210}z_{510} + z_{15}z_{310}z_{510} + 2z_{110}z_{35}z_{510} \\
&+ z_{27}z_{36}z_{510} - 2z_{35}z_{37}z_{510} - z_{17}z_{38}z_{510} + 2z_{36}z_{45}z_{510} \\
&- z_{26}z_{46}z_{510} + z_{16}z_{48}z_{510} + 2z_{13}z_{510}^2 + 2z_{110}z_{210}z_{56} \\
&- z_{19}z_{310}z_{56} - z_{27}z_{310}z_{56} - z_{210}z_{37}z_{56} - z_{110}z_{39}z_{56} \\
&+ 2z_{37}z_{39}z_{56} + 2z_{18}z_{410}z_{56} + z_{26}z_{410}z_{56} + z_{35}z_{410}z_{56} \\
&- z_{38}z_{47}z_{56} + z_{46}z_{48}z_{56} - 2z_{36}z_{49}z_{56} - z_{34}z_{510}z_{56} - z_{18}z_{310}z_{57} \\
&- z_{310}z_{35}z_{57} + z_{210}z_{36}z_{57} + z_{110}z_{38}z_{57} - z_{37}z_{38}z_{57} \\
&+ z_{36}z_{48}z_{57} - 2z_{110}^2z_{58} + 2z_{17}z_{310}z_{58} + 2z_{110}z_{37}z_{58} - 2z_{37}^2z_{58} \\
&- 3z_{16}z_{410}z_{58} - z_{46}^2z_{58} + 3z_{36}z_{47}z_{58} + z_{16}z_{310}z_{59} \\
&- 2z_{110}z_{36}z_{59} + z_{36}z_{37}z_{59} - z_{36}z_{46}z_{59} - 2z_{18}z_{19}z_{610} \\
&- z_{110}z_{25}z_{610} - z_{26}z_{27}z_{610} + z_{17}z_{28}z_{610} + 2z_{16}z_{29}z_{610} - z_{19}z_{35}z_{610} \\
&+ 2z_{25}z_{37}z_{610} - z_{15}z_{39}z_{610} - z_{18}z_{45}z_{610} - 2z_{35}z_{45}z_{610} \\
&+ z_{25}z_{46}z_{610} - z_{15}z_{48}z_{610} - z_{12}z_{510}z_{610} + 2z_{24}z_{56}z_{610} + z_{23}z_{57}z_{610} \\
&- 3z_{14}z_{58}z_{610} - 3z_{13}z_{59}z_{610} + z_{210}z_{26}z_{67} - 2z_{110}z_{28}z_{67} - z_{25}z_{310}z_{67} \\
&- 2z_{29}z_{36}z_{67} + z_{28}z_{37}z_{67} + z_{19}z_{38}z_{67} + z_{18}z_{39}z_{67} \\
&+ 2z_{35}z_{39}z_{67} - z_{38}z_{45}z_{67} + z_{28}z_{46}z_{67} - 2z_{18}z_{48}z_{67} - z_{35}z_{48}z_{67} \\
&- 2z_{23}z_{510}z_{67} + 3z_{34}z_{58}z_{67} + 2z_{110}z_{19}z_{68} - z_{17}z_{210}z_{68} \\
&- z_{110}z_{27}z_{68} - z_{19}z_{37}z_{68} + 2z_{27}z_{37}z_{68} - z_{17}z_{39}z_{68} + z_{15}z_{410}z_{68} \\
&+ z_{45}z_{46}z_{68} - z_{18}z_{47}z_{68} - z_{26}z_{47}z_{68} - 2z_{35}z_{47}z_{68} + 2z_{16}z_{49}z_{68} \\
&+ 2z_{14}z_{510}z_{68} - z_{34}z_{57}z_{68} - 2z_{24}z_{67}z_{68} + z_{110}z_{26}z_{69} + z_{19}z_{36}z_{69} \\
&- z_{26}z_{37}z_{69} - z_{16}z_{39}z_{69} + z_{18}z_{46}z_{69} + z_{35}z_{46}z_{69} - z_{34}z_{56}z_{69} \\
&+ z_{12}z_{610}z_{69} + z_{23}z_{67}z_{69} + z_{14}z_{68}z_{69} + z_{13}z_{69}^2 \\
&+ 2z_{18}^2z_{710} + z_{26}^2z_{710} - 3z_{16}z_{28}z_{710} + 2z_{18}z_{35}z_{710} + 2z_{35}^2z_{710} \\
&- 3z_{25}z_{36}z_{710} + 2z_{15}z_{38}z_{710} - 3z_{23}z_{56}z_{710} + 6z_{13}z_{58}z_{710} \\
&- 3z_{12}z_{68}z_{710} - 2z_{110}z_{18}z_{78} + z_{16}z_{210}z_{78} - z_{15}z_{310}z_{78} \\
&- 2z_{27}z_{36}z_{78} + 2z_{18}z_{37}z_{78} + 2z_{35}z_{37}z_{78} + z_{17}z_{38}z_{78} \\
&- z_{36}z_{45}z_{78} + z_{26}z_{46}z_{78} - 2z_{16}z_{48}z_{78} - 2z_{13}z_{510}z_{78} \\
&+ 2z_{34}z_{56}z_{78} + 2z_{12}z_{610}z_{78} + z_{23}z_{67}z_{78} - z_{14}z_{68}z_{78} \\
&+ 2z_{13}z_{78}^2 - 2z_{18}z_{36}z_{79} + z_{26}z_{36}z_{79} - z_{35}z_{36}z_{79} + z_{16}z_{38}z_{79}
\end{aligned}
$$

$$
\begin{aligned}
& - 3z_{13}z_{68}z_{79} + z_{110}z_{15}z_{810} + z_{17}z_{18}z_{810} - z_{16}z_{27}z_{810} \\
& + z_{17}z_{35}z_{810} - z_{15}z_{37}z_{810} + z_{16}z_{45}z_{810} + z_{14}z_{56}z_{810} \\
& + 2z_{13}z_{57}z_{810} + z_{12}z_{67}z_{810} - z_{110}z_{16}z_{89} - z_{17}z_{36}z_{89} \\
& + 2z_{16}z_{37}z_{89} - z_{16}z_{46}z_{89} - 3z_{13}z_{67}z_{89} - z_{16}z_{18}z_{910} \\
& + z_{16}z_{26}z_{910} - 2z_{16}z_{35}z_{910} + z_{15}z_{36}z_{910} - 3z_{13}z_{56}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{25} = {} & 2z_{110}z_{19}z_{510} - 2z_{17}z_{210}z_{510} + z_{27}z_{37}z_{510} - z_{17}z_{39}z_{510} \\
& + z_{15}z_{410}z_{510} + 2z_{110}z_{45}z_{510} + 2z_{45}z_{46}z_{510} - z_{26}z_{47}z_{510} \\
& - 2z_{35}z_{47}z_{510} + z_{16}z_{49}z_{510} + 2z_{14}z_{510}^2 + z_{19}z_{410}z_{56} \\
& + z_{410}z_{45}z_{56} - z_{210}z_{47}z_{56} + z_{39}z_{47}z_{56} - z_{110}z_{49}z_{56} - z_{46}z_{49}z_{56} \\
& + 2z_{110}z_{210}z_{57} - 2z_{19}z_{310}z_{57} - z_{27}z_{310}z_{57} + z_{37}z_{39}z_{57} \\
& + z_{18}z_{410}z_{57} + z_{26}z_{410}z_{57} - z_{310}z_{45}z_{57} + z_{210}z_{46}z_{57} \\
& - 2z_{38}z_{47}z_{57} + z_{110}z_{48}z_{57} + 2z_{46}z_{48}z_{57} - z_{36}z_{49}z_{57} - z_{34}z_{510}z_{57} \\
& - z_{17}z_{410}z_{58} + 2z_{110}z_{47}z_{58} - z_{37}z_{47}z_{58} + z_{46}z_{47}z_{58} - 2z_{110}^2z_{59} \\
& + 3z_{17}z_{310}z_{59} - z_{37}^2z_{59} - 2z_{16}z_{410}z_{59} - 2z_{110}z_{46}z_{59} - 2z_{46}^2z_{59} \\
& + 3z_{36}z_{47}z_{59} - 2z_{19}^2z_{610} - z_{27}^2z_{610} + 3z_{17}z_{29}z_{610} - 2z_{19}z_{45}z_{610} \\
& - 2z_{45}^2z_{610} + 3z_{25}z_{47}z_{610} - 2z_{15}z_{49}z_{610} + 3z_{24}z_{57}z_{610} \\
& - 6z_{14}z_{59}z_{610} + z_{210}z_{27}z_{67} - 2z_{110}z_{29}z_{67} - z_{29}z_{37}z_{67} \\
& + 2z_{19}z_{39}z_{67} - z_{25}z_{410}z_{67} + z_{39}z_{45}z_{67} - z_{29}z_{46}z_{67} \\
& + 2z_{28}z_{47}z_{67} - z_{19}z_{48}z_{67} - 2z_{45}z_{48}z_{67} - z_{18}z_{49}z_{67} + z_{35}z_{49}z_{67} \\
& - 2z_{24}z_{510}z_{67} + 3z_{34}z_{59}z_{67} - 2z_{19}z_{47}z_{68} + z_{27}z_{47}z_{68} \\
& - z_{45}z_{47}z_{68} + z_{17}z_{49}z_{68} + 2z_{110}z_{19}z_{69} - z_{17}z_{210}z_{69} \\
& + z_{27}z_{37}z_{69} - 2z_{17}z_{39}z_{69} + z_{15}z_{410}z_{69} + 2z_{19}z_{46}z_{69} \\
& + 2z_{45}z_{46}z_{69} - 2z_{26}z_{47}z_{69} - z_{35}z_{47}z_{69} + z_{16}z_{49}z_{69} \\
& + 2z_{14}z_{510}z_{69} - 2z_{34}z_{57}z_{69} - z_{24}z_{67}z_{69} + 2z_{14}z_{69}^2 \\
& + 2z_{18}z_{19}z_{710} - z_{110}z_{25}z_{710} + z_{26}z_{27}z_{710} - 2z_{17}z_{28}z_{710} \\
& - z_{16}z_{29}z_{710} + z_{19}z_{35}z_{710} - z_{25}z_{37}z_{710} + z_{15}z_{39}z_{710} + z_{18}z_{45}z_{710} \\
& + 2z_{35}z_{45}z_{710} - 2z_{25}z_{46}z_{710} + z_{15}z_{48}z_{710} - z_{12}z_{510}z_{710} - z_{24}z_{56}z_{710} \\
& - 2z_{23}z_{57}z_{710} + 3z_{14}z_{58}z_{710} + 3z_{13}z_{59}z_{710} - 2z_{12}z_{69}z_{710} \\
& - z_{110}z_{27}z_{78} + z_{19}z_{37}z_{78} + z_{37}z_{45}z_{78} - z_{27}z_{46}z_{78} + z_{18}z_{47}z_{78} \\
& - z_{17}z_{48}z_{78} + z_{34}z_{57}z_{78} - z_{24}z_{67}z_{78} - z_{12}z_{710}z_{78} + z_{14}z_{78}^2 \\
& - 2z_{110}z_{18}z_{79} + z_{16}z_{210}z_{79} + z_{110}z_{26}z_{79} - z_{15}z_{310}z_{79} \\
& - z_{19}z_{36}z_{79} - z_{27}z_{36}z_{79} + z_{35}z_{37}z_{79} + 2z_{17}z_{38}z_{79} - 2z_{36}z_{45}z_{79} \\
& - z_{18}z_{46}z_{79} + 2z_{26}z_{46}z_{79} - z_{16}z_{48}z_{79} - 2z_{13}z_{510}z_{79} \\
& + z_{34}z_{56}z_{79} + 3z_{12}z_{610}z_{79} + 2z_{23}z_{67}z_{79} - 3z_{14}z_{68}z_{79} \\
& - z_{13}z_{69}z_{79} + z_{13}z_{78}z_{79} + z_{17}z_{19}z_{810} - z_{17}z_{27}z_{810} \\
& + 2z_{17}z_{45}z_{810} - z_{15}z_{47}z_{810} + 3z_{14}z_{57}z_{810} - z_{110}z_{17}z_{89} \\
& + z_{17}z_{37}z_{89} - 2z_{17}z_{46}z_{89} + z_{16}z_{47}z_{89} - 3z_{14}z_{67}z_{89} + z_{110}z_{15}z_{910} \\
& - z_{16}z_{19}z_{910} + z_{17}z_{26}z_{910} - z_{17}z_{35}z_{910} - z_{16}z_{45}z_{910}
\end{aligned}
$$

$$+ z_{15}z_{46}z_{910} - 2z_{14}z_{56}z_{910} - z_{13}z_{57}z_{910} + z_{12}z_{67}z_{910},$$

$$\begin{aligned}\psi_{34} = &-2z_{210}z_{26}z_{510} + 2z_{110}z_{28}z_{510} + z_{25}z_{310}z_{510} + 2z_{210}z_{35}z_{510}\\
&+ z_{29}z_{36}z_{510} - z_{19}z_{38}z_{510} - 2z_{35}z_{39}z_{510} + 2z_{38}z_{45}z_{510} - z_{28}z_{46}z_{510}\\
&+ z_{18}z_{48}z_{510} + 2z_{23}z_{510}^2 + 2z_{210}^2z_{56} - 2z_{29}z_{310}z_{56} - 2z_{210}z_{39}z_{56}\\
&+ 2z_{39}^2z_{56} + 3z_{28}z_{410}z_{56} + z_{48}^2z_{56} - 3z_{38}z_{49}z_{56} - z_{28}z_{310}z_{57}\\
&+ 2z_{210}z_{38}z_{57} - z_{38}z_{39}z_{57} + z_{38}z_{48}z_{57} - 2z_{110}z_{210}z_{58} + z_{19}z_{310}z_{58}\\
&+ z_{27}z_{310}z_{58} + z_{210}z_{37}z_{58} + z_{110}z_{39}z_{58} - 2z_{37}z_{39}z_{58} - z_{18}z_{410}z_{58}\\
&- 2z_{26}z_{410}z_{58} + z_{35}z_{410}z_{58} + 2z_{38}z_{47}z_{58} - z_{46}z_{48}z_{58} + z_{36}z_{49}z_{58}\\
&- z_{34}z_{510}z_{58} + z_{26}z_{310}z_{59} - z_{310}z_{35}z_{59} - z_{210}z_{36}z_{59} - z_{110}z_{38}z_{59}\\
&+ z_{36}z_{39}z_{59} - z_{38}z_{46}z_{59} - z_{210}z_{25}z_{610} - z_{19}z_{28}z_{610} + z_{26}z_{29}z_{610}\\
&- z_{29}z_{35}z_{610} + z_{25}z_{39}z_{610} - z_{28}z_{45}z_{610} - z_{24}z_{58}z_{610} - 2z_{23}z_{59}z_{610}\\
&- z_{210}z_{28}z_{67} - z_{29}z_{38}z_{67} + 2z_{28}z_{39}z_{67} - z_{28}z_{48}z_{67} + z_{19}z_{210}z_{68}\\
&- 2z_{210}z_{27}z_{68} + z_{110}z_{29}z_{68} + z_{29}z_{37}z_{68} - 2z_{19}z_{39}z_{68} + z_{27}z_{39}z_{68}\\
&+ z_{25}z_{410}z_{68} - 2z_{28}z_{47}z_{68} + z_{45}z_{48}z_{68} + z_{18}z_{49}z_{68} + z_{26}z_{49}z_{68}\\
&- 2z_{35}z_{49}z_{68} + 2z_{24}z_{510}z_{68} - z_{34}z_{59}z_{68} + 2z_{210}z_{26}z_{69} - z_{110}z_{28}z_{69}\\
&- z_{25}z_{310}z_{69} - z_{29}z_{36}z_{69} + 2z_{19}z_{38}z_{69} - 2z_{26}z_{39}z_{69} + 2z_{35}z_{39}z_{69}\\
&- z_{38}z_{45}z_{69} + 2z_{28}z_{46}z_{69} - z_{18}z_{48}z_{69} - 2z_{23}z_{510}z_{69}\\
&+ 2z_{34}z_{58}z_{69} - z_{24}z_{68}z_{69} + 2z_{23}z_{69}^2 + z_{18}z_{28}z_{710} - z_{26}z_{28}z_{710}\\
&+ 2z_{28}z_{35}z_{710} - z_{25}z_{38}z_{710} + 3z_{23}z_{58}z_{710} - z_{18}z_{210}z_{78} + z_{28}z_{37}z_{78}\\
&- z_{27}z_{38}z_{78} + z_{18}z_{39}z_{78} - z_{26}z_{48}z_{78} + z_{35}z_{48}z_{78} - z_{34}z_{58}z_{78}\\
&+ z_{24}z_{68}z_{78} + z_{23}z_{78}^2 - z_{28}z_{36}z_{79} - z_{18}z_{38}z_{79} + 2z_{26}z_{38}z_{79}\\
&- z_{35}z_{38}z_{79} - 3z_{23}z_{68}z_{79} - z_{18}z_{19}z_{810} + z_{15}z_{210}z_{810} - 2z_{26}z_{27}z_{810}\\
&+ 2z_{17}z_{28}z_{810} + z_{16}z_{29}z_{810} + z_{27}z_{35}z_{810} + z_{25}z_{37}z_{810}\\
&- 2z_{15}z_{39}z_{810} + z_{26}z_{45}z_{810} - 2z_{35}z_{45}z_{810} + z_{25}z_{46}z_{810} - z_{15}z_{48}z_{810}\\
&- z_{12}z_{510}z_{810} + 3z_{24}z_{56}z_{810} + 3z_{23}z_{57}z_{810} - 2z_{14}z_{58}z_{810} - z_{13}z_{59}z_{810}\\
&+ 2z_{12}z_{69}z_{810} + z_{12}z_{78}z_{810} + z_{110}z_{18}z_{89} - 2z_{16}z_{210}z_{89} + z_{15}z_{310}z_{89}\\
&+ z_{27}z_{36}z_{89} + z_{26}z_{37}z_{89} - 2z_{35}z_{37}z_{89} - 2z_{17}z_{38}z_{89} + z_{16}z_{39}z_{89}\\
&+ z_{36}z_{45}z_{89} - 2z_{26}z_{46}z_{89} + z_{35}z_{46}z_{89} + z_{16}z_{48}z_{89} + 2z_{13}z_{510}z_{89}\\
&- 3z_{34}z_{56}z_{89} - z_{12}z_{610}z_{89} - 3z_{23}z_{67}z_{89} + 2z_{14}z_{68}z_{89} - z_{13}z_{69}z_{89}\\
&- z_{13}z_{78}z_{89} + z_{18}^2z_{910} + 2z_{26}^2z_{910} - 3z_{16}z_{28}z_{910} - 2z_{26}z_{35}z_{910} + 2z_{35}^2z_{910}\\
&- 2z_{25}z_{36}z_{910} + 3z_{15}z_{38}z_{910} - 6z_{23}z_{56}z_{910} + 3z_{13}z_{58}z_{910} - 3z_{12}z_{68}z_{910},\end{aligned}$$

$$\begin{aligned}\psi_{35} = &-2z_{210}z_{27}z_{510} + 2z_{110}z_{29}z_{510} + z_{29}z_{37}z_{510} - z_{19}z_{39}z_{510} + z_{25}z_{410}z_{510}\\
&+ 2z_{210}z_{45}z_{510} - z_{28}z_{47}z_{510} + 2z_{45}z_{48}z_{510} + z_{18}z_{49}z_{510}\\
&- 2z_{35}z_{49}z_{510} + 2z_{24}z_{510}^2 + z_{29}z_{410}z_{56} - 2z_{210}z_{49}z_{56} + z_{39}z_{49}z_{56} - z_{48}z_{49}z_{56}\\
&+ 2z_{210}^2z_{57} - 3z_{29}z_{310}z_{57} + z_{39}^2z_{57} + 2z_{28}z_{410}z_{57} + 2z_{210}z_{48}z_{57}\\
&+ 2z_{48}^2z_{57} - 3z_{38}z_{49}z_{57} - z_{27}z_{410}z_{58} + z_{410}z_{45}z_{58} + z_{210}z_{47}z_{58}\\
&+ z_{47}z_{48}z_{58} + z_{110}z_{49}z_{58} - z_{37}z_{49}z_{58} - 2z_{110}z_{210}z_{59} + z_{19}z_{310}z_{59}\end{aligned}$$

$$
\begin{aligned}
&+ 2z_{27}z_{310}z_{59} - z_{37}z_{39}z_{59} - z_{18}z_{410}z_{59} - z_{26}z_{410}z_{59} - z_{310}z_{45}z_{59}\\
&- z_{210}z_{46}z_{59} + z_{38}z_{47}z_{59} - z_{110}z_{48}z_{59} - 2z_{46}z_{48}z_{59} + 2z_{36}z_{49}z_{59}\\
&- z_{34}z_{510}z_{59} - z_{19}z_{29}z_{610} + z_{27}z_{29}z_{610} - 2z_{29}z_{45}z_{610} + z_{25}z_{49}z_{610}\\
&- 3z_{24}z_{59}z_{610} - z_{210}z_{29}z_{67} + z_{29}z_{39}z_{67} - 2z_{29}z_{48}z_{67} + z_{28}z_{49}z_{67}\\
&- z_{29}z_{47}z_{68} - z_{19}z_{49}z_{68} + 2z_{27}z_{49}z_{68} - z_{45}z_{49}z_{68} + z_{19}z_{210}z_{69}\\
&- z_{27}z_{39}z_{69} + z_{39}z_{45}z_{69} + z_{29}z_{46}z_{69} + z_{19}z_{48}z_{69} - z_{26}z_{49}z_{69}\\
&+ z_{34}z_{59}z_{69} + z_{24}z_{69}^2 - z_{210}z_{25}z_{710} - z_{27}z_{28}z_{710} + z_{18}z_{29}z_{710}\\
&+ z_{29}z_{35}z_{710} + z_{28}z_{45}z_{710} - z_{25}z_{48}z_{710} + 2z_{24}z_{58}z_{710} + z_{23}z_{59}z_{710}\\
&- 2z_{210}z_{27}z_{78} + z_{110}z_{29}z_{78} + 2z_{29}z_{37}z_{78} - z_{19}z_{39}z_{78} + z_{25}z_{410}z_{78}\\
&- z_{28}z_{47}z_{78} - 2z_{27}z_{48}z_{78} + 2z_{45}z_{48}z_{78} + 2z_{18}z_{49}z_{78} - z_{35}z_{49}z_{78}\\
&+ 2z_{24}z_{510}z_{78} - 2z_{34}z_{59}z_{78} + 2z_{24}z_{78}^2 - z_{18}z_{210}z_{79} + 2z_{210}z_{26}z_{79}\\
&- z_{110}z_{28}z_{79} - z_{25}z_{310}z_{79} - 2z_{29}z_{36}z_{79} + z_{19}z_{38}z_{79} + z_{27}z_{38}z_{79} + z_{35}z_{39}z_{79}\\
&- 2z_{38}z_{45}z_{79} + z_{28}z_{46}z_{79} - 2z_{18}z_{48}z_{79} + z_{26}z_{48}z_{79} - 2z_{23}z_{510}z_{79}\\
&+ z_{34}z_{58}z_{79} - 3z_{24}z_{68}z_{79} + z_{23}z_{69}z_{79} - z_{23}z_{78}z_{79} - z_{19}^2z_{810} - 2z_{27}^2z_{810}\\
&+ 3z_{17}z_{29}z_{810} + 2z_{27}z_{45}z_{810} - 2z_{45}^2z_{810} + 2z_{25}z_{47}z_{810} - 3z_{15}z_{49}z_{810}\\
&+ 6z_{24}z_{57}z_{810} - 3z_{14}z_{59}z_{810} + 3z_{12}z_{79}z_{810} + z_{110}z_{19}z_{89} - 2z_{17}z_{210}z_{89}\\
&+ 2z_{27}z_{37}z_{89} - z_{17}z_{39}z_{89} + z_{15}z_{410}z_{89} - z_{37}z_{45}z_{89} - z_{27}z_{46}z_{89}\\
&+ 2z_{45}z_{46}z_{89} - z_{26}z_{47}z_{89} - z_{35}z_{47}z_{89} - z_{17}z_{48}z_{89} + 2z_{16}z_{49}z_{89}\\
&+ 2z_{14}z_{510}z_{89} - 3z_{34}z_{57}z_{89} - 3z_{24}z_{67}z_{89} + z_{14}z_{69}z_{89} - z_{12}z_{710}z_{89}\\
&+ z_{14}z_{78}z_{89} - 2z_{13}z_{79}z_{89} + z_{18}z_{19}z_{910} + z_{15}z_{210}z_{910} + 2z_{26}z_{27}z_{910}\\
&- z_{17}z_{28}z_{910} - 2z_{16}z_{29}z_{910} - z_{27}z_{35}z_{910} - z_{25}z_{37}z_{910} + z_{15}z_{39}z_{910}\\
&- z_{26}z_{45}z_{910} + 2z_{35}z_{45}z_{910} - z_{25}z_{46}z_{910} + 2z_{15}z_{48}z_{910}\\
&- z_{12}z_{510}z_{910} - 3z_{24}z_{56}z_{910} - 3z_{23}z_{57}z_{910} + z_{14}z_{58}z_{910}\\
&+ 2z_{13}z_{59}z_{910} - z_{12}z_{69}z_{910} - 2z_{12}z_{78}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{45} = &-z_{210}z_{37}z_{510} + z_{110}z_{39}z_{510} - z_{35}z_{410}z_{510} + z_{310}z_{45}z_{510}\\
&+ z_{210}z_{46}z_{510} - z_{110}z_{48}z_{510} + z_{34}z_{510}^2 - z_{210}z_{410}z_{56} + 2z_{39}z_{410}z_{56}\\
&- z_{410}z_{48}z_{56} - z_{310}z_{49}z_{56} + z_{210}z_{310}z_{57} - z_{310}z_{39}z_{57} - z_{38}z_{410}z_{57}\\
&+ 2z_{310}z_{48}z_{57} + z_{110}z_{410}z_{58} - 2z_{37}z_{410}z_{58} + z_{410}z_{46}z_{58} + z_{310}z_{47}z_{58}\\
&- z_{110}z_{310}z_{59} + z_{310}z_{37}z_{59} + z_{36}z_{410}z_{59} - 2z_{310}z_{46}z_{59} - z_{210}z_{27}z_{610}\\
&+ z_{110}z_{29}z_{610} + 2z_{29}z_{37}z_{610} - 2z_{19}z_{39}z_{610} + 2z_{25}z_{410}z_{610} - z_{39}z_{45}z_{610}\\
&- z_{29}z_{46}z_{610} - z_{28}z_{47}z_{610} + z_{19}z_{48}z_{610} + 2z_{45}z_{48}z_{610} + z_{18}z_{49}z_{610}\\
&- z_{35}z_{49}z_{610} + z_{24}z_{510}z_{610} - 3z_{34}z_{59}z_{610} + z_{210}^2z_{67} - 3z_{29}z_{310}z_{67}\\
&+ 2z_{39}^2z_{67} + 3z_{28}z_{410}z_{67} - 2z_{39}z_{48}z_{67} + 2z_{48}^2z_{67} - 2z_{38}z_{49}z_{67}\\
&- z_{19}z_{410}z_{68} + z_{27}z_{410}z_{68} - z_{39}z_{47}z_{68} + z_{47}z_{48}z_{68} + z_{37}z_{49}z_{68}\\
&- z_{46}z_{49}z_{68} - z_{110}z_{210}z_{69} + 2z_{19}z_{310}z_{69} + z_{27}z_{310}z_{69} - 2z_{37}z_{39}z_{69}\\
&- z_{18}z_{410}z_{69} - 2z_{26}z_{410}z_{69} + 2z_{39}z_{46}z_{69} + z_{38}z_{47}z_{69} - 2z_{46}z_{48}z_{69}\\
&+ z_{36}z_{49}z_{69} - z_{24}z_{610}z_{69} + 2z_{34}z_{69}^2 + z_{210}z_{26}z_{710} - z_{110}z_{28}z_{710}\\
&- 2z_{25}z_{310}z_{710} - z_{29}z_{36}z_{710} - z_{28}z_{37}z_{710} + z_{19}z_{38}z_{710} + z_{18}z_{39}z_{710}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2z_{35}z_{39}z_{710} - z_{38}z_{45}z_{710} + 2z_{28}z_{46}z_{710} - 2z_{18}z_{48}z_{710} - z_{35}z_{48}z_{710}\\
&- z_{23}z_{510}z_{710} + 3z_{34}z_{58}z_{710} - z_{24}z_{68}z_{710} + 2z_{23}z_{69}z_{71} + z_{110}z_{210}z_{78}\\
&- z_{19}z_{310}z_{78} - 2z_{27}z_{310}z_{78} + 2z_{37}z_{39}z_{78} + 2z_{18}z_{410}z_{78} + z_{26}z_{410}z_{78}\\
&- z_{38}z_{47}z_{78} - 2z_{37}z_{48}z_{78} + 2z_{46}z_{48}z_{78} - z_{36}z_{49}z_{78} + 2z_{24}z_{610}z_{78}\\
&- 2z_{34}z_{69}z_{78} - z_{23}z_{710}z_{78} + 2z_{34}z_{78}^2 - z_{18}z_{310}z_{79} + z_{26}z_{310}z_{79}\\
&+ z_{37}z_{38}z_{79} - z_{36}z_{39}z_{79} - z_{38}z_{46}z_{79} + z_{36}z_{48}z_{79} - z_{23}z_{610}z_{79}\\
&- 2z_{34}z_{68}z_{79} - z_{110}z_{19}z_{810} + z_{17}z_{210}z_{810} - 2z_{27}z_{37}z_{810} + 2z_{17}z_{39}z_{810}\\
&- 2z_{15}z_{410}z_{810} + z_{37}z_{45}z_{810} + z_{27}z_{46}z_{810} - 2z_{45}z_{46}z_{810} + z_{26}z_{47}z_{810}\\
&+ z_{35}z_{47}z_{810} - z_{17}z_{48}z_{810} - z_{16}z_{49}z_{810} - z_{14}z_{510}z_{810} + 3z_{34}z_{57}z_{810}\\
&+ 3z_{24}z_{67}z_{810} - 2z_{14}z_{69}z_{810} + 2z_{12}z_{710}z_{810} + z_{14}z_{78}z_{810} + z_{13}z_{79}z_{810}\\
&+ z_{110}^2z_{89} - 3z_{17}z_{310}z_{89} + 2z_{37}^2z_{89} + 3z_{16}z_{410}z_{89} - 2z_{37}z_{46}z_{89}\\
&+ 2z_{46}^2z_{89} - 2z_{36}z_{47}z_{89} + 3z_{14}z_{610}z_{89} - 6z_{34}z_{67}z_{89} - 3z_{13}z_{710}z_{89}\\
&+ z_{110}z_{18}z_{910} - z_{16}z_{210}z_{910} + 2z_{15}z_{310}z_{910} + z_{27}z_{36}z_{910} + z_{26}z_{37}z_{910}\\
&- 2z_{35}z_{37}z_{910} - z_{17}z_{38}z_{910} - z_{16}z_{39}z_{910} + z_{36}z_{45}z_{910} - 2z_{26}z_{46}z_{910}\\
&+ z_{35}z_{46}z_{910} + 2z_{16}z_{48}z_{910} + z_{13}z_{510}z_{910} - 3z_{34}z_{56}z_{910} - 2z_{12}z_{610}z_{910}\\
&- 3z_{23}z_{67}z_{910} + z_{14}z_{68}z_{910} + z_{13}z_{69}z_{910} - 2z_{13}z_{78}z_{910}.
\end{aligned}
$$

Lemma 3.2. *For every* $A \in GL_5,\ B \in GL_8,\ \tilde{Z} \in \mathrm{Alt}_{10}$, *we have*

$$\tilde{Z} \mapsto (\det A)^4(\det B)\ {}^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}$$

and

$$\Psi((\det A)^4(\det B){}^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}) = (\det A)^{10}(\det B)^3\ {}^tA^{-1}\Psi(\tilde{Z})A^{-1}. \tag{3.4}$$

Proof. It is enough to prove the equivariance (3.4) in the case when A is one of the fundamental matrices,

$$A_u = \begin{pmatrix} 1 & \varepsilon & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad A_d = \mathrm{diag}(a, 1, 1, 1, 1),$$

or permutation matrices.

For the diagonal or permutation matrices, verifying (3.4) is easy. Note that, for $A_d = \mathrm{diag}(a_1, a_2, a_3, a_4, a_5)$, we have

$$\det A_d = a_1a_2a_3a_4a_5 \text{ and } z_{12} \mapsto (\det A_d)^3 a_1^{-1}a_4a_5,$$

$$z_{13} \mapsto (\det A_d)^3 a^{-1}a_3a_5, \cdots, z_{910} \mapsto (\det A_d)^3 a_1a_2a_5^{-1}.$$

Hence, we have $\psi_{ij} \mapsto (\det A_d)^{10}a_i^{-1}a_j^{-1}$, $(1 \le i < j \le 5)$ and then

$$\Psi((\det A_d){}^t\Lambda_2(A_d)\tilde{Z}\Lambda_2(A_d)^{-1}) = (\det A_d)^{10\,t}A_d^{-1}\Psi(\tilde{Z})A_d^{-1}.$$

For A_u, we consider the action of A_u. Since $\det A_u = 1$, and

$$\Psi(\tilde{Z}) \mapsto \Psi({}^t\Lambda_2(A_u)\tilde{Z}\Lambda_2(A_u)^{-1}),$$

then we have $\psi_{2j} \mapsto \psi_{2j} - \varepsilon\psi_{1j}$ $(3 \le j \le 5)$, $\psi_{1j} \mapsto \psi_{1j}$ and $\psi_{lk} \mapsto \psi_{lk}$ $(3 \le l < k \le 5)$. Hence, we have $\Psi({}^t\Lambda_2(A_u)^{-1}\tilde{Z}\Lambda_2(A_u)^{-1}) = {}^tA_u^{-1}\Psi(\tilde{Z})A_u^{-1}$. □

Remark B. We can construct the above polynomials ψ_{ij} $(1 \leq i < j \leq 5)$ in the same program as in Remark A. This case is much more complicated than the case of $\varphi'_{ij}s$. Here we note that we consider constructing the polynomial ψ_{45}. First, we consider the polynomial corresponding to the weight $(\det A)^{10} a_4^{-1} a_5^{-1}$. This polynomial is constructed of 205-term monomials. After the action of the generators of GL_5, we have uniquely the polynomial ψ_{45} constructed of 148-term monomials. Hence from the explicit form of ψ_{45}, we can construct the other polynomials ψ_{ij} by the action of GL_5.

Step 5. From (3.2) and (3.3), $\Phi(\tilde{Z})\Delta(X \cdot Y) \mapsto (\det A)^8 (\det B)^2 A \Phi(\tilde{Z})\Delta(X \cdot Y)A^{-1}$ and hence $F_1(x) = \operatorname{tr} \Phi(\tilde{Z})\Delta(X \cdot Y)$ is a relative invariant of degree 24 corresponding to the character $(\det A)^8 (\det B)^2$.

From (3.1) and (3.4), $(X \cdot Y)\Psi(\tilde{Z}) \mapsto (\det A)^{10} (\det B)^3 A (X \cdot Y)\Psi(\tilde{Z})A^{-1}$ and hence $F_2(x) = \operatorname{tr}((X \cdot Y)\Psi(\tilde{Z}))$ is a relative invariant of degree 26 corresponding to the character $(\det A)^{10} (\det B)^3$.

For a generic point $x_0 = (e_1 \wedge e_2\ ,\ 2e_1 \wedge e_3\ ,\ 2e_2 \wedge e_3\ ,\ e_1 \wedge e_5\ ,\ e_1 \wedge e_4 - e_2 \wedge e_5\ ,\ e_2 \wedge e_4 - e_3 \wedge e_5\ ,\ e_3 \wedge e_4\ ,\ e_4 \wedge e_5\ ,\ e'_2 + e'_8)$, we have

$$X_0 \cdot Y_0 = \begin{pmatrix} 0 & & 2 & & \\ & 0 & & & \\ -2 & & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \text{ and } \Delta(X_0 \cdot Y_0) = \begin{pmatrix} 0 & & & & \\ & 4 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

$$Z_0 = \begin{pmatrix} 1 & & & & & & & \\ & 2 & & & & & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ & & 2 & & & & & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & & & & 1 \end{pmatrix},\ \tilde{Z}_0 = \begin{pmatrix} 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & 0 & & & -4 & & & -4 & \\ & & & 0 & & & & & & \\ & & & & 0 & & & & & \\ & & 4 & & & 0 & 4 & & & \\ & & & & & -4 & 0 & & -4 & \\ & & & & & & & 0 & & \\ & & 4 & & & & 4 & & 0 & \\ & & & & & & & & & 0 \end{pmatrix},$$

$$\Phi(\tilde{Z}_0) = \begin{pmatrix} 0 & & & & \\ & 16 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},\ \Psi(\tilde{Z}_0) = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 192 \\ & & & -192 & 0 \end{pmatrix},$$

and hence

$$F_1(X_1^0, \cdots, X_8^0, Y_0) = \operatorname{tr}(\Phi(\tilde{Z}_0)\ \Delta(X_0 \cdot Y_0))$$

$$= \operatorname{tr}\left(\begin{pmatrix} 0 & & & & \\ & 16 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} 0 & & & & \\ & 4 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \right) = 64 \neq 0, \tag{3.5}$$

(3.6)

$$F_2(X_1^0,\cdots,X_8^0,Y_0) = \operatorname{tr}((X_0\cdot Y_0)\Psi(\tilde{Z}_0))$$
$$= \operatorname{tr}\left(\begin{pmatrix} 0 & & 2 & & \\ & 0 & & & \\ -2 & & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 192 \\ & & & -192 & 0 \end{pmatrix}\right) = -384 \neq 0.$$

Thus they are not identically zero.

Since $\deg F_1 = 24 < \deg F_2 = 26$ and $\deg_Y F_1 = 4 > \deg_Y F_2 = 1$, we have $F_1 \nmid F_2$ and $F_2 \nmid F_1$. Hence, if F_1 or F_2 is not irreducible, it is a power of an irreducible invariant. Since the character corresponding to F_2 is $(\det A)^{10}(\det B)^3$, we have the irreducibility of F_2. If F_1 is not irreducible, we have $F_1 = G^2$ for some relative invariant corresponding to the character $(\det A)^4(\det B)$. For

$$\tilde{x} = ((z_1^{(1)}e_1\wedge e_2, 2z_2^{(2)}e_1\wedge e_3, 2z_5^{(3)}e_2\wedge e_3, z_4^{(4)}e_1\wedge e_5, z_3^{(5)}e_1\wedge e_4$$
$$- z_7^{(5)}e_2\wedge e_5, z_6^{(6)}e_2\wedge e_4 - z_9^{(6)}e_3\wedge e_5, z_8^{(7)}e_3\wedge e_4, z_{10}^{(8)}e_4\wedge e_5), e_2'+e_8') \in V,$$

we have

$$F_1(\tilde{x}) = 64(z_1^{(1)}z_2^{(2)}z_5^{(3)}z_4^{(4)}z_5^{(7)}z_6^{(6)}z_8^{(7)}z_{10}^{(8)})^2 z_2^{(2)}z_{10}^{(8)}$$

and hence $F_1(x)$ is not of the form $G(x)^2$ for some polynomial $G(x)$. This shows the irreducibility of $F_1(x)$.

Theorem 3.3. *The prehomogeneous vector space* $(GL_5\times GL_8, \Lambda_2\otimes\Lambda_1 + 1\otimes\Lambda_1^*, \mathrm{Alt}_5^{\oplus 8}\oplus V(8)^*)$ *has* 2 *basic relative invariants:*

(1) $F_1(x) = \operatorname{tr}\Phi(\tilde{Z})\Delta(X\cdot Y) \longleftrightarrow (\det A)^8(\det B)^2$, $\deg .F_1 = 24$,

(2) $F_2(x) = \operatorname{tr}(X\cdot Y)\Psi(\tilde{Z}) \longleftrightarrow (\det A)^{10}(\det B)^3$, $\deg .F_2 = 26$.

3.2. **Explicit construction of the irreducible relative invariants of** $(GL_1^2 \times \mathrm{Spin}_{10}\times SL_{14}, \Lambda'\otimes\Lambda_1 + 1\otimes\Lambda_1^*)$ **with** $H\sim SL_2\times SO_2$**,** $N=2$**.** We may assume that $GL_1\times\mathrm{Spin}_{10}\times GL_{14}$ acts on $V = V(16)^{\oplus 14}\oplus M(14,1)$ by $x = ((X_1,X_2,\cdots,X_{14}),Y) \mapsto ((\Lambda'(A)X_1,\Lambda'(A)X_2,\cdots,\Lambda'(A)X_{14})^tB, \alpha^tB^{-1}Y)$ for $x = ((X_1,X_2,\cdots,X_{14}),Y) \in V$ and $g = (\alpha, A, B)\in G$.

For $Y = {}^t(y_1,\cdots,y_{14})$, $X\cdot Y := X_1\ y_1 + \cdots + X_{14}\ y_{14} \in V(16)$. Then we have $X\cdot Y \longmapsto \alpha\ \Lambda'(A)\ X\cdot Y$. Now we define the mapping

$$\eta: V(16)\longrightarrow V(10),\ X = (x_0, x_{12},\cdots,x_{2345}) \longmapsto \eta(X) = (\eta_1(x),\ldots,\eta_{10}(x))$$

as follows:

$$\begin{array}{lcllll}
\eta_1(x) &=& -x_{12}x_{1345} & +x_{13}x_{1245} & -x_{14}x_{1235} & +x_{15}x_{1234}, \\
\eta_2(x) &=& x_{23}x_{1245} & -x_{24}x_{1235} & +x_{25}x_{1234} & -x_{12}x_{2345}, \\
\eta_3(x) &=& -x_{34}x_{1235} & +x_{35}x_{1234} & -x_{13}x_{2345} & +x_{23}x_{1345} \\
\eta_4(x) &=& x_{45}x_{1234} & -x_{14}x_{2345} & +x_{24}x_{1345} & -x_{34}x_{1245}, \\
\eta_5(x) &=& -x_{15}x_{2345} & +x_{25}x_{1345} & -x_{35}x_{1245} & +x_{45}x_{1235}, \\
\eta_6(x) &=& x_0x_{2345} & -x_{23}x_{45} & +x_{24}x_{35} & -x_{25}x_{34}, \\
\eta_7(x) &=& -x_0x_{1345} & +x_{34}x_{15} & -x_{35}x_{14} & +x_{13}x_{45}, \\
\eta_8(x) &=& x_0x_{1245} & -x_{45}x_{12} & +x_{14}x_{25} & -x_{24}x_{15}, \\
\eta_9(x) &=& -x_0x_{1235} & +x_{15}x_{23} & -x_{25}x_{13} & +x_{35}x_{12}, \\
\eta_{10}(x) &=& x_0x_{1234} & -x_{12}x_{34} & +x_{13}x_{24} & -x_{14}x_{23}
\end{array}$$

(cf. [7], [8]). Then we have

$$\eta(\alpha\ \Lambda'(A)(X\cdot Y)) = \alpha^2\ \chi(A)\ \eta(X\cdot Y), \tag{3.7}$$

where χ is the vector representation of Spin_{10}. Since the infinitesimal representation of χ (resp. Λ') is given by (5.28) (resp. (5.38)) in [1], we have

$$\chi(\mathrm{Spin}_{10}) = SO(10,K) = \{A \in SL_{10}\ ;\ {}^tAKA = K\} \text{ for } K = \left(\begin{array}{c|c} O & I_5 \\ \hline I_5 & O \end{array}\right).$$

On the other hand, we put

$$\begin{aligned}\tilde{X}_i = {}^t(&x_0^{(i)}, x_{12}^{(i)}, x_{13}^{(i)}, x_{14}^{(i)}, x_{15}^{(i)}, x_{23}^{(i)}, x_{24}^{(i)}, x_{25}^{(i)},\\ &x_{34}^{(i)}, x_{35}^{(i)}, x_{45}^{(i)}, x_{1234}^{(i)}, x_{1235}^{(i)}, x_{1245}^{(i)}, x_{1345}^{(i)}, x_{2345}^{(i)})\\ \in V&(16)\end{aligned}$$

and consider that $Z = \left[\tilde{X_1}, \cdots, \tilde{X_{14}}\right] \in M(16,14)$.

Let $Z^{(i,j)}$ be the 14×14-matrix obtained from $\tilde{Z}$ by substracting the i-th and the j-th rows and $\tilde{Z} := (z_{ij}) \in \mathrm{Alt}_{16}$ with $z_{ij} := (-1)^{i+j}\det Z^{(i,j)}$. Then, by Lemma 2.3, we have

$$\tilde{Z} \longmapsto (\det\Lambda'(A))\ (\det B)\ {}^t\Lambda'(A)^{-1}\ \tilde{Z}\ \Lambda'(A)^{-1}. \tag{3.8}$$

Note that $\det\Lambda'(A) = 1$. Now we shall construct $\Phi(\tilde{Z}) \in \mathrm{Sym}_{10}$ such that

$$\Phi(\tilde{Z}) \longmapsto (\det B)^2\ {}^t\chi(A)^{-1}\ \Phi(\tilde{Z})\ \chi(A)^{-1}. \tag{3.9}$$

If we put

$$\begin{aligned}\Phi(\tilde{Z}) &= (\varphi(\tilde{Z})_{i,j})_{1\le i<j\le 10}\\ &= \Phi(\tilde{Z}) = \left(\begin{array}{ccc|ccc} \varphi_{11} & \cdots & \varphi_{15} & \varphi_{16} & \cdots & \varphi_{110}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \varphi_{15} & \cdots & \varphi_{55} & \varphi_{56} & \cdots & \varphi_{510}\\ \hline \varphi_{16} & \cdots & \varphi_{56} & \varphi_{66} & \cdots & \varphi_{610}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \varphi_{110} & \cdots & \varphi_{510} & \varphi_{610} & \cdots & \varphi_{1010} \end{array}\right) \in \mathrm{Sym}_{10}\end{aligned}$$

with the following entries, then we have Lemma 3.4.

$\varphi_{11} := -z_{215}^2 + 2z_{1415}z_{23} - 2z_{1315}z_{24} + 2z_{1215}z_{25} - z_{314}^2 + 2z_{214}z_{315} + 2z_{1314}z_{34} - 2z_{1214}z_{35} - z_{413}^2 + 2z_{313}z_{414} - 2z_{213}z_{415} + 2z_{1213}z_{45} - z_{512}^2 + 2z_{412}z_{513} - 2z_{312}z_{514} + 2z_{212}z_{515}$,

$\varphi_{22} := -z_{216}^2 + 2z_{1416}z_{26} - 2z_{1316}z_{27} + 2z_{1216}z_{28} - z_{614}^2 + 2z_{214}z_{616} + 2z_{1314}z_{67} - 2z_{1214}z_{68} - z_{713}^2 + 2z_{613}z_{714} - 2z_{213}z_{716} + 2z_{1213}z_{78} - z_{812}^2 + 2z_{712}z_{813} - 2z_{612}z_{814} + 2z_{212}z_{816}$,

$\varphi_{33} := -z_{1012}^2 + 2z_{1216}z_{310} + 2z_{1016}z_{312} - z_{316}^2 + 2z_{1516}z_{36} - 2z_{1316}z_{39} - 2z_{1215}z_{610} - 2z_{1015}z_{612} - z_{615}^2 + 2z_{315}z_{616} + 2z_{1315}z_{69} + 2z_{1213}z_{910} + 2z_{1013}z_{912} - z_{913}^2 + 2z_{613}z_{915} - 2z_{313}z_{916}$,

$\varphi_{44} := -z_{1112}^2 + 2z_{1216}z_{411} + 2z_{1116}z_{412} - z_{416}^2 + 2z_{1516}z_{47} - 2z_{1416}z_{49} - 2z_{1215}z_{711} - 2z_{1115}z_{712} - z_{715}^2 + 2z_{415}z_{716} + 2z_{1415}z_{79} + 2z_{1214}z_{911} + 2z_{1114}z_{912} - z_{914}^2 + 2z_{714}z_{915} - 2z_{414}z_{916}$,

$\varphi_{55} := -z_{1014}^2 - z_{1113}^2 + 2z_{1013}z_{1114} + 2z_{1011}z_{1314} - 2z_{1416}z_{510} + 2z_{1316}z_{511} + 2z_{1116}z_{513} - 2z_{1016}z_{514} - z_{516}^2 + 2z_{1516}z_{58} + 2z_{1415}z_{810} - 2z_{1315}z_{811} - 2z_{1115}z_{813} + 2z_{1015}z_{814} - z_{815}^2 + 2z_{515}z_{816}$,

$$\varphi_{66} := -z_{116}^2+2z_{1116}z_{16}-2z_{1016}z_{17}-z_{611}^2+2z_{111}z_{616}+2z_{1011}z_{67}-z_{710}^2+2z_{610}z_{711}-2z_{110}z_{716}+2z_{79}z_{810}-2z_{69}z_{811}+2z_{19}z_{816}-z_{89}^2+2z_{78}z_{910}-2z_{68}z_{911}+2z_{18}z_{916},$$

$$\varphi_{77} := -z_{115}^2+2z_{1115}z_{13}-2z_{1015}z_{14}-z_{311}^2+2z_{111}z_{315}+2z_{1011}z_{34}-z_{410}^2+2z_{310}z_{411}-2z_{110}z_{415}+2z_{49}z_{510}-2z_{39}z_{511}+2z_{19}z_{515}-z_{59}^2+2z_{45}z_{910}-2z_{35}z_{911}+2z_{15}z_{915},$$

$$\varphi_{88} := -z_{114}^2+2z_{1114}z_{12}-z_{211}^2+2z_{111}z_{214}+2z_{28}z_{411}-2z_{18}z_{414}-z_{48}^2-2z_{27}z_{511}+2z_{17}z_{514}-z_{57}^2+2z_{47}z_{58}-2z_{25}z_{711}+2z_{15}z_{714}+2z_{45}z_{78}+2z_{24}z_{811}-2z_{14}z_{814}$$

$$\varphi_{99} := -z_{113}^2+2z_{1013}z_{12}-z_{210}^2+2z_{110}z_{213}+2z_{28}z_{310}-2z_{18}z_{313}-z_{38}^2-2z_{26}z_{510}+2z_{16}z_{513}-z_{56}^2+2z_{36}z_{58}-2z_{25}z_{610}+2z_{15}z_{613}+2z_{35}z_{68}+2z_{23}z_{810}-2z_{13}z_{813},$$

$$\varphi_{1010} := -z_{112}^2+2z_{19}z_{212}-z_{29}^2-2z_{17}z_{312}-z_{37}^2+2z_{27}z_{39}+2z_{16}z_{412}-z_{46}^2+2z_{36}z_{47}-2z_{26}z_{49}+2z_{14}z_{612}+2z_{34}z_{67}-2z_{24}z_{69}-2z_{13}z_{712}+2z_{23}z_{79}+2z_{12}z_{912},$$

$$\varphi_{12} := -z_{215}z_{216}+z_{1416}z_{23}-z_{1316}z_{24}+z_{1216}z_{25}+z_{1415}z_{26}-z_{1315}z_{27}+z_{1215}z_{28}+z_{214}z_{316}+z_{1314}z_{37}-z_{1214}z_{38}-z_{213}z_{416}-z_{1314}z_{46}+z_{1213}z_{48}+z_{212}z_{516}+z_{1214}z_{56}-z_{1213}z_{57}-z_{514}z_{612}+z_{414}z_{613}-z_{314}z_{614}+z_{214}z_{615}+z_{513}z_{712}-z_{413}z_{713}+z_{313}z_{714}-z_{213}z_{715}-z_{512}z_{812}+z_{412}z_{813}-z_{312}z_{814}+z_{212}z_{815},$$

$$\varphi_{13} := z_{1215}z_{210}+z_{1015}z_{212}+z_{1516}z_{23}-z_{1315}z_{29}-z_{1214}z_{310}-z_{1014}z_{312}-z_{216}z_{315}+z_{314}z_{316}-z_{1316}z_{34}+z_{1216}z_{35}+z_{1415}z_{36}+z_{1314}z_{39}+z_{1213}z_{410}+z_{1013}z_{412}-z_{313}z_{416}-z_{1315}z_{46}-z_{1012}z_{512}+z_{312}z_{516}+z_{1215}z_{56}-z_{1213}z_{59}-z_{515}z_{612}+z_{415}z_{613}-z_{315}z_{614}+z_{215}z_{615}+z_{513}z_{912}-z_{413}z_{913}+z_{313}z_{914}-z_{213}z_{915},$$

$$\varphi_{14} := z_{1215}z_{211}+z_{1115}z_{212}+z_{1516}z_{24}-z_{1415}z_{29}-z_{1214}z_{311}-z_{1114}z_{312}-z_{1416}z_{34}+z_{1415}z_{37}+z_{1213}z_{411}+z_{1113}z_{412}+z_{316}z_{414}-z_{216}z_{415}-z_{413}z_{416}+z_{1216}z_{45}-z_{1315}z_{47}+z_{1314}z_{49}-z_{1112}z_{512}+z_{412}z_{516}+z_{1215}z_{57}-z_{1214}z_{59}-z_{515}z_{712}+z_{415}z_{713}-z_{315}z_{714}+z_{215}z_{715}+z_{514}z_{912}-z_{414}z_{913}+z_{314}z_{914}-z_{214}z_{915},$$

$$\varphi_{15} := -z_{1415}z_{210}+z_{1315}z_{211}+z_{1115}z_{213}-z_{1015}z_{214}+z_{1516}z_{25}-z_{1314}z_{311}-z_{1114}z_{313}+z_{1014}z_{314}-z_{1416}z_{35}+z_{1415}z_{38}+z_{1314}z_{410}+z_{1113}z_{413}-z_{1013}z_{414}+z_{1316}z_{45}-z_{1315}z_{48}-z_{1214}z_{510}+z_{1213}z_{511}-z_{1112}z_{513}-z_{416}z_{513}+z_{1012}z_{514}+z_{316}z_{514}-z_{216}z_{515}+z_{512}z_{516}+z_{1215}z_{58}-z_{515}z_{812}+z_{415}z_{813}-z_{315}z_{814}+z_{215}z_{815},$$

$$\varphi_{16} := z_{1316}z_{14}-z_{1216}z_{15}-z_{1415}z_{16}+z_{115}z_{216}-z_{1116}z_{23}+z_{1015}z_{27}-z_{114}z_{316}-z_{1014}z_{37}+z_{113}z_{416}-z_{1113}z_{46}-z_{112}z_{516}+z_{1112}z_{56}-z_{1012}z_{57}+z_{511}z_{612}-z_{411}z_{613}+z_{311}z_{614}-z_{211}z_{615}+z_{410}z_{713}-z_{310}z_{714}+z_{210}z_{715}-z_{513}z_{79}-z_{412}z_{810}+z_{59}z_{812}+z_{39}z_{814}-z_{29}z_{815}-z_{48}z_{913}+z_{38}z_{914}-z_{28}z_{915},$$

$$\varphi_{17} := -z_{1315}z_{14}+z_{13}z_{1415}+z_{1215}z_{15}-z_{115}z_{215}+z_{1115}z_{23}-z_{1015}z_{24}-z_{311}z_{314}+z_{114}z_{315}+z_{211}z_{315}+z_{1014}z_{34}-z_{1113}z_{34}+z_{1112}z_{35}+z_{313}z_{411}-z_{410}z_{413}+z_{310}z_{414}-z_{113}z_{415}-z_{210}z_{415}-z_{1012}z_{45}+z_{412}z_{510}-z_{312}z_{511}+z_{49}z_{513}-z_{39}z_{514}+z_{112}z_{515}+z_{29}z_{515}-z_{512}z_{59}+z_{45}z_{913}-z_{35}z_{914}+z_{25}z_{915},$$

$$\varphi_{18} := z_{1314}z_{14}-z_{12}z_{1415}-z_{1214}z_{15}+z_{115}z_{214}-z_{211}z_{215}-z_{1114}z_{23}+z_{1113}z_{24}-z_{1112}z_{25}+z_{214}z_{311}-z_{114}z_{314}-z_{213}z_{411}+z_{113}z_{414}-z_{38}z_{414}+z_{28}z_{415}+z_{413}z_{48}+z_{212}z_{511}-z_{47}z_{513}-z_{112}z_{514}+z_{37}z_{514}-z_{27}z_{515}+z_{512}z_{57}-z_{412}z_{58}-z_{45}z_{713}+z_{35}z_{714}-z_{25}z_{715}+z_{45}z_{812}-z_{34}z_{814}+z_{24}z_{815},$$

$$\varphi_{19} := -z_{13}z_{1314}+z_{12}z_{1315}+z_{1213}z_{15}-z_{115}z_{213}+z_{210}z_{215}+z_{1014}z_{23}-z_{1013}z_{24}+z_{1012}z_{25}-z_{214}z_{310}+z_{114}z_{313}-z_{28}z_{315}+z_{314}z_{38}+z_{213}z_{410}-z_{113}z_{413}-z_{313}z_{48}-z_{212}z_{510}+z_{112}z_{513}+z_{46}z_{513}-z_{36}z_{514}+z_{26}z_{515}-z_{512}z_{56}+z_{312}z_{58}+z_{45}z_{613}-z_{35}z_{614}+z_{25}z_{615}-z_{35}z_{812}+z_{34}z_{813}-z_{23}z_{815},$$

$$\varphi_{110} := -z_{12}z_{1215}+z_{1214}z_{13}-z_{1213}z_{14}+z_{115}z_{212}-z_{215}z_{29}-z_{114}z_{312}+z_{27}z_{315}-z_{314}z_{37}+z_{214}z_{39}+z_{113}z_{412}+z_{36}z_{414}-z_{26}z_{415}-z_{413}z_{46}+z_{313}z_{47}-z_{213}z_{49}-z_{112}z_{512}+z_{412}z_{56}-z_{312}z_{57}+z_{212}z_{59}-z_{45}z_{612}+z_{34}z_{614}-z_{24}z_{615}+z_{35}z_{712}-z_{34}z_{713}+z_{23}z_{715}-z_{25}z_{912}+z_{24}z_{913}-z_{23}z_{914},$$

$$\varphi_{23} := z_{1216}z_{210}+z_{1016}z_{212}+z_{1516}z_{26}-z_{1316}z_{29}-z_{216}z_{316}+z_{1416}z_{36}-z_{1316}z_{37}+z_{1216}z_{38}-z_{1214}z_{610}-z_{1014}z_{612}-z_{614}z_{615}+z_{215}z_{616}+z_{314}z_{616}+z_{1315}z_{67}-z_{1215}z_{68}+z_{1314}z_{69}+z_{1213}z_{710}+z_{1013}z_{712}+z_{613}z_{715}-z_{313}z_{716}-z_{1012}z_{812}-z_{612}z_{815}+z_{312}z_{816}-$$

$z_{1213}z_{89} + z_{813}z_{912} - z_{713}z_{913} + z_{613}z_{914} - z_{213}z_{916}$,

$\varphi_{24} := z_{1216}z_{211} + z_{1116}z_{212} + z_{1516}z_{27} - z_{1416}z_{29} - z_{216}z_{416} + z_{1416}z_{46} - z_{1316}z_{47} + z_{1216}z_{48} - z_{1214}z_{611} - z_{1114}z_{612} + z_{414}z_{616} + z_{1415}z_{67} + z_{1213}z_{711} + z_{1113}z_{712} - z_{615}z_{714} + z_{713}z_{715} + z_{215}z_{716} - z_{413}z_{716} - z_{1215}z_{78} + z_{1314}z_{79} - z_{1112}z_{812} - z_{712}z_{815} + z_{412}z_{816} - z_{1214}z_{89} + z_{814}z_{912} - z_{714}z_{913} + z_{614}z_{914} - z_{214}z_{916}$,

$\varphi_{25} := -z_{1416}z_{210} + z_{1316}z_{211} + z_{1116}z_{213} - z_{1016}z_{214} + z_{1516}z_{28} - z_{216}z_{516} + z_{1416}z_{56} - z_{1316}z_{57} + z_{1216}z_{58} - z_{1314}z_{611} - z_{1114}z_{613} + z_{1014}z_{614} + z_{514}z_{616} + z_{1415}z_{68} + z_{1314}z_{710} + z_{1113}z_{713} - z_{1013}z_{714} - z_{513}z_{716} - z_{1315}z_{78} - z_{1214}z_{810} + z_{1213}z_{811} - z_{1112}z_{813} + z_{715}z_{813} + z_{1012}z_{814} - z_{615}z_{814} - z_{812}z_{815} + z_{215}z_{816} + z_{512}z_{816}$,

$\varphi_{26} := -z_{1416}z_{16} + z_{1316}z_{17} - z_{1216}z_{18} + z_{116}z_{216} - z_{1116}z_{26} + z_{1016}z_{27} + z_{611}z_{614} - z_{114}z_{616} - z_{211}z_{616} - z_{1014}z_{67} + z_{1113}z_{67} - z_{1112}z_{68} - z_{613}z_{711} + z_{710}z_{713} - z_{610}z_{714} + z_{113}z_{716} + z_{210}z_{716} + z_{1012}z_{78} - z_{712}z_{810} + z_{612}z_{811} - z_{79}z_{813} + z_{69}z_{814} - z_{112}z_{816} - z_{29}z_{816} + z_{812}z_{89} - z_{78}z_{913} + z_{68}z_{914} - z_{28}z_{916}$,

$\varphi_{27} := z_{13}z_{1416} - z_{1315}z_{17} + z_{1215}z_{18} - z_{116}z_{215} - z_{1016}z_{24} + z_{1115}z_{26} + z_{211}z_{316} - z_{1113}z_{37} + z_{1112}z_{38} - z_{210}z_{416} - z_{1014}z_{46} - z_{1012}z_{48} + z_{29}z_{516} + z_{414}z_{610} - z_{314}z_{611} + z_{114}z_{615} - z_{514}z_{69} - z_{413}z_{710} + z_{313}z_{711} + z_{510}z_{712} - z_{113}z_{715} - z_{312}z_{811} + z_{49}z_{813} + z_{112}z_{815} - z_{512}z_{89} - z_{57}z_{913} + z_{56}z_{914} + z_{25}z_{916}$,

$\varphi_{28} := -z_{12}z_{1416} + z_{1314}z_{17} - z_{1214}z_{18} + z_{116}z_{214} - z_{211}z_{216} - z_{1114}z_{26} + z_{1113}z_{27} - z_{1112}z_{28} + z_{28}z_{416} - z_{27}z_{516} + z_{214}z_{611} - z_{114}z_{614} + z_{514}z_{67} - z_{414}z_{68} - z_{213}z_{711} - z_{58}z_{712} + z_{57}z_{713} + z_{113}z_{714} - z_{56}z_{714} - z_{25}z_{716} + z_{413}z_{78} - z_{512}z_{78} + z_{212}z_{811} + z_{48}z_{812} - z_{47}z_{813} - z_{112}z_{814} + z_{46}z_{814} + z_{24}z_{816}$,

$\varphi_{29} := z_{12}z_{1316} - z_{1314}z_{16} + z_{1213}z_{18} - z_{116}z_{213} + z_{210}z_{216} + z_{1014}z_{26} - z_{1013}z_{27} + z_{1012}z_{28} - z_{28}z_{316} + z_{26}z_{516} - z_{214}z_{610} + z_{58}z_{612} + z_{114}z_{613} - z_{57}z_{613} + z_{56}z_{614} + z_{25}z_{616} - z_{513}z_{67} + z_{314}z_{68} + z_{512}z_{68} + z_{213}z_{710} - z_{113}z_{713} - z_{313}z_{78} - z_{212}z_{810} - z_{38}z_{812} + z_{112}z_{813} + z_{37}z_{813} - z_{36}z_{814} - z_{23}z_{816}$,

$\varphi_{210} := -z_{12}z_{1216} + z_{1214}z_{16} - z_{1213}z_{17} + z_{116}z_{212} - z_{216}z_{29} + z_{27}z_{316} - z_{26}z_{416} - z_{114}z_{612} - z_{48}z_{612} + z_{47}z_{613} - z_{46}z_{614} - z_{24}z_{616} - z_{314}z_{67} + z_{413}z_{67} - z_{412}z_{68} + z_{214}z_{69} + z_{113}z_{712} + z_{38}z_{712} - z_{37}z_{713} + z_{36}z_{714} + z_{23}z_{716} + z_{312}z_{78} - z_{213}z_{79} - z_{112}z_{812} + z_{212}z_{89} - z_{28}z_{912} + z_{27}z_{913} - z_{26}z_{914}$,

$\varphi_{34} := -z_{1012}z_{1112} + z_{1216}z_{311} + z_{1116}z_{312} + z_{1516}z_{37} - z_{1416}z_{39} + z_{1216}z_{410} + z_{1016}z_{412} - z_{316}z_{416} + z_{1516}z_{46} - z_{1316}z_{49} - z_{1215}z_{611} - z_{1115}z_{612} + z_{415}z_{616} + z_{1415}z_{69} - z_{1215}z_{710} - z_{1015}z_{712} - z_{615}z_{715} + z_{315}z_{716} + z_{1315}z_{79} + z_{1214}z_{910} + z_{1213}z_{911} + z_{1014}z_{912} + z_{1113}z_{912} - z_{913}z_{914} + z_{614}z_{915} + z_{713}z_{915} - z_{314}z_{916} - z_{413}z_{916}$,

$\varphi_{35} := z_{1012}z_{1014} - z_{1013}z_{1112} + z_{1011}z_{1213} - z_{1416}z_{310} + z_{1316}z_{311} + z_{1116}z_{313} - z_{1016}z_{314} + z_{1516}z_{38} + z_{1216}z_{510} + z_{1016}z_{512} - z_{316}z_{516} + z_{1516}z_{56} - z_{1316}z_{59} + z_{1415}z_{610} - z_{1315}z_{611} - z_{1115}z_{613} + z_{1015}z_{614} + z_{515}z_{616} - z_{1215}z_{810} - z_{1015}z_{812} - z_{615}z_{815} + z_{315}z_{816} + z_{1315}z_{89} + z_{1314}z_{910} + z_{1113}z_{913} - z_{1013}z_{914} + z_{813}z_{915} - z_{513}z_{916}$,

$\varphi_{36} := -z_{1016}z_{112} - z_{110}z_{1216} - z_{1516}z_{16} + z_{1316}z_{19} + z_{116}z_{316} - z_{1116}z_{36} + z_{1016}z_{37} - z_{1112}z_{610} + z_{1011}z_{612} + z_{611}z_{615} - z_{115}z_{616} - z_{311}z_{616} - z_{1015}z_{67} + z_{1113}z_{69} + z_{1012}z_{710} - z_{610}z_{715} + z_{310}z_{716} - z_{1013}z_{79} + z_{69}z_{815} - z_{39}z_{816} + z_{713}z_{910} - z_{812}z_{910} - z_{613}z_{911} - z_{810}z_{912} + z_{89}z_{913} + z_{68}z_{915} + z_{113}z_{916} - z_{38}z_{916}$,

$\varphi_{37} := z_{1015}z_{112} + z_{110}z_{1215} + z_{13}z_{1516} - z_{1315}z_{19} + z_{1112}z_{310} - z_{1011}z_{312} - z_{116}z_{315} + z_{311}z_{316} - z_{1016}z_{34} + z_{1115}z_{36} - z_{1113}z_{39} - z_{1012}z_{410} - z_{310}z_{416} - z_{1015}z_{46} + z_{1013}z_{49} + z_{39}z_{516} + z_{415}z_{610} - z_{315}z_{611} + z_{115}z_{615} - z_{515}z_{69} - z_{413}z_{910} + z_{512}z_{910} + z_{313}z_{911} + z_{510}z_{912} - z_{59}z_{913} - z_{113}z_{915} + z_{56}z_{915} + z_{35}z_{916}$,

$\varphi_{38} := -z_{1014}z_{112} - z_{110}z_{1214} - z_{12}z_{1516} + z_{1314}z_{19} - z_{1112}z_{210} + z_{1011}z_{212} + z_{1113}z_{29} - z_{216}z_{311} + z_{116}z_{314} - z_{1114}z_{36} + z_{38}z_{416} - z_{1013}z_{47} - z_{37}z_{516} + z_{215}z_{611} - z_{115}z_{614} + z_{515}z_{67} - z_{415}z_{68} - z_{512}z_{710} + z_{59}z_{713} - z_{56}z_{715} - z_{35}z_{716} + z_{410}z_{812} + z_{46}z_{815} + z_{34}z_{816} - z_{413}z_{89} - z_{213}z_{911} - z_{58}z_{912} + z_{113}z_{914}$,

$\varphi_{39} := z_{1013}z_{112} + z_{110}z_{1213} + z_{13}z_{1316} - z_{1315}z_{16} + z_{1012}z_{210} - z_{1016}z_{23} + z_{1015}z_{26} - z_{1013}z_{29} + z_{216}z_{310} - z_{116}z_{313} - z_{316}z_{38} + z_{36}z_{516} - z_{215}z_{610} + z_{512}z_{610} + z_{510}z_{612} + z_{115}z_{613} - z_{59}z_{613} + z_{56}z_{615} + z_{35}z_{616} + z_{315}z_{68} - z_{513}z_{69} - z_{312}z_{810} - z_{310}z_{812} + z_{39}z_{813} - z_{36}z_{815} + z_{313}z_{89} + z_{213}z_{910} - z_{113}z_{913},$

$\varphi_{310} := -z_{1012}z_{112} - z_{1216}z_{13} + z_{1215}z_{16} - z_{1213}z_{19} + z_{116}z_{312} + z_{316}z_{37} - z_{216}z_{39} - z_{36}z_{416} - z_{412}z_{610} - z_{115}z_{612} - z_{410}z_{612} + z_{49}z_{613} - z_{46}z_{615} - z_{34}z_{616} - z_{315}z_{67} + z_{215}z_{69} + z_{413}z_{69} + z_{312}z_{710} + z_{310}z_{712} - z_{39}z_{713} + z_{36}z_{715} - z_{313}z_{79} - z_{212}z_{910} + z_{113}z_{912} - z_{210}z_{912} + z_{29}z_{913} - z_{26}z_{915} + z_{23}z_{916},$

$\varphi_{45} := -z_{1112}z_{1113} + z_{1012}z_{1114} + z_{1011}z_{1214} - z_{1416}z_{410} + z_{1316}z_{411} + z_{1116}z_{413} - z_{1016}z_{414} + z_{1516}z_{48} + z_{1216}z_{511} + z_{1116}z_{512} - z_{416}z_{516} + z_{1516}z_{57} - z_{1416}z_{59} + z_{1415}z_{710} - z_{1315}z_{711} - z_{1115}z_{713} + z_{1015}z_{714} + z_{515}z_{716} - z_{1215}z_{811} - z_{1115}z_{812} - z_{715}z_{815} + z_{415}z_{816} + z_{1415}z_{89} + z_{1314}z_{911} + z_{1114}z_{913} - z_{1014}z_{914} + z_{814}z_{915} - z_{514}z_{916},$

$\varphi_{46} := -z_{1116}z_{112} - z_{111}z_{1216} - z_{1516}z_{17} + z_{1416}z_{19} + z_{116}z_{416} - z_{1116}z_{46} + z_{1016}z_{47} - z_{1112}z_{611} - z_{411}z_{616} - z_{1115}z_{67} + z_{1114}z_{69} + z_{1012}z_{711} + z_{615}z_{711} + z_{1011}z_{712} - z_{710}z_{715} - z_{115}z_{716} + z_{410}z_{716} - z_{1014}z_{79} + z_{79}z_{815} - z_{49}z_{816} + z_{714}z_{910} - z_{614}z_{911} - z_{812}z_{911} - z_{811}z_{912} + z_{89}z_{914} + z_{78}z_{915} + z_{114}z_{916} - z_{48}z_{916},$

$\varphi_{47} := z_{1115}z_{112} + z_{111}z_{1215} + z_{14}z_{1516} - z_{1415}z_{19} + z_{1112}z_{311} - z_{1116}z_{34} + z_{1115}z_{37} - z_{1114}z_{39} - z_{1012}z_{411} + z_{316}z_{411} - z_{1011}z_{412} - z_{116}z_{415} - z_{410}z_{416} - z_{1015}z_{47} + z_{1014}z_{49} + z_{49}z_{516} + z_{415}z_{710} - z_{315}z_{711} + z_{115}z_{715} - z_{515}z_{79} - z_{414}z_{910} + z_{314}z_{911} + z_{512}z_{911} + z_{511}z_{912} - z_{59}z_{914} - z_{114}z_{915} + z_{57}z_{915} + z_{45}z_{916},$

$\varphi_{48} := -z_{1114}z_{112} - z_{111}z_{1214} - z_{14}z_{1416} + z_{1415}z_{17} - z_{1112}z_{211} + z_{1116}z_{24} - z_{1115}z_{27} + z_{1114}z_{29} - z_{216}z_{411} + z_{116}z_{414} + z_{416}z_{48} - z_{47}z_{516} + z_{215}z_{711} - z_{512}z_{711} - z_{511}z_{712} - z_{115}z_{714} + z_{59}z_{714} - z_{57}z_{715} - z_{45}z_{716} - z_{415}z_{78} + z_{514}z_{79} + z_{412}z_{811} + z_{411}z_{812} - z_{49}z_{814} + z_{47}z_{815} - z_{414}z_{89} - z_{214}z_{911} + z_{114}z_{914},$

$\varphi_{49} := z_{1113}z_{112} + z_{111}z_{1213} - z_{12}z_{1516} + z_{1314}z_{19} + z_{1012}z_{211} + z_{1011}z_{212} - z_{1014}z_{29} - z_{1114}z_{36} + z_{216}z_{410} - z_{116}z_{413} - z_{1013}z_{47} - z_{316}z_{48} + z_{46}z_{516} + z_{512}z_{611} - z_{59}z_{614} + z_{57}z_{615} + z_{45}z_{616} + z_{515}z_{67} - z_{215}z_{710} + z_{115}z_{713} + z_{315}z_{78} - z_{311}z_{812} - z_{37}z_{815} + z_{34}z_{816} + z_{314}z_{89} + z_{214}z_{910} - z_{58}z_{912} - z_{114}z_{913},$

$\varphi_{410} := -z_{1112}z_{112} - z_{1216}z_{14} + z_{1215}z_{17} - z_{1214}z_{19} + z_{116}z_{412} - z_{416}z_{46} + z_{316}z_{47} - z_{216}z_{49} - z_{412}z_{611} - z_{411}z_{612} + z_{49}z_{614} - z_{47}z_{615} - z_{415}z_{67} + z_{414}z_{69} + z_{312}z_{711} - z_{115}z_{712} + z_{311}z_{712} - z_{39}z_{714} + z_{37}z_{715} - z_{34}z_{716} + z_{215}z_{79} - z_{314}z_{79} - z_{212}z_{911} + z_{114}z_{912} - z_{211}z_{912} + z_{29}z_{914} - z_{27}z_{915} + z_{24}z_{916},$

$\varphi_{56} := -z_{1116}z_{113} + z_{1016}z_{114} - z_{111}z_{1316} + z_{110}z_{1416} - z_{1516}z_{18} + z_{116}z_{516} - z_{1116}z_{56} + z_{1016}z_{57} + z_{1114}z_{610} - z_{1113}z_{611} - z_{1011}z_{614} - z_{511}z_{616} - z_{1115}z_{68} - z_{1014}z_{710} + z_{1013}z_{711} + z_{1011}z_{713} + z_{510}z_{716} + z_{1015}z_{78} - z_{715}z_{810} + z_{615}z_{811} - z_{115}z_{816} - z_{59}z_{816} + z_{815}z_{89} + z_{814}z_{910} - z_{813}z_{911} - z_{811}z_{913} + z_{810}z_{914} - z_{58}z_{916},$

$\varphi_{57} := z_{1115}z_{113} - z_{1015}z_{114} + z_{111}z_{1315} - z_{110}z_{1415} + z_{15}z_{1516} - z_{1114}z_{310} + z_{1113}z_{311} + z_{1011}z_{314} - z_{1116}z_{35} + z_{1115}z_{38} + z_{1014}z_{410} - z_{1013}z_{411} - z_{1011}z_{413} + z_{1016}z_{45} - z_{1015}z_{48} - z_{416}z_{510} + z_{316}z_{511} - z_{116}z_{515} + z_{516}z_{59} + z_{415}z_{810} - z_{315}z_{811} + z_{115}z_{815} - z_{515}z_{89} - z_{514}z_{910} + z_{513}z_{911} + z_{511}z_{913} - z_{510}z_{914} + z_{58}z_{915},$

$\varphi_{58} := -z_{1114}z_{113} + z_{1014}z_{114} - z_{111}z_{1314} - z_{1416}z_{15} + z_{1415}z_{18} + z_{1114}z_{210} - z_{1113}z_{211} - z_{1011}z_{214} + z_{1116}z_{25} - z_{1115}z_{28} - z_{216}z_{511} + z_{116}z_{514} - z_{516}z_{57} + z_{416}z_{58} + z_{514}z_{710} - z_{513}z_{711} - z_{511}z_{713} + z_{510}z_{714} - z_{58}z_{715} - z_{515}z_{78} - z_{414}z_{810} + z_{215}z_{811} + z_{413}z_{811} + z_{411}z_{813} - z_{115}z_{814} - z_{410}z_{814} + z_{48}z_{815} - z_{45}z_{816},$

$\varphi_{59} := z_{1113}z_{113} - z_{1013}z_{114} + z_{110}z_{1314} + z_{1316}z_{15} - z_{1315}z_{18} - z_{1014}z_{210} + z_{1013}z_{211} + z_{1011}z_{213} - z_{1016}z_{25} + z_{1015}z_{28} + z_{216}z_{510} - z_{116}z_{513} + z_{516}z_{56} - z_{316}z_{58} - z_{514}z_{610} + z_{513}z_{611} + z_{511}z_{613} - z_{510}z_{614} + z_{58}z_{615} + z_{515}z_{68} - z_{215}z_{810} + z_{314}z_{810} - z_{313}z_{811} + z_{115}z_{813} - z_{311}z_{813} + z_{310}z_{814} - z_{38}z_{815} + z_{35}z_{816},$

$\varphi_{510} := -z_{1112}z_{113} + z_{1012}z_{114} + z_{111}z_{1213} - z_{110}z_{1214} - z_{12}z_{1516} - z_{1114}z_{36} - z_{1013}z_{47} +$

$z_{116}z_{512}-z_{416}z_{56}+z_{316}z_{57}-z_{216}z_{59}-z_{413}z_{611}+z_{410}z_{614}-z_{48}z_{615}+z_{45}z_{616}-z_{415}z_{68}-z_{314}z_{710}+z_{311}z_{713}+z_{38}z_{715}-z_{35}z_{716}+z_{315}z_{78}-z_{115}z_{812}+z_{215}z_{89}+z_{214}z_{910}-z_{213}z_{911}-z_{58}z_{912}-z_{211}z_{913}+z_{210}z_{914},$

$\varphi_{67} := z_{115}z_{116}-z_{1116}z_{13}+z_{1016}z_{14}-z_{1115}z_{16}+z_{1015}z_{17}-z_{111}z_{316}-z_{1011}z_{37}+z_{110}z_{416}+z_{1011}z_{46}-z_{19}z_{516}-z_{411}z_{610}+z_{311}z_{611}-z_{111}z_{615}+z_{511}z_{69}+z_{410}z_{710}-z_{310}z_{711}+z_{110}z_{715}-z_{510}z_{79}-z_{49}z_{810}+z_{39}z_{811}-z_{19}z_{815}+z_{59}z_{89}-z_{48}z_{910}+z_{57}z_{910}+z_{38}z_{911}-z_{56}z_{911}-z_{18}z_{915}-z_{15}z_{916},$

$\varphi_{68} := -z_{114}z_{116}+z_{1116}z_{12}+z_{1114}z_{16}-z_{1014}z_{17}+z_{111}z_{216}+z_{1011}z_{27}-z_{18}z_{416}+z_{17}z_{516}-z_{211}z_{611}+z_{111}z_{614}-z_{511}z_{67}+z_{411}z_{68}-z_{57}z_{710}+z_{210}z_{711}+z_{56}z_{711}-z_{110}z_{714}+z_{15}z_{716}-z_{410}z_{78}+z_{59}z_{78}+z_{58}z_{79}+z_{47}z_{810}-z_{29}z_{811}-z_{46}z_{811}+z_{19}z_{814}-z_{14}z_{816}-z_{48}z_{89}-z_{28}z_{911}+z_{18}z_{914},$

$\varphi_{69} := z_{113}z_{116}-z_{1016}z_{12}-z_{1113}z_{16}+z_{1013}z_{17}-z_{110}z_{216}-z_{1011}z_{26}+z_{18}z_{316}-z_{16}z_{516}+z_{211}z_{610}+z_{57}z_{610}-z_{56}z_{611}-z_{111}z_{613}-z_{15}z_{616}+z_{510}z_{67}-z_{311}z_{68}-z_{59}z_{68}-z_{58}z_{69}-z_{210}z_{710}+z_{110}z_{713}+z_{310}z_{78}+z_{29}z_{810}-z_{37}z_{810}+z_{36}z_{811}-z_{19}z_{813}+z_{13}z_{816}+z_{38}z_{89}+z_{28}z_{910}-z_{18}z_{913},$

$\varphi_{610} := -z_{112}z_{116}+z_{1112}z_{16}-z_{1012}z_{17}+z_{19}z_{216}-z_{17}z_{316}+z_{16}z_{416}-z_{47}z_{610}+z_{46}z_{611}+z_{111}z_{612}+z_{14}z_{616}+z_{311}z_{67}-z_{410}z_{67}+z_{49}z_{68}-z_{211}z_{69}+z_{48}z_{69}+z_{37}z_{710}-z_{36}z_{711}-z_{110}z_{712}-z_{13}z_{716}-z_{39}z_{78}+z_{210}z_{79}-z_{38}z_{79}+z_{19}z_{812}-z_{29}z_{89}-z_{27}z_{910}+z_{26}z_{911}+z_{18}z_{912}+z_{12}z_{916},$

$\varphi_{78} := z_{114}z_{115}-z_{1115}z_{12}-z_{1114}z_{13}+z_{1014}z_{14}-z_{111}z_{215}-z_{1011}z_{24}+z_{211}z_{311}-z_{111}z_{314}-z_{210}z_{411}-z_{38}z_{411}+z_{110}z_{414}+z_{18}z_{415}+z_{410}z_{48}-z_{47}z_{510}+z_{29}z_{511}+z_{37}z_{511}-z_{19}z_{514}-z_{17}z_{515}-z_{49}z_{58}+z_{57}z_{59}-z_{45}z_{710}+z_{35}z_{711}-z_{15}z_{715}-z_{34}z_{811}+z_{14}z_{815}+z_{45}z_{89}+z_{25}z_{911}-z_{15}z_{914},$

$\varphi_{79} := -z_{113}z_{115}+z_{1015}z_{12}+z_{1113}z_{13}-z_{1013}z_{14}+z_{110}z_{215}+z_{1011}z_{23}-z_{211}z_{310}+z_{111}z_{313}-z_{18}z_{315}+z_{311}z_{38}+z_{210}z_{410}-z_{110}z_{413}-z_{310}z_{48}-z_{29}z_{510}+z_{46}z_{510}-z_{36}z_{511}+z_{19}z_{513}+z_{16}z_{515}+z_{39}z_{58}-z_{56}z_{59}+z_{45}z_{610}-z_{35}z_{611}+z_{15}z_{615}+z_{34}z_{810}-z_{13}z_{815}-z_{35}z_{89}-z_{25}z_{910}+z_{15}z_{913},$

$\varphi_{710} := z_{112}z_{115}-z_{1112}z_{13}+z_{1012}z_{14}-z_{19}z_{215}-z_{111}z_{312}+z_{17}z_{315}-z_{311}z_{37}+z_{211}z_{39}+z_{36}z_{411}+z_{110}z_{412}-z_{16}z_{415}-z_{410}z_{46}+z_{310}z_{47}-z_{210}z_{49}-z_{19}z_{512}+z_{49}z_{56}-z_{39}z_{57}+z_{29}z_{59}+z_{34}z_{611}-z_{14}z_{615}-z_{45}z_{69}-z_{34}z_{710}+z_{13}z_{715}+z_{35}z_{79}+z_{24}z_{910}-z_{23}z_{911}-z_{15}z_{912}-z_{12}z_{915},$

$\varphi_{89} := z_{113}z_{114}-z_{1014}z_{12}-z_{1113}z_{12}+z_{210}z_{211}-z_{111}z_{213}-z_{110}z_{214}-z_{28}z_{311}+z_{18}z_{314}-z_{28}z_{410}+z_{18}z_{413}+z_{38}z_{48}+z_{27}z_{510}+z_{26}z_{511}-z_{17}z_{513}-z_{16}z_{514}+z_{56}z_{57}-z_{37}z_{58}-z_{46}z_{58}+z_{25}z_{611}-z_{15}z_{614}-z_{45}z_{68}+z_{25}z_{710}-z_{15}z_{713}-z_{35}z_{78}-z_{24}z_{810}-z_{23}z_{811}+z_{14}z_{813}+z_{13}z_{814},$

$\varphi_{810} := -z_{112}z_{114}+z_{1112}z_{12}+z_{111}z_{212}+z_{19}z_{214}-z_{211}z_{29}+z_{27}z_{311}-z_{17}z_{314}-z_{26}z_{411}-z_{18}z_{412}+z_{16}z_{414}-z_{38}z_{47}+z_{46}z_{48}+z_{28}z_{49}+z_{17}z_{512}-z_{47}z_{56}+z_{37}z_{57}-z_{27}z_{59}-z_{24}z_{611}+z_{14}z_{614}+z_{45}z_{67}+z_{23}z_{711}+z_{15}z_{712}-z_{13}z_{714}+z_{34}z_{78}-z_{25}z_{79}-z_{14}z_{812}+z_{24}z_{89}+z_{12}z_{914},$

$\varphi_{910} := z_{112}z_{113}-z_{1012}z_{12}-z_{110}z_{212}-z_{19}z_{213}+z_{210}z_{29}-z_{27}z_{310}+z_{18}z_{312}+z_{17}z_{313}+z_{37}z_{38}-z_{28}z_{39}+z_{26}z_{410}-z_{16}z_{413}-z_{36}z_{48}-z_{16}z_{512}+z_{46}z_{56}-z_{36}z_{57}+z_{26}z_{59}+z_{24}z_{610}-z_{15}z_{612}-z_{14}z_{613}-z_{35}z_{67}-z_{34}z_{68}+z_{25}z_{69}-z_{23}z_{710}+z_{13}z_{713}+z_{13}z_{812}-z_{23}z_{89}-z_{12}z_{913}.$

Lemma 3.4. *For every* $A \in \mathrm{Spin}_{10}$, $B \in GL_{14}$, $\tilde{Z} \in \mathrm{Alt}_{16}$, *we have* $\tilde{Z} \mapsto (\det B)\tilde{Z}$, $\tilde{Z} \mapsto {}^t\Lambda'(A)^{-1}\tilde{Z}\Lambda'(A)^{-1}$ *and hence*

$$\begin{gathered} \Phi((\det B)\tilde{Z}) \mapsto (\det B)^2\Phi(\tilde{Z}), \\ \Phi({}^t\Lambda'(A)^{-1}\tilde{Z}\Lambda'(A)^{-1}) = {}^t\chi(A)^{-1}\Phi(\tilde{Z})\chi(A)^{-1}, \end{gathered} \tag{3.10}$$

where χ is the vector representation of Spin_{10} in (3.9). Hence we have, by the action of $(\alpha, A, B) \in GL_1 \times \mathrm{Spin}_{10} \times GL_{14}$,

$$\Phi(\tilde{Z}) \mapsto (\det B)^t \chi(A)^{-1} \Phi(\tilde{Z}) \chi(A)^{-1}.$$

Proof. We may prove the equivariance (3.10) for $A's$ of one of the following three forms:

$$A_u = \left(\begin{array}{ccccc|ccccc} 1 & \varepsilon & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 1 & & & & \\ & & & & & -\varepsilon & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right),$$

$$A_{u'} = \left(\begin{array}{ccccc|ccccc} 1 & & & & & & \varepsilon & & & \\ & 1 & & & & -\varepsilon & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 1 & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right);$$

$$A_{u''} = \left(\begin{array}{ccccc|ccccc} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & \varepsilon & & & & 1 & & & & \\ -\varepsilon & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right),$$

$$A_d = \mathrm{diag}(a_1, a_2, a_3, a_4, a_5, a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}, a_5^{-1});$$

and permutation type matrices such as

$$A_u = \left(\begin{array}{ccccc|ccccc} 0 & 1 & & & & & & & & \\ 1 & 0 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 0 & 1 & & & \\ & & & & & 1 & 0 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right).$$

Checking (3.10), for diagonal or permutation type matrices is easy. Note that, for A_d, $z_{12} \mapsto a_3a_4a_5z_{12}, z_{13} \mapsto a_2a_4a_5z_{13}, \ldots, z_{23} \mapsto a_1^{-1}a_4a_5z_{23}, z_{24} \mapsto a_1^{-1}a_3a_5z_{24}, \ldots, z_{1516} \mapsto a_3^{-1}a_4^{-1}a_5^{-1}z_{1516}$. Hence $\varphi_{ij} \mapsto a_ia_j\varphi_{ij}$ for $1 \le i \le j \le 5$, $\varphi_{lk} \mapsto a_l^{-1}a_k^{-1}\varphi_{lk}$ for $6 \le l \le k \le 10$, $\varphi_{lj} \mapsto a_l^{-1}a_j\varphi_{lj}$ for $6 \le l \le 10, 1 \le j \le 5$. Then we have $\Phi({}^t\Lambda'(A_d)^{-1}\tilde{Z}\Lambda'(A_d)^{-1}) = {}^t\chi(A_d)^{-1}\Phi(\tilde{Z})\chi(A_d)^{-1}$. For A_u, we consider the action of A_u. Since $\det A_u = 1$, $\Phi(\tilde{Z}) \mapsto \Phi({}^t\Lambda'(A_u)\tilde{Z}\Lambda'(A_u)^{-1})$. Then we have $\varphi_{11} \mapsto \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^2\varphi_{22}$, $\varphi_{1j} \mapsto \varphi_{1j} + \varepsilon\varphi_{2j}$ $(1 \le j \le 5)$, $\varphi_{77} \mapsto \varphi_{77} - 2\varepsilon\varphi_{67} + \varepsilon^2\varphi_{66}$, $\varphi_{67} \mapsto \varphi_{67} - \varepsilon\varphi_{66}$, $\varphi_{7k} \mapsto \varphi_{7k} - \varepsilon\varphi_{6k}$ $(8 \le k \le 10)$, $\varphi_{17} \mapsto \varphi_{17} + \varepsilon(\varphi_{27} - \varphi_{16}) + \varepsilon^2\varphi_{26}$, $\varphi_{1j} \mapsto \varphi_{1j} + \varepsilon\varphi_{2j}$ $(j \in \{6, 8, 9, 10\})$, $\varphi_{k7} \mapsto \varphi_{k7} - \varepsilon\varphi_{k6}$ $(k \in \{2, 3, 4, 5\})$ and other φ_{ij}'s are invariants. Thus, we have $\Phi({}^t\Lambda'(A_u)^{-1}\tilde{Z}\Lambda'(A_u)^{-1}) = {}^t\chi(A_u)^{-1}\Phi(\tilde{Z}){}^t\chi(A_u)^{-1}$.

In the same way, we have $\Phi({}^t\Lambda'(A_{u'})^{-1}\tilde{Z}\Lambda'(A_{u'})^{-1}) = {}^t\chi(A_{u'})^{-1}\Phi(\tilde{Z})\chi(A_{u'})^{-1}$ and $\Phi({}^t\Lambda'(A_{u''})^{-1}\tilde{Z}\Lambda'(A_{u''})^{-1}) = {}^t\chi(A_{u''})^{-1}\Phi(\tilde{Z})\chi(A_{u''})^{-1}$. □

Remark C. We can construct the above polynomials φ_{ij} $(1 \le i \le j \le 10)$ in the same program as in Remark A in §§3.1. We have

$$K\Phi(\tilde{Z}) \mapsto (\det B)^2K{}^t\chi(A)^{-1}\Phi(\tilde{Z})\chi(A)^{-1} = (\det B)^2\chi(A)K\Phi(\tilde{Z})\chi(A)^{-1}$$

with $K = \left(\begin{array}{c|c} 0 & I_5 \\ \hline I_5 & 0 \end{array}\right)$ and hence $F_1(x) = \operatorname{tr} K\Phi(\tilde{Z})$ is a relative invariant of degree 28 corresponding to the character $(\det B)^2$. Moreover, by (3.7) and Lemma 3.4, we put $F_2(x) = \langle\eta(X\cdot Y)|\Phi(\tilde{Z})|\eta(X\cdot Y)\rangle := {}^t\eta(X\cdot Y)\Phi(\tilde{Z})\eta(X\cdot Y)$ and hence $F_2(x)$ is a relative invariant of degree 36 corresponding to the character $\alpha^4(\det B)^2$.

For a generic point

$$\begin{aligned} x_0 = (&e_1e_5 + e_2e_3e_4e_5\ ,\ e_2e_5\ ,\ e_3e_5\ ,\ e_4e_5\ ,\ -e_1e_3e_4e_5\ , e_1e_2e_4e_5\ , -e_1e_2e_3e_5\ , \\ &-1 + e_1e_2e_3e_4\ ,\ e_1e_2\ ,\ e_1e_3\ ,\ e_1e_4\ , -e_3e_4\ ,\ e_2e_4\ ,\ -e_2e_3\ ,\ e_4' + e_{14}'), \end{aligned}$$

we have

$$X_0 \cdot Y_0 = {}^t(0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)\,,$$

$$\eta(X_0 \cdot Y_0) = {}^t(0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)\,,$$

$$\tilde{Z}_0 = (e_1 \wedge e_5 - e_1 \wedge e_{16}) \in \mathrm{Alt}_{16},$$

$$\Phi(\tilde{Z}_0) = \Phi(\tilde{Z}_0) = \left(\begin{array}{ccccc|ccccc} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \in \mathrm{Sym}_{10}\,.$$

Then we have $F_2(X_1^{(0)}, \cdots, X_{14}^{(0)}, Y_0) = \langle \eta(X_0 \cdot Y_0) \mid \Phi(\tilde{Z}_0) \mid \eta(X_0 \cdot Y_0)\rangle = -1 \neq 0$ and $F_1(X_1^{(0)}, \cdots, X_{14}^{(0)}, Y_0) = \mathrm{tr}\, K\Phi(\tilde{Z}_0) = -4 \neq 0$ and hence they are not identically zero.

In [6] or [8], the irreducible relative invariant $f(x)$ of degree 4 of a prehomogeneous vector space $(\mathrm{Spin}_{10} \times GL_2, \Lambda' \otimes \Lambda_1, V(16) \oplus V(16))$ is constructed. Through the theory of castling transformation, above polynomial $F_1(x)$ is corresponding to the irreducible relative invariant of a prehomogeneous vector space $(\mathrm{Spin}_{10} \times GL_2, \Lambda' \otimes \Lambda_1, V(16) \oplus V(16))$ studied in [6]. By Lemma 1.2, we can know the irreducibility of $F_1(x)$. Note that we can also check the degree of $F_1(x)$ by compairing the degree of $f(x)$ by Lemma 1.2.

If the polynomial $F_2(x)$ is not irreducible, then $F_2(x) = F_1(x)G(x)$ for some relative invariant $G(x)$ of degree 8 corresponding to the character α^4, or $F_2(x) = H(x)^2$ for some relative invariant $H(x)$ of degree 18 corresponding to the character $\alpha^2(\det B)$. For $x_1 = {}^t(tX_1^{(0)}, X_2^{(0)}, \ldots, X_{14}^{(0)}, Y_1) \in V(16)^{\oplus 14} \oplus V(14)^*$ with $Y_1 = {}^t(y_1, 0, 0, y_4, 0, 0, 0, y_8, 0, 0, y_{11}, 0, 0, y_{14})$, we have $F_1(x_1) = -4t^2$ and $F_2(x_1) = 2t^2 y_1^2 y_{11}^2 y_{14} - y_{14}^2 y_4^2 - 2t^2 y_1^2 y_8^2$. Then we have $F_1(x_1) \nmid F_2(x_1)$ and hence $F_1(x) \nmid F_2(x)$. Moreover, $F_2(x_1)$ cannot be of the form $F_2(x_1) = H(x_1)^2$, and hence $F_2(x)$ cannot be of the form $F_2(x) = H(x)^2$. Thus we have obtained the irreducibility of $F_2(x)$.

Theorem 3.5. *The prehomogeneous vector space* $(GL_1 \times \mathrm{Spin}_{10} \times GL_{14}, \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, V(16)^{\oplus 14} \oplus V(14)^*)$ *has 2 basic relative invariants:*

(1) $F_1(x) = \mathrm{tr}\, K\Phi(\tilde{Z}) \longleftrightarrow (\det B)^2$, $\deg.F_1 = 28$,
(2) $F_2(x) = \langle \eta(X \cdot Y) | \Phi(\tilde{Z}) | \eta(X \cdot Y)\rangle \longleftrightarrow \alpha^4(\det B)^2$, $\deg.F_2 = 36$.

Acknowledgment

The authors would like to express their hearty thanks to Professor H. Ochiai for his invaluable comments for this paper.

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[12] *Mathematica*, Wolfram Research Inc.

The Department of Mathematics, Josai University, Saitama, 305-0295, Japan
E-mail address: kogiso@math.josai.ac.jp

The Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571, Japan
E-mail address: subaru@first.tsukuba.ac.jp

The Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571, Japan
E-mail address: miyuki@math.tsukuba.ac.jp

The Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571, Japan
E-mail address: kimurata@math.tsukuba.ac.jp