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AN OVERLAPPING DOMAIN DECOMPOSITION PRECONDITIONER FOR A CLASS OF DISCONTINUOUS GALERKIN APPROXIMATIONS OF ADVECTION-DIFFUSION PROBLEMS

CAROLINE LASSER AND ANDREA TOSELLI

Abstract. We consider a scalar advection-diffusion problem and a recently proposed discontinuous Galerkin approximation, which employs discontinuous finite element spaces and suitable bilinear forms containing interface terms that ensure consistency. For the corresponding sparse, nonsymmetric linear system, we propose and study an additive, two-level overlapping Schwarz preconditioner, consisting of a coarse problem on a coarse triangulation and local solvers associated to a family of subdomains. This is a generalization of the corresponding overlapping method for approximations on continuous finite element spaces. Related to the lack of continuity of our approximation spaces, some interesting new features arise in our generalization, which have no analog in the conforming case. We prove an upper bound for the number of iterations obtained by using this preconditioner with GMRES, which is independent of the number of degrees of freedom of the original problem and the number of subdomains. The performance of the method is illustrated by several numerical experiments for different test problems using linear finite elements in two dimensions.

1. Introduction

We consider the following scalar advection-diffusion problem with Dirichlet conditions:

$$\begin{aligned} \mathcal{L}u \;=\; -\nabla\cdot(a\nabla u) + b\cdot\nabla u + cu \;&=\; f, \quad \text{in } \Omega\,, \\ u \;&=\; 0, \quad \text{on } \Gamma\,, \end{aligned} \tag{1.1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$, and Γ its boundary. Problem (1.1) describes a large class of diffusion-transport-reaction processes.

Discontinuous Galerkin (DG) approximations have a very long history and have recently become more and more popular for the approximation of problems involving convection phenomena; we refer to [5] for a comprehensive review of these methods. Here, we consider the discontinuous finite element method proposed in [9].

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As for many DG methods, the approximate solution belongs to a space of discontinuous finite element functions, i.e., it is piecewise polynomial of a certain degree on a given triangulation, being in general discontinuous across the elements. Suitable bilinear forms, which also contain interface contributions, are then employed, in order to ensure consistency. The corresponding systems of algebraic equations are sparse but often too large to be handled by direct solvers. In addition, they are nonsymmetric, since the bilinear forms contain advection and interface terms.

Fixing a family of polynomial degrees on the elements, we construct and analyze a Schwarz preconditioner for linear systems obtained from DG discretizations, to be used with a Krylov-type method, like GMRES. Our two-level Schwarz preconditioner is built from a coarse solver and a number of smaller local solvers, associated with a partition of the domain Ω. While the coarse level is designed to reduce the low-energy components of the error, the fine level splits the original problem into a number of smaller problems, not only to reduce the problem size but also to enable efficient parallel computing. We then generalize the additive Schwarz theory for nonsymmetric problems, developed by Cai and Widlund in [2] and [3], to the class of DG approximations in question. Our main result is an upper bound for the convergence rate of the preconditioned system, which is independent of the size of the fine mesh and the number of local problems.

Even though the linear systems that we consider come from hp finite element approximations and the proposed preconditioner can be employed for general hp approximations, the bounds that we prove are for the h version only, since they depend in general on the polynomial degree p.

We only know of one previous work on domain decomposition (DD) preconditioners for DG approximations. In [8], a two-level Schwarz preconditioner has been proposed and analyzed for a different type of DG approximations for the Poisson problem. As opposed to our approach, the method in [8] gives rise to a symmetric positive-definite problem, and the conjugate gradient method can be employed. In [8] an explicit bound for the condition number for a nonoverlapping preconditioner is obtained, which grows linearly with the number of degrees of freedom in each subdomain. The method that we present here is similar to that in [8], but designed for a different DG approximation, which is suitable for advection-reaction-diffusion equations. The coarse space that we consider is also different, and we believe more appropriate for the case of overlapping methods. We use GMRES as an iterative solver and prove an upper bound for the number of iterations obtained when a two-level overlapping preconditioner is employed. Due to the available error estimates for GMRES and the nonsymmetry of our problem, bounds that are explicit in the relative overlap cannot be obtained in general, similarly to the case of conforming approximations; see [2, 3]. Our numerical results show, however, that, as expected, the rate of convergence improves when the overlap increases.

The rest of the paper is organized as follows. In section 2, we introduce our model problem and the discontinuous finite element spaces. After defining the bilinear form and the corresponding discrete problem in section 3, we describe our overlapping Schwarz method in section 4. Section 5 provides technical tools, like estimates for the coarse space and the local spaces, as well as a stable decomposition. Section 6 contains the convergence result. We finally illustrate the performance of our algorithm in section 7 by several numerical experiments for the case of linear finite elements in two dimensions.

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2. Model problem and finite element spaces

We consider problem (1.1) and make some further hypotheses. We assume that $a = \{a_{i,j}\}_{i,j=1}^d$ is a symmetric positive-definite matrix,

$$\xi^T a(x)\xi \geq \alpha_0 > 0, \quad \xi \in \mathbb{R}^d, \quad x \in \Omega,$$

b and c are a vector field in $W^{1,\infty}(\Omega)$ and a function in $L^\infty(\Omega)$, respectively, such that

$$(c - \frac{1}{2}\nabla \cdot b)(x) \ \geq \ \gamma_0 \ > \ 0\,, \quad x \in \Omega\,, \tag{2.1}$$

and the right-hand side f is a function in $L^2(\Omega)$. The existence of a unique solution of (1.1) is shown in [9]. We note that we have considered only the case of strongly imposed homogeneous Dirichlet boundary conditions for simplicity, but that more general ones can be employed, such as Neumann, Robin, or weakly imposed Dirichlet conditions. Our analysis remains valid in these cases.

In the following, the norm, seminorm, and inner product of a Hilbert space $\mathcal{H}$ are denoted by $\|\cdot\|_{\mathcal{H}}, |\cdot|_{\mathcal{H}}$, and $(\cdot,\cdot)_{\mathcal{H}}$, respectively.

In our analysis we will use some regularity properties for second order elliptic problems, and tacitly assume that the domain Ω and the subdomains considered satisfy them. Such properties are certainly valid for general polygonal and polyhedral domains with angles between their edges (or faces) smaller than 2π. In particular we will assume that the Poisson problem on Ω (and consequently Problem (1.1) and its adjoint) with Dirichlet or Neumann conditions has $H^{\eta+3/2}$ regularity, for all $\eta < \eta_\Omega$, where $\eta_\Omega > 0$ depends on Ω and the particular type of boundary conditions considered; see [6, Cor. 18.15 and Cor. 23.5].

We next introduce a conforming, shape-regular triangulation $\mathcal{T}_h$ of Ω, consisting of open simplices with diameter $O(h)$. For a nonnegative integer k, we denote by $\mathcal{P}_k(\kappa)$ the space of polynomials of total degree k on $\bar{\kappa}$, and define the vector of local polynomial degrees $\mathbf{p} = (p_\kappa : \kappa \in \mathcal{T}_h)$. We consider the finite element space

$$S^{\mathbf{p}}(\Omega, \mathcal{T}_h) = \{u \in L^2(\Omega) : u|_{\bar{\kappa}} \in \mathcal{P}_{p_\kappa}(\kappa)\}\,.$$

Let p be the maximum of the polynomial degrees in $\mathbf{p}$.

Given a domain $D \subseteq \Omega$, which is the union of some elements in $\mathcal{T}_h$, we define the product space

$$H^1(D, \mathcal{T}_h) = \{u \in L^2(D) : \ u|_\kappa \in H^1(\kappa)\,,\ \kappa \in \mathcal{T}_h\,,\ \kappa \subset D\}\,,$$

and equip $H^1(D, \mathcal{T}_h)$ with the broken Sobolev norm and seminorm, given by

$$\|u\|^2_{H^1(D,\mathcal{T}_h)} = \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset D}} \|u\|^2_{H^1(\kappa)}\,, \quad |u|^2_{H^1(D,\mathcal{T}_h)} = \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset D}} |u|^2_{H^1(\kappa)}\,.$$

$H^1_0(\Omega, \mathcal{T}_h)$ and $S^{\mathbf{p}}_0(\Omega, \mathcal{T}_h)$ denote the subspaces of functions in $H^1(\Omega, \mathcal{T}_h)$ and $S^{\mathbf{p}}(\Omega, \mathcal{T}_h)$, respectively, vanishing on Γ.

Our FE approximation space is chosen as

$$V^h = S^{\mathbf{p}}_0(\Omega, \mathcal{T}_h)\,.$$

We denote by $\mathcal{E}$ the set of all open $(d-1)$-dimensional faces (edges, for $d = 2$) of the elements $\mathcal{T}_h$, and define the set of interior faces $\mathcal{E}_{int} = \{e \in \mathcal{E} : e \subset \Omega\}$ and the interior interface Γ_{int}, such that $\bar{\Gamma}_{int} = \bigcup_{e\in\mathcal{E}_{int}} \bar{e}$.

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For $\kappa \in \mathcal{T}_h$, we denote the unit outward normal to $\partial\kappa$ at $x \in \partial\kappa$ by $\mu_\kappa(x)$ and partition the part of its boundary that is also contained in Γ_{int} into two sets:

$$\begin{aligned} \partial_-\kappa &= \{x \in \partial\kappa \cap \Gamma_{int} : b(x)\cdot\mu_\kappa(x) < 0\} \qquad \text{(inflow part)}\,, \\ \partial_+\kappa &= \{x \in \partial\kappa \cap \Gamma_{int} : b(x)\cdot\mu_\kappa(x) \geq 0\} \qquad \text{(outflow part)}\,. \end{aligned}$$

Given $v \in H^1(\Omega, \mathcal{T}_h)$, its restriction to $\bar{D} \subset \bar{\Omega}$ is denoted by $v_D = v|_{\bar{D}}$. Then, for $x \in \partial_-\kappa$ there exists a unique neighbor κ' with $x \in \partial\kappa'$, and we set

$$v_\kappa^+(x) = v_\kappa(x)\,, \quad v_\kappa^-(x) = v_{\kappa'}(x)\,, \quad \lfloor v \rfloor_\kappa = v_\kappa^+ - v_\kappa^-\,.$$

Given an interior face $e \in \mathcal{E}_{int}$, there are two elements κ_i, κ_j, with, e.g., $i > j$, that share this face. We define

$$[v]_e = v|_{\partial\kappa_i \cap e} - v|_{\partial\kappa_j \cap e}, \quad \langle v \rangle_e = \frac{1}{2}(v|_{\partial\kappa_i \cap e} + v|_{\partial\kappa_j \cap e})\,,$$

and ν as the unit normal which points from κ_i to κ_j. We note that μ and ν point in different directions in general, and that $\lfloor\cdot\rfloor$ and $[\cdot]$ are distinct. While μ and $\lfloor\cdot\rfloor$ depend on the sign of the advective normal flux on an element boundary, ν and $[\cdot]$ depend on the element numbering. Similarly, for $e = \partial\kappa \cap \Gamma$, we set

$$[v]_e = v|_e.$$

Finally, we introduce a discontinuity-penalization function σ defined on Γ_{int}: for a face $e = \partial\kappa \cap \partial\kappa' \in \mathcal{E}_{int}$, we denote the diameter of e by h_e and define

$$\sigma_e \;=\; \sigma_0\, \frac{\bar{a}_\kappa p_\kappa^2 + \bar{a}_{\kappa'} p_{\kappa'}^2}{2h_e}\,,$$

where $\bar{a} = ||a||$ and σ_0 is a suitably chosen positive constant.

3. Bilinear form and discrete problem

For $u, v \in V^h$, we consider the bilinear form

$$\begin{aligned} B(u,v) \;=\; & \sum_{\kappa\in\mathcal{T}_h} \int_\kappa a\nabla u \cdot \nabla v dx + \sum_{\kappa\in\mathcal{T}_h} \int_\kappa (b\cdot\nabla u + cu) v dx \\ & - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial_-\kappa\cap\Gamma_{int}} (b\cdot\mu)\lfloor u \rfloor v^+ ds + \int_{\Gamma_{int}} \sigma[u][v] ds \\ & + \int_{\Gamma_{int}} \left([u]\langle (a\nabla v)\cdot\nu\rangle - \langle (a\nabla u)\cdot\nu\rangle [v]\right) ds\,, \end{aligned}$$

which has been proposed in [9]. Our DG approximation of (1.1) is then defined as the unique $u \in V^h$ such that

$$B(u,v) = (f,v)_{L^2(\Omega)}\,, \quad v \in V^h\,. \tag{3.1}$$

Problem (3.1) can be written in matrix form as

$$Bu = f, \tag{3.2}$$

where we have used the same notation for a function $u \in V^h$ and the corresponding vector of degrees of freedom, and a bilinear form, e.g., $B(\cdot,\cdot)$, and its matrix representation in the space V^h. Similarly, in the following we use the same notation for functional spaces and the corresponding spaces of vectors of degrees of freedom.

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We next define some additional bilinear forms. It can be easily verified that

$$A(u,v) \quad = \quad \sum_{\kappa\in\mathcal{T}_h}\int_\kappa a\nabla u\cdot\nabla v dx + \int_{\Gamma_{int}}\sigma[u][v]ds$$

defines a scalar product in $H_0^1(\Omega,\mathcal{T}_h)$ and a norm $\|\cdot\|_A = A(\cdot,\cdot)^{\frac{1}{2}}$.

Furthermore, let

$$\begin{aligned} D(u,v) &= \sum_{\kappa\in\mathcal{T}_h}\int_\kappa b\cdot\nabla u\, v\, dx - \sum_{\kappa\in\mathcal{T}_h}\int_{\partial_-\kappa\cap\Gamma_{int}}(b\cdot\mu)\lfloor u\rfloor v^+ ds\,, \\ S(u,v) &= \int_{\Gamma_{int}}([u]\langle(a\nabla v)\cdot\nu\rangle - \langle(a\nabla u)\cdot\nu\rangle[v])\, ds\,, \\ C(u,v) &= (cu,v)_{L^2(\Omega)}\,. \end{aligned}$$

We note that $B(u,v) = A(u,v) + D(u,v) + C(u,v) + S(u,v)$.

An important tool in the analysis of Schwarz methods is represented by some Poincaré and Friedrichs type inequalities valid for Sobolev spaces. The following lemma provides two generalizations to the discontinuous space $H^1(D,\mathcal{T}_h)$; see also [1, 8].

Lemma 3.1 (Poincaré-Friedrichs)**.** *Let $D\subseteq\Omega$ be a domain which is the union of some elements in $\mathcal{T}_h$. Then there exists a positive constant C, depending only on the geometry of D (but not on its size) and the shape-regularity constant of $\mathcal{T}_h$, such that, for all $u\in H^1(D,\mathcal{T}_h)$,*

$$\|u\|^2_{L^2(D)} \le CH_D^2\left(|u|^2_{H^1(D,\mathcal{T}_h)} + \sum_{\substack{e\in\mathcal{E}\\ e\subset D}}\int_e h_e^{-1}[u]^2 ds + \sum_{\substack{e\in\mathcal{E}\\ e\subset\partial D}}\int_e h_e^{-1}u^2 ds\right)\,, \tag{3.3}$$

where H_D is the diameter of D. If in addition $\int_D u dx = 0$, then

$$\|u\|^2_{L^2(D)} \le CH_D^2\left(|u|^2_{H^1(D,\mathcal{T}_h)} + \sum_{\substack{e\in\mathcal{E}\\ e\subset D}}\int_e h_e^{-1}[u]^2 ds\right)\,. \tag{3.4}$$

Proof. Here, we only present a proof for the Poincaré-type inequality (3.4). A proof for the Friedrichs inequality (3.3) can be found in [1] for the case of a convex D, and can be easily generalized to our more general case.

We first suppose that D has unit diameter and proceed similarly to [1, Lem. 2.2]. Let $u\in H^1(D,\mathcal{T}_h)$ with $\int_D u dx = 0$ and $v\in H^{\eta+3/2}(D)$, for an $\eta>0$ be the solution of the following Neumann problem:

$$-\Delta v = u,\ \text{in } D,\quad \frac{\partial v}{\partial n} = 0,\ \text{on } \partial D,\quad \int_D v dx = 0.$$

Then there exists a constant $C>0$ such that

$$\|v\|_{H^{\eta+3/2}(D)} \le C\|u\|_{L^2(D)}.$$

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Integration by parts on each κ and summation over all the elements yield

$$\begin{aligned}
\|u\|^2_{L^2(D)} &= (u, -\Delta v)_{L^2(D)} \\
&= (\nabla u, \nabla v)_{L^2(D)} - \sum_{\kappa\subset D} \left(u, \frac{\partial v}{\partial n}\right)_{L^2(\partial\kappa\setminus\partial D)} \\
&\leq \left(|u|^2_{H^1(D,\mathcal{T}_h)} + \sum_{e\subset D} \int_e h_e^{-1}[u]^2 ds\right)^{\frac{1}{2}} \\
&\times \left(|v|^2_{H^1(D,\mathcal{T}_h)} + \sum_{\kappa\subset D} \int_{\partial\kappa\setminus\partial D} h_\kappa \left(\frac{\partial v}{\partial n}\right)^2 ds\right)^{\frac{1}{2}} .
\end{aligned}$$

Using a trace inequality for $\partial v/\partial n$, as in [1], we obtain (3.4).

The corresponding inequalities for the case of a general D can be obtained employing a scaling argument. □

We note that (3.3) is the generalization of the corresponding estimate for functions in $H^1(\Omega)$ with support contained in $\bar{D}$ to discontinuous functions in $H^1(D, \mathcal{T}_h)$. In particular, (3.3) remains valid for a function that is constant in D and vanishes in $\Omega \setminus \overline{D}$, due to the contributions on the edges on ∂D. On the other hand, (3.4) requires the additional restriction $\int_D u dx = 0$, which a constant function does not meet.

The following inverse inequalities are proven in [13, Sect. 4.6.1].

Lemma 3.2 (Local inverse inequalities)**.** *There exists a positive constant C, depending only on the shape-regularity constant of $\mathcal{T}_h$, such that for all $u \in \mathcal{P}_{p_\kappa}(\kappa)$ and for all $\kappa \in \mathcal{T}_h$*

$$\|u\|^2_{L^2(\partial\kappa)} \leq C \frac{p_\kappa^2}{h_\kappa} \|u\|^2_{L^2(\kappa)} , \tag{3.5}$$

$$|u|^2_{H^1(\kappa)} \leq C \frac{p_\kappa^4}{h_\kappa^2} \|u\|^2_{L^2(\kappa)} . \tag{3.6}$$

Using these tools, we obtain the following lemmata.

Lemma 3.3 (Continuity)**.** *There exists $C > 0$ such that*

$$|B(u, v)| \leq C \|u\|_A \|v\|_A , \quad u, v \in V^h .$$

Proof. The bilinear form B consists of five contributions I, II, III, IV, and V, all of which can be bounded by $C\|u\|_A\|v\|_A$. We easily find that

$$\begin{aligned}
|I| &= |\sum_{\kappa\in\mathcal{T}_h} \int_\kappa a\nabla u \cdot \nabla v \, dx| \leq C \|u\|_A \|v\|_A , \\
|IV| &= |\int_{\Gamma_{int}} \sigma[u][v] \, ds| \leq C \|u\|_A \|v\|_A .
\end{aligned}$$

The Cauchy-Schwarz inequality and (3.3) with $D = \Omega$ yield

$$\begin{aligned}
|II| &= |\sum_{\kappa\in\mathcal{T}_h} \int_\kappa (b \cdot \nabla u + cu) v \, dx| \\
&\leq C \sum_{\kappa\in\mathcal{T}_h} \left(|u|_{H^1(\kappa)} \|v\|_{L^2(\kappa)} + \|u\|_{L^2(\kappa)} \|v\|_{L^2(\kappa)}\right) \\
&\leq C \|u\|_A \|v\|_A .
\end{aligned}$$

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Applying the inverse inequality (3.5), Lemma 3.1, and the definition of σ, we find that

$$\begin{aligned}
|III| &= |\sum_{\kappa\in\mathcal{T}_h}\int_{\partial_-\kappa\cap\Gamma_{int}}(b\cdot\mu)\lfloor u\rfloor v^+ ds| \\
&\le C\left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa^{-1}\|\lfloor u\rfloor\|^2_{L^2(\partial_-\kappa\cap\Gamma_{int})}\right)^{\frac12}\left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa\|v^+\|^2_{L^2(\partial_-\kappa\cap\Gamma_{int})}\right)^{\frac12} \\
&\le C\left(\int_{\Gamma_{int}}\sigma[u]^2 ds\right)^{\frac12}\|v\|_{L^2(\Omega)} \;\le\; C\,\|u\|_A\|v\|_A\,.
\end{aligned}$$

Using (3.5), we finally obtain

$$\begin{aligned}
|V| &= |\int_{\Gamma_{int}}([u]\langle(a\nabla v)\cdot\nu\rangle - \langle(a\nabla u)\cdot\nu\rangle[v])\,ds| \\
&\le C\left(\sum_{e\in\mathcal{E}_{int}} h_e^{-1}\|[u]\|^2_{L^2(e)}\cdot\sum_{\substack{\kappa\in\mathcal{T}_h\\ \partial\kappa\subset\Gamma_{int}}} h_\kappa\|\langle a\nabla v\rangle\|^2_{L^2(\partial\kappa)}\right)^{\frac12} \\
&+ C\left(\sum_{\substack{\kappa\in\mathcal{T}_h\\ \partial\kappa\subset\Gamma_{int}}} h_\kappa\|\langle a\nabla u\rangle\|^2_{L^2(\partial\kappa)}\cdot\sum_{e\in\mathcal{E}_{int}} h_e^{-1}\|[v]\|^2_{L^2(e)}\right)^{\frac12} \\
&\le C\left(\int_{\Gamma_{int}}\sigma[u]^2 ds\cdot\sum_{\kappa\in\mathcal{T}_h}\|a\nabla v\|^2_{L^2(\kappa)}\right)^{\frac12} \\
&+ C\left(\sum_{\kappa\in\mathcal{T}_h}\|a\nabla u\|^2_{L^2(\kappa)}\cdot\int_{\Gamma_{int}}\sigma[v]^2 ds\right)^{\frac12} \\
&\le C\,\|u\|_A\|v\|_A\,.
\end{aligned}$$

□

Lemma 3.4 (Coercivity)**.** *We have*

$$B(u,u)\ge\|u\|_A^2\,,\quad u\in H_0^1(\Omega,\mathcal{T}_h)\,.$$

Proof. Indeed,

$$\begin{aligned}
B(u,u) &= \sum_{\kappa\in\mathcal{T}_h}\|\sqrt{a}\nabla u\|^2_{L^2(\kappa)} + \int_{\Gamma_{int}}\sigma[u]^2 ds \\
&+ \sum_{\kappa\in\mathcal{T}_h}\int_\kappa(b\cdot\nabla u + cu)u dx - \sum_{\kappa\in\mathcal{T}_h}\int_{\partial_-\kappa\cap\Gamma_{int}}(b\cdot\mu)\lfloor u\rfloor u^+ ds \\
&=: \|u\|_A^2 + R(u,u).
\end{aligned}$$

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Therefore, we just have to make sure that $R(u,u) \geq 0$. Integration by parts yields

$$\begin{aligned} R(u,u) &= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \left(-\frac{1}{2}(\nabla\cdot b)+c\right) u^2 dx \\ &+ \sum_{\kappa\in\mathcal{T}_h} \left(\int_{\partial\kappa\cap\Gamma_{int}} \frac{1}{2}(b\cdot\mu)u^2 ds - \int_{\partial_-\kappa\cap\Gamma_{int}} (b\cdot\mu)\lfloor u\rfloor u^+ ds \right). \end{aligned}$$

Condition (2.1) ensures that the first sum is positive. To deal with the second sum, we consider an interior face $e \subset \mathcal{E}_{int}$ which is common to the elements κ and κ'. Let e be an inflow edge of, e.g., κ'. Then the second sum can be written as

$$\begin{aligned} &\sum_{e\subset\mathcal{E}_{int}} \int_e \left(\frac{1}{2}(b\cdot\mu_\kappa)(u_\kappa)^2 + \frac{1}{2}(b\cdot\mu_{\kappa'})(u_{\kappa'})^2 - (b\cdot\mu_{\kappa'})(u_{\kappa'}-u_\kappa)u_{\kappa'} \right) ds \\ = &\sum_{e\subset\mathcal{E}_{int}} \int_e \frac{1}{2}\,|b\cdot\mu_{\kappa'}|(u_{\kappa'}-u_\kappa)^2\, ds \;=\; \int_{\Gamma_{int}} \frac{1}{2}\,|b\cdot\mu|[u]^2 ds \;\geq\; 0\,, \end{aligned}$$

where we have used the fact that $e \subset \partial_-\kappa'$ also belongs to $\partial_+\kappa$. □

Using similar arguments as in the proofs of Lemmata 3.3 and 3.4, we can prove the following lemma:

Lemma 3.5. *There exists a constant $C > 0$ such that for all $u, v \in V^h$*

$$\begin{aligned} |D(u,v)| &\leq C\|u\|_{L^2(\Omega)}\|v\|_A\,, \\ |D(u,v)| &\leq C\|u\|_A\|v\|_{L^2(\Omega)}\,. \end{aligned}$$

Finally, we are able to control the interface penalization contribution by requiring that the penalization coefficient is sufficiently large:

Lemma 3.6. *Let $H > 0$ and $\sigma_0 \geq c_0/H$ for some constant $c_0 > 0$. Then there exists $C > 0$, such that for all $u, v \in V^h$*

$$|S(u,v)| \leq C\sqrt{H}\,\|u\|_A\|v\|_A\,.$$

Proof. Since $\sigma^{-1} \leq CHh$, using the inverse inequality (3.5), we obtain

$$\begin{aligned} |S(u,v)| &\leq \left(\sum_{\kappa\in\mathcal{T}_h} \sigma\|[u]\|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} \sigma^{-1}\|\langle a\nabla v\rangle\|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}} \\ &+ \left(\sum_{\kappa\in\mathcal{T}_h} \sigma\|[v]\|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} \sigma^{-1}\|\langle a\nabla u\rangle\|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}} \\ &\leq C\,\|u\|_A\sqrt{H}\left(\sum_{\kappa\in\mathcal{T}_h} h\,\|\langle a\nabla v\rangle\|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}} \\ &+ C\,\|v\|_A\sqrt{H}\left(\sum_{\kappa\in\mathcal{T}_h} h\,\|\langle a\nabla u\rangle\|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}} \\ &\leq C\sqrt{H}\,\|u\|_A\|v\|_A\,. \end{aligned}$$

□

We remark that the restriction imposed by the previous lemma on σ does not appear to be required in practice; see Section 7.

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4. An overlapping Schwarz method

In this section, we introduce our two-level algorithm. It is the generalization of the classical overlapping method with a standard coarse space. We refer to [16] and [14] for further details and some implementation issues. As previously remarked, even though our preconditioner can be employed for general hp approximations, the bounds that we prove are for the h version only, since they depend in general on the polynomial degree p.

We first introduce a shape-regular coarse triangulation of Ω

$$\mathcal{T}_H = \{\Omega_i\}_{1\le i\le N},$$

of diameter $H > h$, and suppose that $\mathcal{T}_h$ is obtained by refining $\mathcal{T}_H$. We next extend each Ω_i to a larger region $\Omega_i' \subset \Omega$, in such a way that Ω_i' is the union of some elements in $\mathcal{T}_h$. Concerning the overlap of the extended subregions, we assume that there exists a constant $\alpha > 0$ such that

$$\operatorname{dist}(\partial\Omega_i' \cap \Omega, \partial\Omega_i) \ge \alpha H\,, \quad 1 \le i \le N\,. \tag{4.1}$$

In addition, we also assume a finite covering property. We start by defining an auxiliary partition, obtained by augmenting the $\{\Omega_i'\}$ by one layer of fine elements. Let

$$\Omega_i'' := \bigcup_{\substack{\kappa\in\mathcal{T}_h\\ \overline{\kappa}\cap\overline{\Omega_i'}\neq\emptyset}} \kappa \supset \Omega_i', \quad i = 1,\dots,N.$$

We assume

Property 4.1 (Finite covering)**.** Every point $x \in \Omega$ belongs to at most N_c subdomains in $\{\Omega_i''\}$.

We note that conforming overlapping methods require a finite covering property that just involves the original partition $\{\Omega_i'\}$ (see [14, Sect. 1.3.1]). Here, in the discontinuous case we need a more restrictive covering to control the skew-symmetric part of the bilinear form (see the proof of Lemma 5.7).

Before proceeding, we remark that more general partitions and coarse meshes can be employed in overlapping methods. In particular, the coarse mesh does not need to be related to the fine one, and the nonoverlapping partition $\{\Omega_i\}$ does not need to be related to the coarse mesh $\mathcal{T}_H$. Indeed, one only needs to assume that the diameter of $\mathcal{T}_H$ and the diameters of the $\{\Omega_i\}$ are of the same size H; see, e.g., [4]. Our results and proofs remain valid in this more general case.

The first problem we need to address is the choice of the local solvers associated with the $\{\Omega_i'\}$. Our FE spaces are discontinuous, and at a first glance there are no traces to match! We then proceed in a purely algebraic way, by first defining some local spaces (or, equivalently, by extracting some blocks from B) and then identifying the corresponding problems, if any, that they represent.

Our local spaces are defined by

$$V_i = \{\, u \in V^h : u_{|_\kappa} = 0, \quad \kappa \in \mathcal{T}_h,\ \kappa \subset \Omega\backslash\Omega_i' \,\}\,, \quad 1 \le i \le N\,. \tag{4.2}$$

We note that a function in V_i is discontinuous and, as opposed to the case of conforming approximations, in general does not vanish on $\partial\Omega_i'$. Let $R_i^T :\ V_i \to V^h$ be the natural injection operator from the subspace V_i into V^h. We recall that the restriction operator $R_i :\ V^h \to V_i$, defined as the transpose of R_i^T with respect to the Euclidean scalar product, puts the degrees of freedom outside $\bar{\Omega}_i'$ equal to zero.

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The matrix block corresponding to the space V_i is obtained by extracting *all* the degrees of freedom relative to the elements contained in Ω_i', and is equal to

$$B_i = R_i B R_i^T : V_i \longrightarrow V_i.$$

It can easily be verified that the matrix B_i is the representation of the following local bilinear form:

$$\begin{aligned}
B_i(u,v) \;=\; & \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset\Omega_i'}} \int_\kappa (a\nabla u\cdot\nabla v + b\cdot\nabla u v + cuv)dx \\
- & \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset\Omega_i'}} \int_{\partial_-\kappa\cap\Omega_i'} (b\cdot\mu)\lfloor u\rfloor v^+ ds + \int_{\Gamma_{int}\cap\Omega_i'} \sigma[u][v]ds \\
+ & \int_{\Gamma_{int}\cap\Omega_i'} \left([u]\langle(a\nabla v)\cdot\nu\rangle - \langle(a\nabla u)\cdot\nu\rangle[v]\right) ds \\
- & \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset\Omega_i'}} \int_{\partial_-\kappa\cap\partial\Omega_i'} (b\cdot\mu)u^+v^+ ds \\
+ & \frac{1}{2}\int_{\Gamma_{int}\cap\partial\Omega_i'} \left(u((a\nabla v)\cdot\nu) - (a\nabla u)\cdot\nu)v\right) ds + \int_{\Gamma_{int}\cap\partial\Omega_i'} \sigma uv ds\,,
\end{aligned}$$

for $u, v \in V_i$. The contributions in the first three lines come from the DG approximation of the operator $\mathcal{L}$ on Ω_i', while the remaining contributions are boundary contributions on $\partial\Omega_i'$, which appear since we have kept the boundary degrees of freedom in the definition of V_i. We first consider the purely hyperbolic case $a = 0$. Following [9], we see that B_i is the approximation of a Dirichlet problem with *weakly* imposed boundary conditions on the inflow part of the boundary $\partial\Omega_i'$ and it is therefore well-posed. This is opposed to the standard overlapping method for conforming approximations, where, by extracting local blocks, strongly imposed Dirichlet conditions on *all* $\partial\Omega_i'$ and thus potentially ill-posed local problems are obtained. In the purely diffusive case $b = 0$, we note the presence of the term $1/2$ in the skew-symmetric boundary contribution, arising from the average of the fluxes. Without this multiplicative factor, B_i would still be the approximation of a Dirichlet problem with weakly imposed boundary conditions on $\partial\Omega_i'$; see [9]. Despite the presence of the term $1/2$, we note however that B_i is positive-definite thanks to the presence of the penalization contribution, and the local problem on Ω_i' is well-posed. In the general transport-diffusion case, the local matrices are still positive-definite, even if they do not represent local Dirichlet problems in general. We will prove that our choice of local problems gives an optimal method, i.e. a method converging independently of h and H.

We also note that, thanks to the choice of the local spaces, the case of zero overlap,

$$\Omega_i' = \Omega_i, \quad 1 \le i \le N,$$

can be considered, as was already noted in [8]. This has no analogue in the conforming case, and is due to the fact that we work with discontinuous FE spaces. Most of our numerical results show that the number of iterations obtained in this case is comparable, even if larger, to that for the overlapping case.

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We now introduce our coarse solver. It is defined on $\mathcal{T}_H$ and is the FE approximation of our original problem on the *continuous, piecewise linear* FE space

$$V_0 = S^1(\Omega, \mathcal{T}_H) \cap H_0^1(\Omega) \subset V^h.$$

If $R_0^T : V_0 \to V^h$ is the natural injection operator from the subspace V_0 into V^h, then our coarse solver is

$$B_0 = R_0 B R_0^T,$$

and it can be easily shown to be positive-definite. Other choices are possible: we could, e.g., consider a DG approximation of the original problem on the coarse mesh $\mathcal{T}_H$ with piecewise linear finite elements; see [8]. Our bounds remain valid in this case.

We are now ready to define our Schwarz preconditioner

$$\hat{B}^{-1} = \sum_{i=0}^{N} R_i^T B_i^{-1} R_i.$$

In order to analyze the spectral properties of the corresponding preconditioned system $\hat{B}^{-1}B$, we write the latter using some projections; see [14]. As is standard practice in Schwarz methods, for $0 \le i \le N$ we define the B-projections $P_i : V^h \to V_i$ by

$$B(P_i u, v) \;=\; B(u, v)\,, \quad v \in V_i\,.$$

It can be easily shown (see [14]) that

$$P_i = (R_i^T B_i^{-1} R_i)\, B,$$

and consequently that the preconditioned matrix $\hat{B}^{-1}B$ is equal to the additive Schwarz operator:

$$P = \sum_{i=0}^{N} P_i\,.$$

In Theorem 6.1 we will show that P is invertible.

We consider the generalized minimum residual method (GMRES) applied to the preconditioned system

$$Pu = g, \tag{4.3}$$

where $g = \hat{B}^{-1} f$. Some convergence bounds for GMRES are proven in [7], to which we refer for a description of the algorithm. We denote by

$$c_P = \inf_{u \neq 0} \frac{A(u, Pu)}{A(u, u)} \quad \text{and} \quad C_P = \sup_{u \neq 0} \frac{\|Pu\|_A}{\|u\|_A}\,,$$

the smallest eigenvalue of the symmetric part and the operator norm of P, respectively. Then, if $c_p > 0$, GMRES applied to (4.3) converges in a finite number of steps, and after m steps the norm of the residual of the preconditioned system $r_m := g - Pu_m$ is bounded by

$$\|r_m\|_A \;\le\; \left(1 - \frac{c_p^2}{C_P^2}\right)^{\frac{m}{2}} \|r_0\|_A\,.$$

5. Technical tools

In this section, we provide all the technical tools needed for the proof of our convergence result contained in Theorem 6.1.

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5.1. **Results for the coarse space.** We start by defining an interpolation operator onto the coarse space. Let y be a node of the coarse mesh $\mathcal{T}_H$, and B_y be the union of the elements in $\mathcal{T}_H$ that share y. The following definition of the quasi-interpolant and the proof of Lemma 5.1 are given for $d = 2$. Our definitions and analysis can easily be adapted to the case $d = 3$, and we refer to [15, p. 10] for a similar interpolant onto a conforming finite element space.

We define

$$Q_H : L^2(\Omega) \to V_0$$

by assigning a nodal value to every vertex a, b, c of every coarse element $K \in \mathcal{T}_H$. We set

$$(Q_H u)(y) = meas(B_y)^{-1} \int_{B_y} u(x)\, dx, \quad y \in \{a, b, c\}\,.$$

The following lemma ensures that Q_H is stable and provides an error bound.

Lemma 5.1 (Coarse mesh quasi-interpolant)**.** *There exists $C > 0$, independent of h and H, such that, for all $u \in H^1(\Omega, \mathcal{T}_h)$,*

$$\|Q_H u - u\|^2_{L^2(\Omega)} \leq CH^2 \|u\|^2_A\,, \tag{5.1}$$

$$\|Q_H u\|^2_A \leq C \|u\|^2_A\,. \tag{5.2}$$

Proof. We consider a coarse element $K \in \mathcal{T}_H$ with vertices a, b, c, and denote by $\widetilde{K}$ the union of K and its neighboring elements. We clearly have

$$\|Q_H u\|_{L^2(K)} \leq C \|u\|_{L^2(\widetilde{K})}, \quad u \in L^2(\Omega)\,.$$

Since $\widetilde{K}$ has a diameter of order H, inequality (3.4) ensures the existence of a positive constant C, independent of h and H, such that for $v \in H^1(\Omega, \mathcal{T}_h)$ with $\int_{\widetilde{K}} v\, dx = 0$

$$\|v\|^2_{L^2(\widetilde{K})} \leq CH^2 \left(|v|^2_{H^1(\widetilde{K}, \mathcal{T}_h)} + \int_{\Gamma_{int} \cap \widetilde{K}} \sigma[v]^2 ds \right)\,.$$

Now let $u \in H^1(\Omega, \mathcal{T}_h)$ and $\bar{u} := u - meas(\widetilde{K})^{-1} \int_{\widetilde{K}} u dx$. Since Q_H reproduces the constant functions on K, we obtain

$$\begin{aligned} \|Q_H u - u\|^2_{L^2(K)} &= \|Q_H \bar{u} - \bar{u}\|^2_{L^2(K)} \leq C \|\bar{u}\|^2_{L^2(\widetilde{K})} \\ &\leq CH^2 \left(|u|^2_{H^1(\widetilde{K}, \mathcal{T}_h)} + \int_{\Gamma_{int} \cap \widetilde{K}} \sigma[u]^2 ds \right)\,. \end{aligned}$$

Summing over all $K \in \mathcal{T}_H$ and using the finite covering property for the partition $\{\tilde{K} : K \in \mathcal{T}_H\}$, we have, for $u \in H^1(\Omega, \mathcal{T}_h)$,

$$\begin{aligned} \|Q_H u - u\|^2_{L^2(\Omega)} &\leq C \sum_{\widetilde{K} \in \mathcal{T}_H} \|Q_H u - u\|^2_{L^2(\widetilde{K})} \\ &\leq CH^2 \sum_{\widetilde{K} \in \mathcal{T}_H} \left(|u|^2_{H^1(\widetilde{K}, \mathcal{T}_h)} + \int_{\Gamma_{int} \cap \widetilde{K}} \sigma[u]^2 ds \right) \\ &\leq CH^2 \|u\|^2_A\,, \end{aligned}$$

which concludes the proof of (5.1).

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Using the inverse inequality (3.6) for an element $K \in \mathcal{T}_H$ and (3.4), we find that

$$\begin{aligned} |Q_H u|^2_{H^1(K)} &= |Q_H \bar{u}|^2_{H^1(K)} \leq C\, H^{-2}\, \|Q_H \bar{u}\|^2_{L^2(K)} \\ &\leq CH^{-2} \left(\|Q_H \bar{u} - \bar{u}\|^2_{L^2(K)} + \|\bar{u}\|^2_{L^2(\widetilde{K})} \right) \\ &\leq C \left(|u|^2_{H^1(\widetilde{K},\mathcal{T}_h)} + \int_{\Gamma_{int}\cap\widetilde{K}} \sigma [u]^2 ds \right). \end{aligned}$$

Since $Q_H u$ is continuous in Ω, $\|Q_H u\|_A$ is equal to the broken H^1-seminorm, and summing over all $K \in \mathcal{T}_H$ concludes the proof of inequality (5.2). □

We note that we have considered the interpolant Q_H instead of the L^2-orthogonal projection, in order to make our analysis valid for the case of a coarse mesh that is not quasi-uniform; see, e.g., [4].

The following lemma contains some bounds for the B-projection P_0.

Lemma 5.2. *There exists $C > 0$ such that, for all $u \in V^h$,*

$$\|P_0 u\|_A \leq C\, \|u\|_A,$$
$$\|P_0 u - u\|_{L^2(\Omega)} \leq C\, H^\gamma\, \|u\|_A,$$

where $\gamma > 1/2$ is related to the regularity constant of the adjoint problem with Dirichlet boundary conditions.

Proof. The coercivity and continuity of B, and the definition of P_0, yield

$$\|P_0 u\|^2_A \;\leq\; B(P_0 u, P_0 u) \;=\; B(u, P_0 u) \;\leq\; C\, \|u\|_A \|P_0 u\|_A\,,$$

which gives the first inequality.

In order to obtain a bound for the error $u - P_0 u$, we consider the auxiliary problem

$$\mathcal{L}^* w = P_0 u - u \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma\,,$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. We have, for any $w_0 \in V_0$,

$$\begin{aligned} \|P_0 u - u\|^2_{L^2(\Omega)} &= (P_0 u - u, \mathcal{L}^* w)_{L^2(\Omega)} = B(P_0 u - u, w) \\ &= B(P_0 u - u, w - w_0) \leq C\, \|P_0 u - u\|_A \|w - w_0\|_A\,. \end{aligned}$$

Since $P_0 u - u \in L^2(\Omega)$, then $w \in H^{\eta+3/2}(\Omega)$ for an $\eta > 0$, and the Sobolev embedding theorem implies $H^{\eta+3/2}(\Omega) \subset C(\overline{\Omega})$. Therefore, $w - w_0$ is continuous, and $\|w - w_0\|_A$ is equal to the broken H^1-seminorm. Standard approximation estimates yield the existence of $w_0 \in V_0$ such that

$$\|w - w_0\|_{H^1(\Omega)} \;\leq\; C\, H^\gamma\, \|w\|_{H^{1+\gamma}(\Omega)}\,,$$

with $\gamma = \eta + 1/2$; see, e.g., [12]. Therefore,

$$\|P_0 u - u\|^2_{L^2(\Omega)} \;\leq\; C\, H^\gamma\, \|P_0 u - u\|_A \|P_0 u - u\|_{L^2(\Omega)}\,,$$

which gives the L^2-bound. □

As for the analogous algorithm in the conforming case ([2, 14]), we need to control the lower-order and skew-symmetric terms of the bilinear form B. Lemmata 3.5, 3.6, and 5.2 set the stage for the proof of the following bounds, which can be carried out as in [14, Lem. 16, Ch. 5.4].

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Lemma 5.3. *There exists a constant $C > 0$, independent of h and H, such that, for all $u \in V^h$,*

$$|C(P_0 u - u, P_0 u)| \le C\, H^\gamma \, \|u\|_A^2 \,, \qquad |D(P_0 u - u, P_0 u)| \le C\, H^\gamma \, \|u\|_A^2 \,,$$

and, if $\sigma_0 \ge c_0/H$,

$$|S(P_0 u - u, P_0 u)| \le C \sqrt{H}\, \|u\|_A^2 \,.$$

5.2. **Local results.** We first define some local seminorms that we need to control the skew-symmetric part of B (see the proof of Lemma 5.7). For $u \in V^h$, $i = 1, \dots, N$, we set

$$\begin{aligned} \|u\|_{A_i}^2 &:= |u|_{H^1(\Omega_i', \mathcal{T}_h)}^2 + \sum_{\substack{e \in \mathcal{E} \\ e \subset \overline{\Omega_i'}}} \int_e h_e^{-1} [u]^2 ds, \\ \|u\|_{A_i'}^2 &:= |u|_{H^1(\Omega_i'', \mathcal{T}_h)}^2 + \sum_{\substack{e \in \mathcal{E} \\ e \subset \overline{\Omega_i'}}} \int_e h_e^{-1} [u]^2 ds. \end{aligned}$$

The following two lemmata are immediate consequences of the finite covering property of the subdomains.

Lemma 5.4. *Let $u = \sum_{i=0}^N u_i$, with $u_i \in V_i$. Then there exists a constant $C > 0$ such that*

$$\|u\|_A^2 \le C \sum_{i=0}^N \|u_i\|_A^2.$$

Lemma 5.5. *Let $u \in V^h$. Then there exists a constant $C > 0$ such that*

$$\sum_{i=1}^N \|u\|_{L^2(\Omega_i')}^2 \le C\, \|u\|_{L^2(\Omega)}^2, \tag{5.3}$$

$$\sum_{i=1}^N \|u\|_{A_i}^2 \le C\, \|u\|_A^2, \tag{5.4}$$

$$\sum_{i=1}^N \|u\|_{A_i'}^2 \le C\, \|u\|_A^2. \tag{5.5}$$

The Friedrichs inequality (3.3) directly yields

Lemma 5.6. *There exists $C > 0$ such that, for all $u \in V^h$ and $1 \le i \le N$,*

$$\|P_i u\|_{L^2(\Omega)} \le C\, H\, \|P_i u\|_A \,.$$

We are now ready to prove the key result of this technical subsection.

Lemma 5.7. *There exists a constant $C > 0$, independent of h and H, such that, for all $u \in V^h$,*

$$\left| \sum_{i=1}^N C(P_i u - u, P_i u) \right| \le C\, H \left(\|u\|_A^2 + \sum_{i=1}^N \|P_i u\|_A^2 \right), \tag{5.6}$$

$$\left| \sum_{i=1}^N D(P_i u - u, P_i u) \right| \le C\, H \left(\|u\|_A^2 + \sum_{i=1}^N \|P_i u\|_A^2 \right), \tag{5.7}$$

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and, if $\sigma_0 \geq c_0/H$,

$$\left|\sum_{i=1}^{N} S(P_i u - u, P_i u)\right| \leq C\sqrt{H}\left(\|u\|_A^2 + \sum_{i=1}^{N}\|P_i u\|_A^2\right). \tag{5.8}$$

Proof. We start with (5.6). We can write

$$\begin{aligned}\left|\sum_{i=1}^{N} C(P_i u - u, P_i u)\right| &\leq C\sum_{i=1}^{N}\int_{\Omega_i'} |P_i u - u|\,|P_i u| dx \\ &\leq C\sum_{i=1}^{N}\|P_i u - u\|_{L^2(\Omega_i')}\|P_i u\|_{L^2(\Omega_i')} \\ &\leq C\left(\sum_{i=1}^{N}\|P_i u - u\|_{L^2(\Omega_i')}^2\right)^{1/2}\left(\sum_{i=1}^{N}\|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2}.\end{aligned} \tag{5.9}$$

The first term on the right-hand side can be bounded by using the triangle inequality, (5.3), and the Friedrichs inequality (3.3) with $D = \Omega$. We find that

$$\begin{aligned}&\left(\sum_{i=1}^{N}\|P_i u - u\|_{L^2(\Omega_i')}^2\right)^{1/2} \\ &\leq \sqrt{2}\left(\sum_{i=1}^{N}\|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2} + \sqrt{2}\left(\sum_{i=1}^{N}\|u\|_{L^2(\Omega_i')}^2\right)^{1/2} \\ &\leq \sqrt{2}\left(\sum_{i=1}^{N}\|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2} + C\|u\|_{L^2(\Omega)} \\ &\leq \sqrt{2}\left(\sum_{i=1}^{N}\|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2} + C\|u\|_A.\end{aligned} \tag{5.10}$$

Combining (5.9) and (5.10), and using Lemma 5.6, we obtain

$$\begin{aligned}&\left|\sum_{i=1}^{N} C(P_i u - u, P_i u)\right| \\ &\leq C\left(\left(\sum_{i=1}^{N}\|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2} + \|u\|_A\right)\left(\sum_{i=1}^{N}\|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2} \\ &\leq C\left(H\left(\sum_{i=1}^{N}\|P_i u\|_A^2\right)^{1/2} + \|u\|_A\right) H\left(\sum_{i=1}^{N}\|P_i u\|_A^2\right)^{1/2} \\ &\leq CH\left(\sum_{i=1}^{N}\|P_i u\|_A^2 + \|u\|_A^2\right).\end{aligned}$$

In order to prove (5.6), we first note that

$$|D(P_i u - u, P_i u)| \leq C\|P_i u - u\|_{A_i}\|P_i u\|_{L^2(\Omega_i')}, \quad i = 1, \ldots, N.$$

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We can then write

$$\begin{aligned}\left|\sum_{i=1}^{N} D(P_i u - u, P_i u)\right| &\le C \sum_{i=1}^{N} \|P_i u - u\|_{A_i} \|P_i u\|_{L^2(\Omega_i')} \\ &\le C \left(\sum_{i=1}^{N} \|P_i u - u\|_{A_i}^2\right)^{1/2} \left(\sum_{i=1}^{N} \|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2} \\ &\le C \left(\left(\sum_{i=1}^{N} \|P_i u\|_{A_i}^2\right)^{1/2} + \left(\sum_{i=1}^{N} \|u\|_{A_i}^2\right)^{1/2}\right) \left(\sum_{i=1}^{N} \|P_i u\|_{L^2(\Omega_i')}^2\right)^{1/2}.\end{aligned}$$

Using (5.4), Lemma 5.6, and the fact that

$$\|P_i u\|_{A_i} = \|P_i u\|_A, \quad i = 1, \dots, N,$$

we obtain (5.7).

In order to prove (5.8), we first need a bound for each term $S(P_i u - u, P_i u)$. Proceeding as for Lemma 3.6, we find

$$|S(P_i u - u, P_i u)| \le C\sqrt{H} \|P_i u - u\|_{A_i'} \|P_i u\|_{A_i}, \quad i = 1, \dots, N. \tag{5.11}$$

We can then write

$$\begin{aligned}\left|\sum_{i=1}^{N} S(P_i u - u, P_i u)\right| &\le C\sqrt{H} \sum_{i=1}^{N} \|P_i u - u\|_{A_i'} \|P_i u\|_{A_i} \\ &\le C\sqrt{H} \left(\sum_{i=1}^{N} \|P_i u - u\|_{A_i'}^2\right)^{1/2} \left(\sum_{i=1}^{N} \|P_i u\|_{A_i}^2\right)^{1/2}.\end{aligned}$$

Using (5.5) and the fact that

$$\|P_i u\|_{A_i} = \|P_i u\|_{A_i'} = \|P_i u\|_A, \quad i = 1, \dots, N,$$

we obtain (5.8). □

5.3. **A stable decomposition.** The following lemma ensures that, for every function in the discontinuous space V^h, a stable decomposition can be found for the family of subspaces $\{V_i\}$.

Lemma 5.8 (Decomposition)**.** *There exists a constant $C_0 > 0$, independent of h and H, such that for all $u \in V^h$ there exists $\{u_i \in V_i\}_{0 \le i \le N}$ with $u = \sum_{i=0}^{N} u_i$ and*

$$\sum_{i=0}^{N} \|u_i\|_A^2 \;\le\; C_0^2 \, \|u\|_A^2 \,.$$

Proof. We denote by $C(\Omega, \mathcal{T}_h) = \{u \in L^2(\Omega) : u|_{\bar\kappa} \in C(\bar\kappa)\,, \kappa \in \mathcal{T}_h\}$ the space of piecewise continuous functions. We define the operator

$$I^h : C(\Omega, \mathcal{T}_h) \to V^h\,,$$

where for each element $\kappa \in \mathcal{T}_h$, the restriction $I^h|_{\bar\kappa}$ to $\bar\kappa$ is equal to the nodal interpolation operator onto $\mathcal{P}_{p_\kappa}(\kappa)$.

For $u \in V^h$, we define

$$\begin{cases} u_0 = Q_H u, \\ u_i = I^h(\theta_i(u - u_0))\,, \quad 1 \le i \le N\,, \end{cases}$$

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where $\{\theta_i\}_{1\le i\le N}$ is a piecewise linear partition of unity relative to the family $\{\Omega_i'\}_{1\le i\le N}$; see, e.g., [14]. We recall, in particular, that $\theta_i \in [0,1]$, $\mathrm{supp}(\theta_i) \subset \bar{\Omega}_i'$, for $1 \le i \le N$, and $\sum_{i=1}^N \theta_i(x) = 1$ for all $x \in \Omega$. Furthermore, our assumption (4.1) on the overlap of the extended subdomains ensures that $\|\nabla\theta_i\|_{L^\infty(\Omega)} \le CH^{-1}$, where C depends on α. By construction, $u_i \in V_i$ for $0 \le i \le N$, and $u = \sum_{i=0}^N u_i$.

Let $w = u - u_0$. The same arguments used in the proof of the decomposition lemma for standard conforming finite elements [14, Chapter 5.3] yield, for $\kappa \in \mathcal{T}_h$ and $1 \le i \le N$,

$$|u_i|^2_{H^1(\kappa)} \;\le\; 2\,|w|^2_{H^1(\kappa)} + CH^{-2}\,\|w\|^2_{L^2(\kappa)}\,.$$

The finite covering property ensures that on summing over i we obtain

$$\sum_{i=1}^N |u_i|^2_{H^1(\kappa)} \;\le\; C\,|w|^2_{H^1(\kappa)} + CH^{-2}\,\|w\|^2_{L^2(\kappa)}\,.$$

We next sum over all the elements κ, and obtain

$$\sum_{i=1}^N |u_i|^2_{H^1(\Omega,\mathcal{T}_h)} \;\le\; C\,|w|^2_{H^1(\Omega,\mathcal{T}_h)} + CH^{-2}\,\|w\|^2_{L^2(\Omega)}\,.$$

Furthermore, we have, for all $1 \le i \le N$,

$$\|[\theta_i w]\|_{L^\infty(\Gamma_{int})} \;\le\; \|[w]\|_{L^\infty(\Gamma_{int})}\,,$$

where we have used the fact that θ_i is continuous and that $\|\theta_i\|_{L^\infty(\Omega)} \le 1$. Since $w \in V^h$, we obtain

$$\int_{\Gamma_{int}} \sigma[u_i]^2 ds \le \int_{\Gamma_{int}} \sigma[w]^2 ds\,.$$

The finite covering of the subdomains yields

$$\sum_{i=1}^N \int_{\Gamma_{int}} \sigma[u_i]^2 ds \;\le\; C \int_{\Gamma_{int}} \sigma[w]^2 ds\,.$$

Summing the H^1-seminorms and jump terms, we obtain

$$\sum_{i=1}^N \|u_i\|^2_A \;\le\; C\,\|w\|^2_A + CH^{-2}\,\|w\|_{L^2(\Omega)}\,,$$

and the proof is concluded by applying Lemma 5.1. □

Remark 5.1. The proof of the previous lemma can be carried out also in the case of zero overlap: $\Omega_i' = \Omega_i$. In this case the partition of unity $\{\theta_i\}$ consists of the (discontinuous) characteristic functions of the subdomains $\{\Omega_i\}$. However, C_0^2 depends on H/h in this case; see also [8] for a similar algorithm.

6. The convergence result

We have now completed all the preparations required to obtain a lower bound for c_P and an upper bound for C_P. We remark that the following proof is similar to those in [2], [3], and [14, Ch. 5.4].

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Theorem 6.1. *There exist constants $C > 0$, $H_0 > 0$, $c(H_0) > 0$, such that, for all $u \in V^h$,*

$$A(Pu, Pu) \;\le\; C\, A(u,u)\,,$$
$$c(H_0)A(u,u) \;\le\; A(u, Pu), \quad H \le H_0.$$

Proof. First we observe that Lemma 5.4 implies

$$\|Pu\|_A^2 \;=\; \left\| \sum_{i=0}^N P_i u \right\|_A^2 \;\le\; C \sum_{i=0}^N \|P_i u\|_A^2\,. \tag{6.1}$$

Since B is coercive and continuous, we find that

$$\begin{aligned} \sum_{i=0}^N \|P_i u\|_A^2 \;&\le\; \sum_{i=0}^N B(P_i u, P_i u) = \sum_{i=0}^N B(u, P_i u) = B(u, \sum_{i=0}^N P_i u) \\ &\le\; C\,\|u\|_A \left\| \sum_{i=0}^N P_i u \right\|_A \;\le\; C\,\|u\|_A \left(\sum_{i=0}^N \|P_i u\|_A^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2), we obtain $\|Pu\|_A^2 \le C\,\|u\|_A^2$, which proves our upper bound.

In order to prove the lower bound, we first provide a bound for $\sum_{i=0}^N \|P_i u\|_A^2$. The coercivity and continuity of B, Lemma 5.8, and the Cauchy-Schwarz inequality yield

$$\begin{aligned} \|u\|_A^2 \;&\le\; B(u,u) \;=\; \sum_{i=0}^N B(u, u_i) \;=\; \sum_{i=0}^N B(P_i u, u_i) \\ &\le\; C \sum_{i=0}^N \|P_i u\|_A \|u_i\|_A \;\le\; C \left(\sum_{i=0}^N \|P_i u\|_A^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=0}^N \|u_i\|_A^2 \right)^{\frac{1}{2}} \\ &\le\; C \left(\sum_{i=0}^N \|P_i u\|_A^2 \right)^{\frac{1}{2}} \cdot C_0\,\|u\|_A\,, \end{aligned}$$

which gives

$$\sum_{i=0}^N \|P_i u\|_A^2 \;\ge\; C\,\|u\|_A^2. \tag{6.3}$$

Using the definition of P_i and B, we have

$$\begin{aligned} 0 \;&=\; B(P_i u - u, P_i u) \\ &=\; A(P_i u - u, P_i u) + C(P_i u - u, P_i u) + D(P_i u - u, P_i u) + S(P_i u - u, P_i u)\,, \end{aligned}$$

and consequently,

$$\begin{aligned} &A(u, Pu) \\ &\quad=\; \sum_{i=0}^N \left[A(P_i u, P_i u) + C(P_i u - u, P_i u) + D(P_i u - u, P_i u) + S(P_i u - u, P_i u) \right]. \end{aligned}$$

Using 6.3, we find that

$$A(u, Pu) \ge C\|u\|_A^2 - \left| \sum_{i=0}^N \left[C(P_i u - u, P_i u) + D(P_i u - u, P_i u) + S(P_i u - u, P_i u) \right] \right|.$$

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Upper bounds for the terms in the sum can be found using Lemmata 5.3 and 5.7, for $i = 0$ and $i > 0$, respectively. We obtain, after applying (6.3),

$$A(u, Pu) \geq \left[C - C_1 \max(H^\gamma, \sqrt{H}, H) - C_2 \max(H^\gamma, \sqrt{H}, H)\right] \|u\|_A^2.$$

The proof of the lower bound is completed by choosing H sufficiently small. □

Remark 6.1. Our analysis is valid for FE spaces of arbitrary polynomial degree on each element, but the constants C, H_0, and c in Theorem 6.1 depend on p in general.

7. Numerical results

We present some numerical results to illustrate the performance of our overlapping Schwarz algorithm for piecewise linear finite elements in two dimensions. We have tested the two-level preconditioner introduced in the previous sections, as well as the one-level preconditioner built on the same partitions, and we are interested in the performance of the two methods when varying h, H, and the overlap. We consider problem (1.1) in $\Omega = (0,1)^2$ with weakly imposed Dirichlet boundary conditions; see, e.g., [9]. Our test cases are a Poisson problem, an advection-diffusion equation with constant coefficients, and an advection-diffusion equation with a rotating flow field.

We use a two-level subdivision of Ω, consisting of a fine triangulation $\mathcal{T}_h$, obtained by dividing Ω into h^{-2} squares that are then cut into two triangles, and a coarse triangulation consisting of H^{-2} squares $\{\Omega_i\}$, which are possibly extended in order to form a partition $\{\widetilde{\Omega_i}\}$ by adding $q \in \{0, 1, 2, \ldots\}$ layers of h-level triangles in all directions. We set $\Omega_i' = \widetilde{\Omega_i} \cap \Omega$. The overlap is $\delta = qh$, $\delta \geq 0$.

Though our theory requires H to be sufficiently small and the penalization parameter σ_0 to be of order H^{-1}, our experiments show that in practice these restrictions are not required. We have chosen $\sigma_0 = 1$ and solved the coarse and local problems exactly by using Gaussian elimination.

We remark that all our theoretical estimates employ the A-induced scalar product, but that our GMRES implementation employs the standard Euclidean product. Our theoretical results are still valid in this case:

The inverse estimates (3.5) and (3.6) yield positive constants d_0 and d_1, independent of h, such that

$$d_0 h^d \|x\|_2^2 \leq \|x\|_A^2 \leq d_1 h^{d-2} \|x\|_2^2, \quad x \in \mathbb{R}^n \ ;$$

see for example [10, Sect. 7.7]. Therefore, the use of the Euclidean norm increases the iteration counts only by an additive term of order $\log_{10}(h)$, which is hard to observe in our computational experiments; see also [11, Sect. 5].

In our experiments we stop GMRES as soon as $\|r_i\|_2 \leq 10^{-6}\|r_0\|_2$ or after 100 iterations. Our numerical results have been obtained with *Matlab 5.3*.

7.1. Poisson equation.

We first consider the Poisson equation with inhomogeneous Dirichlet conditions:

$$-\Delta u = xe^y \quad \text{in } \Omega\,, \quad u = -xe^y \quad \text{on } \Gamma\,.$$

and partitions into $N \times N$ squares ($H = 1/N$), with $N = 2, 4, 8, 16, 32$.

Table 7.1 shows the iteration counts for the one- and two-level algorithms, as functions of h, H, and the inverse of the relative overlap. We have also considered

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TABLE 7.1. Poisson's equation: Iteration counts for GMRES with the one-level and two-level preconditioners, respectively, versus h, H, and the relative overlap.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	2	17	-	16	14	12
16	4	24	-	-	22	17
32	2	22	21	17	14	12
32	4	33	-	30	23	18
32	8	44	-	-	38	29
64	2	30	27	22	17	14
64	4	45	40	32	24	18
64	8	60	-	53	41	30
64	16	84	-	-	73	54
128	4	60	54	44	33	25
128	8	82	72	57	43	31
128	16	100	-	100	78	57
128	32	100	-	-	100	100

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	2	13	-	11	11	11
16	4	13	-	-	13	14
32	2	16	13	12	11	10
32	4	15	-	13	12	13
32	8	13	-	-	13	15
64	2	21	16	14	12	11
64	4	19	15	14	13	13
64	8	16	-	13	13	14
64	16	13	-	-	13	15
128	4	25	18	16	14	13
128	8	35	15	14	13	14
128	16	15	-	13	13	15
128	32	12	-	-	13	15

the case of zero overlap, denoted by $H/\delta = \infty$. We note that both methods appear to be rather insensitive to the size of the original problem when H is fixed, but that, as expected, the iterations for the one-level preconditioner (table on the left-hand side) grow with the number of subdomains. The two-level algorithm (table on the right-hand side), on the other hand, appears to be scalable: the iteration numbers appear to depend only on the relative overlap δ/H and decrease when the relative overlap increases. Since our convergence bound for the two-level preconditioner is not explicit in the overlap, we can only give the heuristic explanation that the subproblems capture more and more of the entire problem when the overlap is increased, and thus convergence is improved.

Finally, we remark that the restrictions that H be sufficiently small and that $\sigma_0 > C/H$ are not required in practice for all problem types we have considered. Again, we can only offer a heuristic explanation: the convergence bounds we proved are based on the known bounds for GMRES, and are therefore not sharp. The restriction $H < H_0$ was already observed not to be necessary in practice for conforming approximations of nonsymmetric, positive-definite, second order problems; see, e.g., [2, 14]. We also emphasize that the accuracy of the DG solution deteriorates if σ_0 is chosen too large, and it is thus essential that the theoretically derived restriction $\sigma_0 > C/H$ does not apply in practice.

The case of zero overlap requires a special discussion. Our results show that the numbers of iterations are slightly higher than those obtained in the case of $\delta > 0$ for both algorithms, except for $h = 1/128$ and $H = 1/8$, where the two-level preconditioner without overlap has iteration counts considerably higher than with overlap. From our numerical results, we are unable to deduce whether the two-level method without overlap is nonoptimal with the number of iterations growing, e.g., as a power of H/h. We refer to the following tables for a clearer behavior of the convergence rate in the nonoverlapping case, and to [8] for a method with the same local solvers but a different coarse space, which exhibits a rate of convergence

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that appears to grow linearly with H/h. However, we believe that, due to the minimal communication between the subdomains and the relatively small iteration counts that we have obtained, the two-level algorithm with zero overlap might be competitive in practice.

7.2. **Advection-diffusion problem with constant coefficients.** We next consider the advection-diffusion equation

$$-\Delta u + b \cdot \nabla u = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \Gamma\,,$$

with constant coefficients and zero Dirichlet boundary conditions. We consider the two cases

$$b \in \{-(k\pi, k\pi) : k = 3, 300\}\,.$$

The right-hand side f is always chosen such that the exact solution is given by $u = xe^{xy}\sin(\pi x)\sin(\pi y)$.

Table 7.2 presents the results for $k = 3$, for the one- and two-level algorithms, respectively. As for the Poisson problem with nonvanishing overlap, the iteration counts decrease when the overlap increases, and are independent of the number of subdomains for the two-level method. The use of a coarse solver clearly improves the convergence properties. Here, for moderate convection, the behavior for zero overlap appears to be more regular. As expected, the iteration counts increase when the number of subdomains increases for the one-level algorithm. On the other hand, if a coarse solver is employed, the number of iterations appears to *decrease* with H only, independently of h.

Our second set of results is for $k = 300$ and is shown in Table 7.3. The iteration counts obtained with the two-level method are significantly higher than in the examples above, which is due to the strong convection (the Reynolds number is approximately 1000). The one-level method performs fairly well, but our coarse space slows down the convergence considerably. Such behavior is partly due to the fact that our coarse solver comes from a nonstabilized approximation of an advection-diffusion problem on a continuous FE space. We believe that a different type of coarse solver needs to be devised for this class of convection-dominated problems. Note that the iterations for the one-level method appear to depend only

TABLE 7.2. Case of $b = -(3\pi, 3\pi)$: iteration counts for GMRES with the one-level and two-level preconditioners, respectively, versus h, H, and the relative overlap.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	25	-	-	15	17
32	4	33	-	21	16	17
32	8	45	-	-	25	22
64	4	49	28	22	16	17
64	8	59	-	36	27	24
64	16	84	-	-	47	39
128	4	43	28	22	16	17
128	8	59	36	27	24	24
128	16	84	-	47	39	39

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	15	-	-	14	16
32	4	16	-	15	14	15
32	8	12	-	-	14	16
64	4	20	16	16	15	15
64	8	14	-	13	13	16
64	16	10	-	-	12	16
128	4	20	16	16	15	15
128	8	14	13	13	16	16
128	16	10	-	12	16	16

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TABLE 7.3. Case of $b = -(300\pi, 300\pi)$: iteration counts for GMRES with the one- and two-level preconditioners, versus h, H, and the relative overlap.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	13	-	-	12	16
32	4	14	-	13	13	16
32	8	22	-	-	16	21
64	4	15	13	13	13	16
64	8	23	-	21	17	20
64	16	38	-	-	26	27
128	4	15	13	13	14	16
128	8	23	21	17	20	20
128	16	38	-	26	27	27

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	32	-	-	21	19
32	4	32	-	28	21	18
32	8	74	-	-	32	23
64	4	32	30	27	21	18
64	8	73	-	47	32	23
64	16	100	-	-	36	27
128	4	33	31	27	21	18
128	8	73	47	32	23	23
128	16	100	-	36	27	28

on H, and grow with $1/H$. For both the one- and two-level methods with zero overlap, they also seem to depend only on H and to grow with $1/H$.

7.3. **Advection-diffusion problem with a rotating flow field and boundary layers.** Finally, we consider an advection-diffusion equation with a rotating wind $b = 0.5\,(y+1, -x-1)$, a constant $c = 10^{-4}$, the right-hand side $f = 0$, and discontinuous Dirichlet boundary data:

$$\begin{aligned} -\nu\Delta u + b\cdot\nabla u + cu &= f, \quad \text{in } \Omega\,, \\ u &= 1 \quad \text{if} \quad (x,y) \in]0.5,1] \times \{-1,1\} \cup \{1\} \times [0,1]\,, \\ u &= 0 \quad \text{elsewhere on } \Gamma\,. \end{aligned}$$

We note that for small values of ν, the solution exhibits internal layers and boundary layers along the four sides of Ω.

TABLE 7.4. Rotating flow field, case of $\nu = 1$: iteration counts for GMRES with the one- and two-level preconditioners, versus h, H, and the relative overlap.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	22	-	-	14	16
32	4	30	-	19	15	17
32	8	39	-	-	23	22
64	4	40	26	20	16	18
64	8	53	-	33	25	24
64	16	72	-	-	42	37
128	4	54	28	21	16	18
128	8	53	33	25	24	26
128	16	72	-	42	37	42

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	13	-	-	13	14
32	4	15	-	13	13	13
32	8	14	-	-	13	15
64	4	19	15	14	13	14
64	8	16	-	14	13	14
64	16	13	-	-	13	15
128	4	24	18	14	13	14
128	8	16	14	13	13	14
128	16	13	-	13	15	14

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TABLE 7.5. Rotating flow field, case of $\nu = 0.01$: iteration counts for GMRES with the one- and two-level preconditioners, versus h, H, and the relative overlap.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	13	-	-	10	13
32	4	16	-	11	10	14
32	8	23	-	-	15	17
64	4	19	13	10	10	14
64	8	28	-	18	15	18
64	16	43	-	-	25	25
128	4	25	13	11	10	14
128	8	28	18	15	18	19
128	16	43	-	25	25	27

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	27	-	-	19	16
32	4	28	-	22	19	16
32	8	33	-	-	20	18
64	4	31	26	23	19	17
64	8	36	-	24	20	17
64	16	23	-	-	17	19
128	4	35	26	23	19	17
128	8	36	24	20	17	17
128	16	23	-	17	19	18

Table 7.4 shows the results for the two methods for a case of small Reynolds number ($\nu = 1$). The remarks made for the example with moderate convection (Table 7.2) apply in this case as well.

We then consider a convection-dominated problem. Table 7.5 shows the results for a much smaller diffusion coefficient ($\nu = 0.01$). As for the the convection-dominated example with constant flow, the results for the one-level method are better than those with a coarse space, even though, due to the smaller Reynolds number (100 compared to 1000), the difference is not as large as in Table 7.3. On the other hand, similarly to the example with moderate constant flow, the iteration counts for the two-level algorithm decrease when the overlap increases and seem to be independent of the number of subdomains. For the case of zero overlap, the one-level algorithm shows increasing iteration counts when H decreases, while for the two-level algorithm the iteration counts do not seem to follow a regular pattern.

7.4. **Concluding remarks for the numerical results.** The numerical experiments confirm our convergence analysis and suggest that the restrictions $H < H_0$ and $\sigma_0 > c_0/H$ are not needed in practice. For low Reynolds numbers the two-level algorithm with zero overlap performs similarly to the analyzed overlapping method, but with larger iteration counts. A comparison of overlapping one- and two-level methods in the case of high Reynolds numbers reveals the need for a coarse space which is better adjusted to the convection-dominated regime.

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Center for Mathematical Sciences, Technische Universität München, D-85748 Garching bei München, Germany

E-mail address: `classer@mathematik.tu-muenchen.de`

Seminar for Applied Mathematics, ETH Zürich, Rämisstr. 101, CH-8092 Zürich, Switzerland

E-mail address: `toselli@sam.math.ethz.ch`

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