MATHEMATICS OF COMPUTATION
Volume 72, Number 243, Pages 1329–1348
S 0025-5718(03)01505-9
Article electronically published on February 3, 2003

THE SPECTRA OF LARGE TOEPLITZ BAND MATRICES WITH A RANDOMLY PERTURBED ENTRY

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ABSTRACT. This paper is concerned with the union $\mathrm{sp}_\Omega^{(j,k)}\, T_n(a)$ of all possible spectra that may emerge when perturbing a large $n \times n$ Toeplitz band matrix $T_n(a)$ in the (j,k) site by a number randomly chosen from some set Ω. The main results give descriptive bounds and, in several interesting situations, even provide complete identifications of the limit of $\mathrm{sp}_\Omega^{(j,k)}\, T_n(a)$ as $n \to \infty$. Also discussed are the cases of small and large sets Ω as well as the "discontinuity of the infinite volume case", which means that in general $\mathrm{sp}_\Omega^{(j,k)}\, T_n(a)$ does not converge to something close to $\mathrm{sp}_\Omega^{(j,k)}\, T(a)$ as $n \to \infty$, where $T(a)$ is the corresponding infinite Toeplitz matrix. Illustrations are provided for tridiagonal Toeplitz matrices, a notable special case.

1. Introduction and main results

For a complex-valued continuous function a on the complex unit circle $\mathbf{T}$, the infinite Toeplitz matrix $T(a)$ and the finite Toeplitz matrices $T_n(a)$ are defined by

$$T(a) = (a_{j-k})_{j,k=1}^{\infty} \quad \text{and} \quad T_n(a) = (a_{j-k})_{j,k=1}^{n},$$

where a_ℓ is the ℓth Fourier coefficient of a,

$$a_\ell = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-i\ell\theta}\, d\theta, \quad \ell \in \mathbf{Z}.$$

Here, we restrict our attention to the case where a is a trigonometric polynomial, $a \in \mathcal{P}$, implying that at most a finite number of the Fourier coefficients are nonzero; equivalently, $T(a)$ is a banded matrix. The matrix $T(a)$ induces a bounded operator on $\ell^2(\mathbf{N})$, and we think of $T_n(a)$ as a bounded operator on $\mathbf{C}^n$ with the ℓ^2 norm.

Let A stand for $T(a)$ or $T_n(a)$. The spectrum $\mathrm{sp}\, A$ is defined as usual, that is, as the set of all $\lambda \in \mathbf{C}$ for which $A - \lambda I$ is not invertible. Given a complex number ω, we denote by $\omega e_j e_k^*$ the matrix that is zero everywhere except in the (j,k) entry, which is ω. For a subset Ω of $\mathbf{C}$, we put

$$\mathrm{sp}_\Omega^{(j,k)}\, A = \bigcup_{\omega \in \Omega} \mathrm{sp}(A + \omega e_j e_k^*).$$

Received by the editor August 3, 2001.
2000 *Mathematics Subject Classification.* Primary 47B35, 65F15; Secondary 15A18, 47B80, 82B44.
Key words and phrases. Toeplitz operator, pseudospectrum, random perturbation.
The work of the second author was supported by UK Engineering and Physical Sciences Research Council Grant GR/M12414.

Thus, $\operatorname{sp}_\Omega^{(j,k)} A$ is the union of all possible spectra that may emerge as the result of a perturbation of A in the (j,k) site by a number randomly chosen in Ω. The purpose of this paper is to study $\operatorname{sp}_\Omega^{(j,k)} T_n(a)$ for fixed j, k, and Ω as $n \to \infty$.

Gohberg identified the spectrum of $T(a)$ in 1952 [12], proving that

$$\operatorname{sp} T(a) = a(\mathbf{T}) \cup \{\lambda \in \mathbf{C} \setminus a(\mathbf{T}) : \operatorname{wind}(a,\lambda) \neq 0\}, \tag{1.1}$$

where $\operatorname{wind}(a,\lambda)$ is the winding number of a (on the counterclockwise oriented unit circle) about λ. Every point in $\operatorname{sp} T(a) \setminus a(\mathbf{T})$ is an eigenvalue of finite multiplicity of $T(a)$ or the transpose of $T(a)$; a point on $a(\mathbf{T})$ may be an eigenvalue or not.

The spectra of the large finite Toeplitz matrices $T_n(a)$ were studied in 1960 by Schmidt and Spitzer [26], who observed that the sets $\operatorname{sp} T_n(a)$ converge in the Hausdorff metric to a limiting set $\Lambda(a)$,

$$\lim_{n\to\infty} \operatorname{sp} T_n(a) = \Lambda(a). \tag{1.2}$$

The set $\Lambda(a)$ is either a singleton or the union of at most finitely many analytic arcs, each pair of which has at most endpoints in common; it can be described as follows. We can write $a(t) = \sum_k a_k t^k$ $(t \in \mathbf{T})$, the sum being finite. For $\varrho \in (0,\infty)$, define $a_\varrho \in \mathcal{P}$ by $a_\varrho(t) = \sum_k a_k \varrho^k t^k$ $(t \in \mathbf{T})$. Then

$$\Lambda(a) = \bigcap_{\varrho > 0} \operatorname{sp} T(a_\varrho). \tag{1.3}$$

From (1.1), (1.2), and (1.3) we see that in general $\operatorname{sp} T_n(a)$ does not converge to $\operatorname{sp} T(a)$. Surprisingly, pseudospectra of Toeplitz matrices behave differently. For $\varepsilon > 0$, the ε-pseudospectrum $\operatorname{sp}_\varepsilon A$ is defined by

$$\operatorname{sp}_\varepsilon A = \bigcup_{\|K\| \le \varepsilon} \operatorname{sp}(A+K),$$

with the union taken over all matrices K (of the same size as A) which induce an operator of norm at most ε (see, e.g., [7], [27], [28]). Landau [20] and Reichel and Trefethen [24] (also see [2] and [7]) showed that for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \operatorname{sp}_\varepsilon T_n(a) = \operatorname{sp}_\varepsilon T(a). \tag{1.4}$$

It is the abyss between (1.2) and (1.4) that makes the question of the limit of $\operatorname{sp}_\Omega^{(j,k)} T_n(a)$ intriguing. To state precise results, we need some more preliminaries. For a sequence $\{M_n\}_{n=1}^\infty$ of nonempty sets $M_n \subset \mathbf{C}$, consider the limiting sets

$$\begin{aligned}\liminf_{n\to\infty} M_n &:= \{\lambda \in \mathbf{C} : \lambda \text{ is the limit of some} \\ &\qquad\quad \text{sequence } \{\lambda_n\}_{n=1}^\infty \text{ with } \lambda_n \in M_n\}, \\ \limsup_{n\to\infty} M_n &:= \{\lambda \in \mathbf{C} : \lambda \text{ is a partial limit of some} \\ &\qquad\quad \text{sequence } \{\lambda_n\}_{n=1}^\infty \text{ with } \lambda_n \in M_n\}.\end{aligned}$$

Notice that if all the sets involved are compact, then the two equalities

$$\liminf_{n\to\infty} M_n = \limsup_{n\to\infty} M_n = M$$

are equivalent to saying that M_n converges to M in the Hausdorff metric (see [14, Sections 3.1.1 and 3.1.2] or [16, Section 28]), in which case we simply write

$$\lim_{n\to\infty} M_n = M,$$

as in (1.2) and (1.4).

If $\lambda \in \mathbf{C} \setminus \Lambda(a)$, then, by (1.3), there is a $\varrho > 0$ such that $T(a_\varrho - \lambda)$ is invertible. We show in Lemma 3.1 that

$$T^{-1}(a_\varrho - \lambda) = \left(\varrho^{j-k} d_{jk}(\lambda)\right)_{j,k=1}^{\infty} \tag{1.5}$$

with analytic functions $d_{jk} : \mathbf{C} \setminus \Lambda(a) \to \mathbf{C}$ that do not depend on ϱ. (Here and in what follows, we write $T^{-1}(\cdot)$ for $(T(\cdot))^{-1}$.) For a set $M \subset \mathbf{C}$, we define $-1/M$ by

$$-1/M = \{\gamma \in \mathbf{C} : 1 + \mu\gamma = 0 \text{ for some } \mu \in M\}.$$

It is well known and easily seen that if $0 \in \Omega$, then

$$\mathrm{sp}_\Omega^{(j,k)} A = \mathrm{sp}\, A \cup \{\lambda \notin \mathrm{sp}\, A : [(A - \lambda I)^{-1}]_{kj} \in -1/\Omega\}, \tag{1.6}$$

where $[(A - \lambda I)^{-1}]_{kj}$ is the (k, j) entry of the resolvent $(A - \lambda I)^{-1}$. Consequently, letting

$$H_\Omega^{jk}(a) = \{\lambda \in \mathbf{C} \setminus \Lambda(a) : d_{kj}(\lambda) \in -1/\Omega\}, \tag{1.7}$$

we obtain

$$\mathrm{sp}_\Omega^{(j,k)} T(a) = \mathrm{sp}\, T(a) \cup H_\Omega^{jk}(a). \tag{1.8}$$

(Note that for λ outside $\mathrm{sp}\, T(a)$ we can take (1.5) with $\varrho = 1$.) Formula (1.8) disposes of the "infinite volume case". In the "finite volume case" we have the following theorem, which is the main result of this paper.

Theorem 1.1. *Let $a \in \mathcal{P}$ and let Ω be a compact subset of $\mathbf{C}$ that contains the origin. If $d_{kj} : \mathbf{C} \setminus \Lambda(a) \to \mathbf{C}$ is identically zero or nowhere locally constant or assumes a constant value c that does not belong to $-1/\Omega$, then*

$$\lim_{n\to\infty} \mathrm{sp}_\Omega^{(j,k)} T_n(a) = \Lambda(a) \cup H_\Omega^{jk}(a). \tag{1.9}$$

We conjecture that for every $a \in \mathcal{P}$ and every (j, k) the function d_{kj} is either identically zero or nowhere locally constant, and hence that (1.9) is always true, but we have not been able to prove this. The results of [6] imply at least the following.

Theorem 1.2. *Let $a \in \mathcal{P}$ be of the form*

$$a(t) = \sum_{k=-p}^{q} a_k t^k, \quad p \geq 0, \quad q \geq 0, \quad a_{-p} a_q \neq 0, \tag{1.10}$$

and let G be a bounded component of $\mathbf{C} \setminus \mathrm{sp}\, T(a)$. The function d_{11} is always nowhere locally constant in G. If $k \geq 1$ or $j \geq 1$, then the function d_{kj} is either identically zero in G or nowhere locally constant in G, provided one of the following conditions is satisfied:

(a) *$\mathbf{C} \setminus \Lambda(a)$ is connected;*
(b) *p or q equals 1;*
(c) *$p + q$ is a prime number and p or q equals 2;*
(d) *$p + q = 4$ or $p + q = 7$.*

We remark that $\mathbf{C} \setminus \Lambda(a)$ is in particular connected if $T(a)$ is tridiagonal (which means that $p + q = 2$) or triangular ($p = 0$ or $q = 0$) or Hermitian. Condition (b) is equivalent to saying that $T(a)$ is a Hessenberg matrix. Notice that we may also without loss of generality assume that $p \leq q$ (otherwise we may pass to the adjoint operator). The case $p + q = 3$ is covered by (a) for $p = 0$ and by (b) for $p = 1$, while the case $p + q = 5$ is contained in (a) for $p = 0$, in (b) for $p = 1$, and in (c) for $p = 2$. We have no result in the case $p + q = 6$ (unless $p = 0$ or $p = 1$).

Since $\operatorname{sp} T(a)$ is in general much larger than $\Lambda(a)$, we see from (1.8) and (1.9) that $\operatorname{sp}_\Omega^{(j,k)}$ generically behaves discontinuously when passing from large finite Toeplitz matrices to infinite Toeplitz matrices. The following result is in the same vein. A set $\Omega \in \mathbf{C}$ is said to be starlike if it contains the line segment $[0,\omega]$ for every point $\omega \in \Omega$. We denote by $\overline{\Omega}$ or $\operatorname{clos}\Omega$ the closure of a set Ω and by Ω° the set of the interior points of Ω. Finally, we put $\varepsilon\Omega = \{\varepsilon\omega : \omega \in \Omega\}$.

Theorem 1.3. *Let $a \in \mathcal{P}$ and let $\Omega \subset \mathbf{C}$ be a nonempty starlike compact set such that $\Omega = \operatorname{clos}\Omega^\circ$. Then*

$$\lim_{n\to\infty} \operatorname{sp}_{\varepsilon\Omega}^{(j,k)} T_n(a) = \Lambda(a) \cup H_{\varepsilon\Omega}^{jk}(a) \tag{1.11}$$

for all $\varepsilon \in (0,\infty)$ with the possible exception of at most finitely many $\varepsilon_1,\dots,\varepsilon_\ell$, where ℓ does not exceed the number of bounded components of $\mathbf{C}\setminus\Lambda(a)$.

We now turn to the case where Ω is a small set, that is, we consider Toeplitz band matrices with a single "impurity". If $a \in \mathcal{P}$ and $T(a)$ is triangular, then the limit of $\operatorname{sp}_\Omega^{(j,j)} T_n(a)$ is strictly larger than $\Lambda(a)$ for every $\Omega \neq \{0\}$. It turns out that the limiting spectra of nontriangular Toeplitz band matrices are not affected by sufficiently small impurities localized in a single site.

Theorem 1.4. *Let $a \in \mathcal{P}$ and suppose $T(a)$ is not triangular. Let Ω be any compact subset of the plane which contains the origin. Then for each (j,k) there exists an $\varepsilon_1 > 0$, depending on j, k, and a, such that*

$$\lim_{n\to\infty} \operatorname{sp}_\Omega^{(j,k)} T_n(a) = \Lambda(a) \tag{1.12}$$

whenever $\Omega \subset \varepsilon_1\overline{\mathbf{D}}$.

In [4] we showed that if $a \in \mathcal{P}$ is not constant, then there is an $\varepsilon_0 > 0$, depending on j, k, and a, such that

$$\operatorname{sp}_\Omega^{(j,k)} T(a) = \operatorname{sp} T(a) \tag{1.13}$$

provided $0 \in \Omega \subset \varepsilon_0\overline{\mathbf{D}}$. Equalities (1.12) and (1.13) tell us that $\lim \operatorname{sp}_\Omega^{(j,k)} T_n(a)$ and $\operatorname{sp}_\Omega^{(j,k)} T(a)$ stabilize at constant values before Ω contracts to zero and that, however, these values are in general different.

Here is a result for large perturbations.

Theorem 1.5. *Let $a \in \mathcal{P}$ and suppose $T(a)$ is not triangular. Let $\Omega \subset \mathbf{C}$ be a compact set. Then there is an $\varepsilon_2 > 0$, depending on a, such that*

$$\lim_{n\to\infty} \operatorname{sp}_\Omega^{(1,1)} T_n(a) = \operatorname{sp}_\Omega^{(1,1)} T(a) \tag{1.14}$$

whenever $\varepsilon_2\overline{\mathbf{D}} \subset \Omega$.

We will show that Theorem 1.5 is in general no longer true with $(1,1)$ replaced by (j,k).

Theorems 1.4 and 1.5 reveal that for small perturbation sets Ω, the asymptotic behavior of $\operatorname{sp}_\Omega^{(j,k)} T_n(a)$ is as in (1.2), while for large Ω the sets $\operatorname{sp}_\Omega^{(1,1)} T_n(a)$ mimic (1.4). In a sense, Theorem 1.5 describes a situation in which there is no discontinuity when passing from large finite matrices to an infinite matrix. On the other hand, Theorem 1.4 shows that in the presence of only very small impurities the passage from finite matrices to the infinite matrix is analogous to (1.2), and thus discontinuous.

Theorem 1.2 is immediate from [6]. Theorems 1.1 and 1.3 will be proved in Section 3, and the proofs to Theorems 1.4 and 1.5 will be given in Section 4. In Section 5 we discuss a concrete example: single entry perturbations to tridiagonal Toeplitz matrices. Section 2 provides several instructive illustrations. Our approach is based on formula (1.6) and the convergence of the finite section method for invertible Toeplitz band matrices. In the case where Ω is a finite set, the asymptotic behavior of $\mathrm{sp}_{\Omega}^{(j,k)}\, T_n(a)$ has been thoroughly studied in [1, 13, 19]. However, these results and techniques are not sufficient to uncover phenomena like those described by the above theorems. Perturbed Toeplitz matrices arise in a variety of settings in applied mathematics and physics, including nonHermitian quantum mechanics [8, 10, 15, 29] (also see [9]), population biology [23], linear systems theory [17, 18], small world networks [21], and eigenvalue perturbation theory for general matrices [22]. In particular, single-entry perturbations are discussed in [10, 21]. For more on the sets $\mathrm{sp}_{\Omega}^{(j,k)}\, T(a)$ (representing the infinite volume case) we refer to our recent paper [4]. Finally, it should be noted that (1.6) is the appropriate tool for investigating random perturbations in a single site. Perturbations in a finite set of sites require more machinery; this is the subject of our article [5], which proves an analog of Theorem 1.4 for perturbations to the upper-left block of a matrix.

2. Illustrations

Figures 1–4 concern the symbol $a(t) = t + \frac{1}{9}\, t^{-1}$, which yields a tridiagonal Toeplitz matrix. The range $a(\mathbf{T})$ is an ellipse, $\mathrm{sp}\, T(a)$ equals $a(\mathbf{T}) \cup E_+$, where E_+ denotes the set of points inside this ellipse, and $\Lambda(a)$ is the line segment between the foci of the ellipse. Details are given in Section 5; related analysis and similar illustrations for the infinite volume case are presented in [4].

Since $\mathbf{C} \setminus \Lambda(a)$ is connected, Theorems 1.1 and 1.2 imply that (1.9) is valid. At the intersection of the jth row and kth column of Figure 1 we see $\Lambda(a) \cup H_{\Omega}^{jk}(a)$ and thus $\lim_{n\to\infty} \mathrm{sp}_{\Omega}^{(j,k)}\, T_n(a)$ for $\Omega = [-5, 5]$.

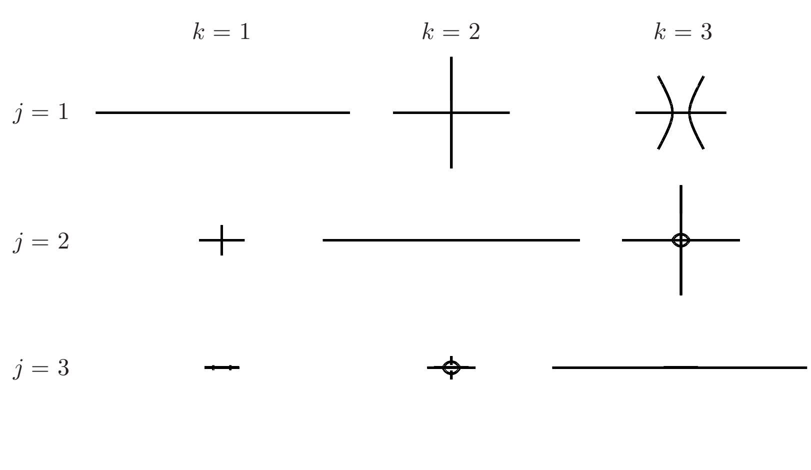

Figure 1. The sets $\Lambda(a) \cup H_{\Omega}^{jk}(a)$ for $a(t) = t + \frac{1}{9}\, t^{-1}$, $\Omega = [-5, 5]$, and the nine possible choices of (j, k) with $j, k \in \{1, 2, 3\}$.

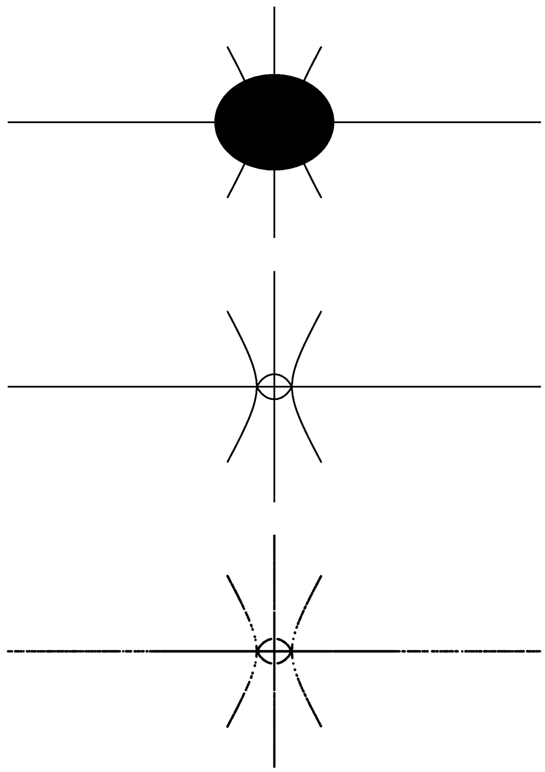

FIGURE 2. Real single-entry perturbations to $T(a)$ and $T_n(a)$ for $a(t) = t + \frac{1}{9}\,t^{-1}$ and $\Omega = [-5, 5]$. The top picture shows the union of $\mathrm{sp}_\Omega^{(j,k)}\,T(a)$ over all (j,k) in the upper 3×3 block; the middle picture represents the union of $\lim_{n\to\infty} \mathrm{sp}_\Omega^{(j,k)}\,T_n(a)$ over the same (j,k). The bottom picture superimposes the eigenvalues of 2000 single-entry perturbations of $T_{20}(a)$, where the perturbed entry is randomly selected from the upper left 3×3 block, and the perturbation itself is a random number uniformly distributed in $[-5, 5]$.

Figures 2 and 3 illustrate the following experiment. We choose one of the entries of the upper $m \times m$ block of $T_n(a)$ randomly with probability $1/m^2$ and then perturb $T_n(a)$ in this entry by a random number uniformly distributed in $\Omega = [-5, 5]$, plotting the n eigenvalues of the perturbed matrix. We repeat this N times and consider the superimposition of the Nn eigenvalues obtained. Equality (1.9) suggests that this superimposition should approximate

$$\begin{aligned} &\bigcup_{1\le j,k\le m} \lim_{n\to\infty} \mathrm{sp}_{[-5,5]}^{(j,k)} T_n(a) \\ &\qquad = \bigcup_{1\le j,k\le m} \Big(\Lambda(a) \cup H_{[-5,5]}^{(j,k)}(a)\Big) \end{aligned} \tag{2.1}$$

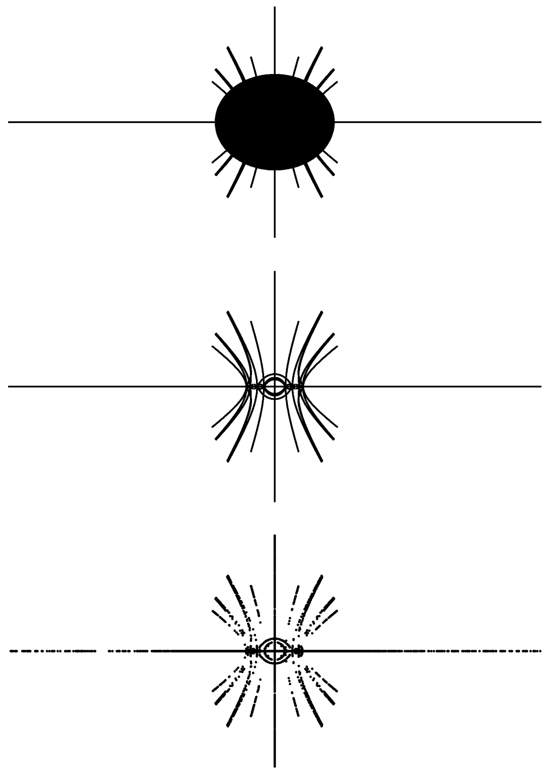

FIGURE 3. This is the analogue of Figure 2 for real single-entry perturbations in the upper 5×5 block.

as $n \to \infty$ and $N \to \infty$. For $m = 3$ and $m = 5$, the sets (2.1) are shown in the middle pictures of Figures 2 and 3, respectively. Notice that, up to a change in scale, the middle picture of Figure 2 is nothing but the union of the nine pictures of Figure 1. The bottom pictures of Figures 2 and 3 depict the result of concrete numerical experiments with $N = 2000$ and $n = 20$. The agreement between the $n \to \infty$ theory and practice is striking even for modest n.

The top pictures of Figures 2 and 3 illustrate

$$\bigcup_{1 \le j,k \le m} \mathrm{sp}_{[-5,5]}^{(j,k)} T(a) = \bigcup_{1 \le j,k \le m} \left(\mathrm{sp}\, T(a) \cup H_{[-5,5]}^{(j,k)}(a) \right). \tag{2.2}$$

Obviously, these top pictures (infinite volume case) differ significantly from the middle pictures (finite volume case). Even more than that, in the finite volume case we discover a remarkable structure in the set (2.2). In the infinite volume case, this structure is hidden behind the black ellipse E_+, so we are only aware of the ends of certain arcs, resembling antennae sprouting from the ellipse.

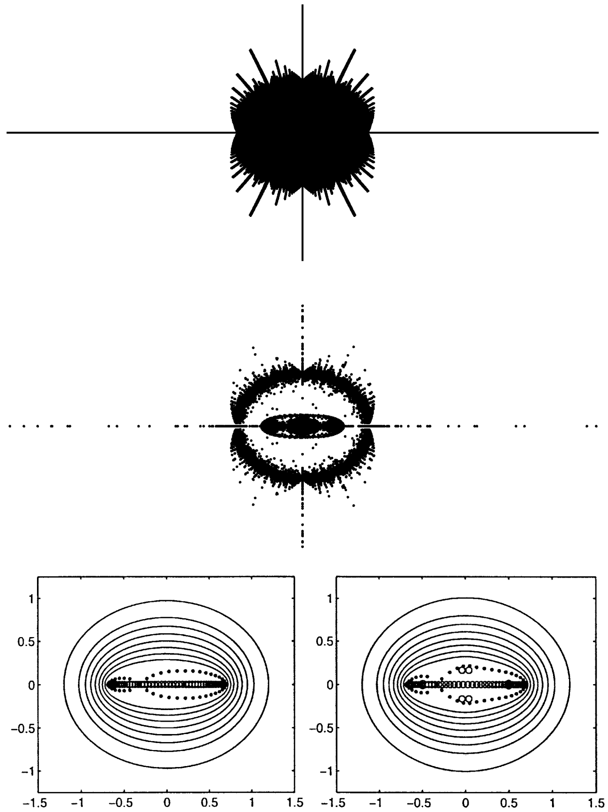

FIGURE 4. Real single-entry perturbations to $T(a)$ and $T_n(a)$ for $a(t) = t + \frac{1}{9}t^{-1}$ and $\Omega = [-5, 5]$. The top picture is the union of $\mathrm{sp}_\Omega^{(j,k)}\, T(a)$ over $(j,k) \in \mathbf{N} \times \mathbf{N}$. The middle picture shows computed eigenvalues of 1000 single-entry perturbations to $T_{50}(a)$, where the perturbed entry is chosen at random anywhere in the matrix, and the perturbation itself is randomly chosen from the uniform distribution on $[-5, 5]$. The interior elliptical region of high eigenvalue concentration in the middle picture is an artifact of finite precision arithmetic; this is revealed by the two bottom pictures, which show the boundaries of the pseudospectra $\mathrm{sp}_\varepsilon\, T_{50}(a)$ (left) and $\mathrm{sp}_\varepsilon\, (T_{50}(a) + \omega e_5 e_4^*)$ with $\omega = -3$ (right) for $\varepsilon = 10^{-1}, 10^{-3}, \ldots, 10^{-15}$. Dots $(\cdot)$ denote computed eigenvalues; circles $(\circ)$ show the true eigenvalue locations.

The top picture of Figure 4 shows

$$\bigcup_{(j,k)\in\mathbf{N}\times\mathbf{N}} \mathrm{sp}_{[-5,5]}^{(j,k)} T(a)$$
$$= \bigcup_{(j,k)\in\mathbf{N}\times\mathbf{N}} \Big(\mathrm{sp}\, T(a) \cup H_{[-5,5]}^{(j,k)}(a)\Big)$$

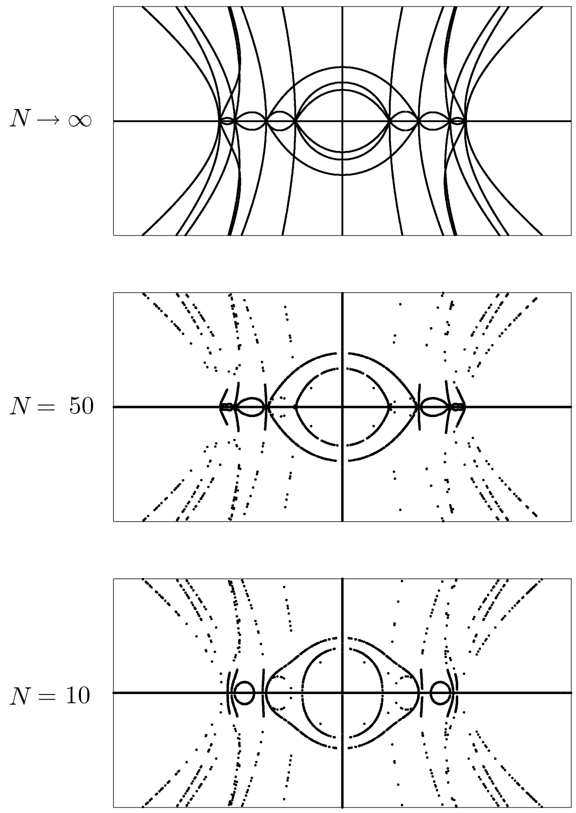

FIGURE 5. Closer inspection of Figure 3. The top plot shows a portion of $\lim_{n\to\infty} \mathrm{sp}_{\Omega}^{(j,k)}\, T_n(a)$ over all (j,k) in the top 5×5 corner for $\Omega = [-5,5]$. The middle image shows eigenvalues of 10,000 random perturbations to a single entry in the top corner of $T_{50}(a)$; the bottom image shows the same for $T_{10}(a)$. (The accurate eigenvalues for $n = 50$ were obtained by transforming the problem to a more numerically stable one via similarity.)

(infinite volume case), while in the middle picture we approximate

$$\bigcup_{(j,k)\in\mathbf{N}\times\mathbf{N}} \lim_{n\to\infty} \mathrm{sp}_{[-5,5]}^{(j,k)} T_n(a) = \bigcup_{(j,k)\in\mathbf{N}\times\mathbf{N}} \Big(\Lambda(a) \cup H_{[-5,5]}^{(j,k)}(a)\Big)$$

(finite volume case) by the union of $\mathrm{sp}(T_{50}(a) + \omega e_j e_k^*)$ for 1000 random choices of $j, k \in \{1, 2, \ldots, 50\}$ and $\omega \in [-5,5]$. It is well known that the eigenvalues of the finite Toeplitz matrices $T_n(a)$ are highly sensitive to perturbations even for modest dimensions [24]. It is interesting that the single entry perturbations investigated here do not generally change the qualitative nature of that eigenvalue instability. This is revealed for a specific example by the pseudospectral plots at the bottom of Figure 4 (computed using [30]). This explains the dark interior ellipse in the center

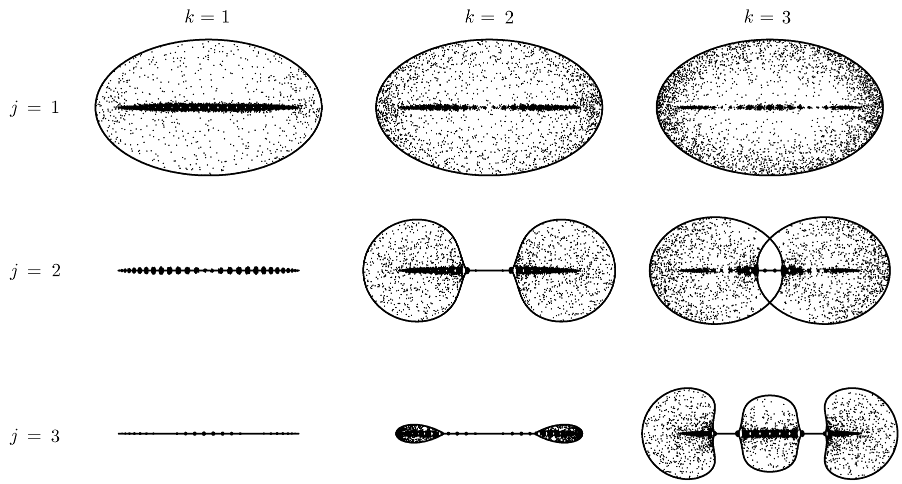

FIGURE 6. Complex single-entry perturbations to $T(a)$ and $T_n(a)$ for $a(t) = t + \frac{1}{4}\,t^{-1}$ and $\Omega = \overline{\mathbf{D}}$, the closed unit disk. The plot in the jth row and kth column shows the superimposed eigenvalues of 1000 perturbations to $T_{30}(a)$ in the (j,k) entry for $j,k \in \{1,2,3\}$. Each perturbation is a random number uniformly distributed in $\overline{\mathbf{D}}$. The boundaries of the regions $\lim_{n\to\infty} \mathrm{sp}_{\overline{\mathbf{D}}}^{(j,k)}\, T_n(a) = \Lambda(a) \cup H_\Omega^{jk}(a)$ are drawn as solid curves.

plot of Figure 4: many of these computed eigenvalues are inaccurate due to rounding errors. Generic perturbations of norm 10^{-15} obscure the effects of our larger, single-entry perturbations. The true structure is more delicate, as emphasized by Figure 5, which zooms in on the middle image of Figure 3 for perturbations to the upper left 5×5 corner of $T_n(a)$. We compare the $n \to \infty$ structure to the eigenvalues of perturbations of $T_{10}(a)$ and $T_{50}(a)$. The convergence to the asymptotic limit is compelling, though from an applications perspective, any point in the interior of $a(\mathbf{T})$ will behave like an eigenvalue when n is large.

Figure 6 illustrates the effects of complex single-entry perturbations. In the jth row and the kth column of Figure 6 we see

$$\mathrm{sp}_{\overline{\mathbf{D}}}^{(j,k)} T_{25}(a) \quad \text{for} \quad a(t) = t + \tfrac{1}{4}\,t^{-1}.$$

The dots represent the results of random experiments; the solid lines are the boundaries of $\Lambda(a) \cup H_{\overline{\mathbf{D}}}^{(j,k)}(a)$. While the emergence of wings (or antennae) is typical for real-valued perturbations, one finds that complex perturbations usually lead to "bubbles". For example, we see two bubbles in $\mathrm{sp}_{\overline{\mathbf{D}}}^{(2,2)} T_{25}(a)$, which split into three bubbles in $\mathrm{sp}_{\overline{\mathbf{D}}}^{(3,3)} T_{30}(a)$.

Finally, Figure 7 exhibits a Toeplitz matrix with six diagonals. The symbol a is given by

$$\begin{aligned} a(t) = \;& (1.5 - 1.2\,i)t^{-1} \\ & + (0.34 + 0.84\,i)t + (-0.46 - 0.1\,i)t^2 + (0.17 - 1.17\,i)t^3 + (-1 + 0.77\,i)t^4, \end{aligned}$$

introduced in [3]. Notice again the emergence of many wings, which make the set $\Lambda(a)$ (middle picture) become something reminiscent of a horse in cave paintings.

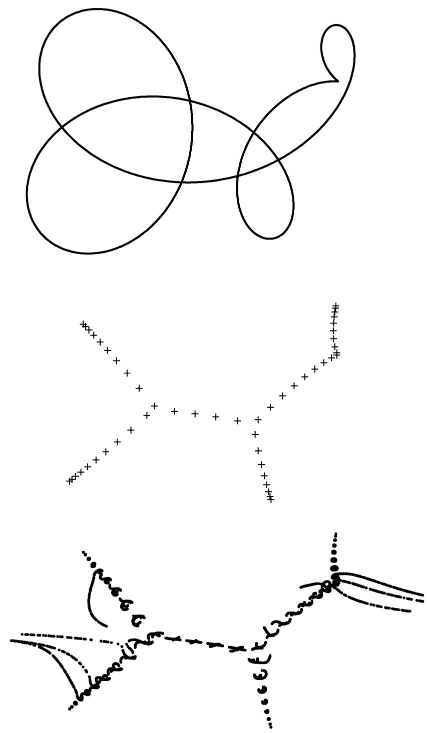

FIGURE 7. Real single-entry perturbations to $T_n(a)$. The range $a(\mathbf{T})$ of the trigonometric polynomial a is seen in the top picture. The middle picture shows $\operatorname{sp} T_{50}(a)$ and provides a very good idea of $\Lambda(a)$. The bottom picture depicts the superimposed eigenvalues of 1000 perturbations of $T_{50}(a)$ in a randomly chosen entry from $(1,1),(2,2),(3,3)$ by a random number uniformly distributed in $[-5,5]$.

For further illustrations, together with analogous plots for the circulant matrix induced by the same symbol, see [3].

3. The limiting set

This section contains proofs for Theorems 1.1 and 1.3. We begin by proving formula (1.5).

Lemma 3.1. *Let $\lambda \in \mathbf{C}\setminus\Lambda(a)$ and $\varrho > 0$. If $T(a_\varrho - \lambda)$ is invertible, then $T^{-1}(a_\varrho - \lambda)$ is of the form (1.5) with numbers $d_{jk}(\lambda)$ that do not depend on ϱ.*

Proof. Suppose first that $T(a)$ is not triangular, that is, let a be of the form (1.10) with $p \geq 1$ and $q \geq 1$. Pick $\lambda \in \mathbf{C} \setminus \Lambda(a)$ and choose $\varrho > 0$ so that $T(a_\varrho - \lambda)$ is invertible. One can write

$$a(t) - \lambda = t^{-p} a_q \prod_{j=1}^{p+q} (t - z_j(\lambda)),$$

whence

$$a_\varrho(t) - \lambda = \varrho^{-p} t^{-p} a_q \prod_{j=1}^{p+q} (\varrho t - z_j(\lambda)).$$

Using (1.1), it is not difficult to check that the invertibility of $T(a_\varrho - \lambda)$ is equivalent to the existence of a labelling of the zeros $z_j(\lambda)$ such that

$$|z_1(\lambda)| \le \cdots \le |z_p(\lambda)| < \varrho < |z_{p+1}(\lambda)| \le \cdots \le |z_{p+q}(\lambda)|. \tag{3.1}$$

Abbreviating $z_j(\lambda)$ to z_j, we have

$$a_\varrho(t) - \lambda = a_q \varphi_-(t) \varphi_+(t),$$

where

$$\varphi_-(t) = \prod_{j=1}^{p} \left(1 - \frac{z_j}{\varrho t}\right), \quad \varphi_+(t) = \prod_{j=p+1}^{p+q} (\varrho t - z_j).$$

Standard computations with Toeplitz matrices (see, e.g., [7]) now give

$$\begin{aligned} T(a_\varrho - \lambda) &= a_q T(\varphi_- \varphi_+) = a_q T(\varphi_-) T(\varphi_+), \\ T^{-1}(a_\varrho - \lambda) &= a_q^{-1} T(\varphi_+^{-1}) T(\varphi_-^{-1}). \end{aligned} \tag{3.2}$$

Clearly,

$$\varphi_-^{-1}(t) = \prod_{j=1}^{p} \left(1 + \frac{z_j}{\varrho t} + \frac{z_j^2}{\varrho^2 t^2} + \cdots\right) =: \sum_{n=0}^{\infty} b_n \varrho^{-n} t^{-n}, \tag{3.3}$$

$$\varphi_+^{-1}(t) = \frac{(-1)^q}{z_{p+1} \cdots z_{p+q}} \prod_{j=p+1}^{p+q} \left(1 + \frac{\varrho t}{z_j} + \frac{\varrho^2 t^2}{z_j^2} + \cdots\right) =: \sum_{n=0}^{\infty} c_n \varrho^n t^n, \tag{3.4}$$

where $b_n = b_n(\lambda)$ and $c_n = c_n(\lambda)$ are independent of ϱ. Convergence of these series is a consequence of (3.1). Thus, by (3.2), $T^{-1}(a_\varrho - \lambda)$ equals

$$\frac{1}{a_q} \begin{pmatrix} c_0 & & & \\ c_1 \varrho & c_0 & & \\ c_2 \varrho^2 & c_1 \varrho & c_0 & \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} b_0 & b_1/\varrho & b_2/\varrho^2 & \cdots \\ & b_0 & b_1/\varrho & \cdots \\ & & b_0 & \cdots \\ & & & \cdots \end{pmatrix},$$

which shows that

$$\left[T^{-1}(a_\varrho - \lambda)\right]_{jk} = \varrho^{j-k} a_q^{-1} (c_{j-1} b_{k-1} + c_{j-2} b_{k-2} + \cdots), \tag{3.5}$$

and thus proves (1.5).

The proof of (1.5) for triangular $T(a)$ is similar. □

If $T(a_\varrho - \lambda)$ is invertible, then so is $T(a_\varrho - \mu)$ for all μ in some open neighborhood of λ. This, in conjunction with Lemma 3.1, implies that the functions $\lambda \mapsto d_{jk}(\lambda)$ are analytic in $\mathbf{C} \setminus \Lambda(a)$.

Lemma 3.2. *Fix a site (j, k) and let $\lambda \in \mathbf{C} \setminus \Lambda(a)$. If $\varrho > 0$ and $T(a_\varrho - \lambda)$ is invertible, then there exist an open neighborhood $U \subset \mathbf{C} \setminus \Lambda(a)$ of λ and a natural number n_0 such that $T(a_\varrho - \mu)$ is invertible for all $\mu \in U$, the matrices $T_n(a_\varrho - \mu)$ are invertible for all $\mu \in U$ and all $n \ge n_0$, and*

$$\left[T_n^{-1}(a_\varrho - \mu)\right]_{jk} \to \left[T^{-1}(a_\varrho - \mu)\right]_{jk} \quad \text{as } n \to \infty$$

uniformly with respect to $\mu \in U$.

Proof. It is a standard result of the theory of projection methods for Toeplitz operators that if $T(a_\varrho - \lambda)$ is invertible, then there exist an open neighborhood V of λ and a natural number m_0 such that $T(a_\varrho - \mu)$ is invertible for all $\mu \in V$,

$$M := \sup_{n \geq m_0} \sup_{\mu \in V} \|T_n^{-1}(a_\varrho - \mu)\| < \infty,$$

and

$$\left[T_n^{-1}(a_\varrho - \mu)\right]_{jk} \to \left[T^{-1}(a_\varrho - \mu)\right]_{jk} \text{ as } n \to \infty$$

for all $\mu \in V$; see, e.g., [7]. Hence, given any $\varepsilon > 0$, we can find a number $n_0 \geq m_0$ and an open neighborhood $U \subset V$ of λ such that

$$\left|\left[T_n^{-1}(a_\varrho - \lambda)\right]_{jk} - \left[T^{-1}(a_\varrho - \lambda)\right]_{jk}\right| < \varepsilon/3,$$

$$\left|\left[T^{-1}(a_\varrho - \mu)\right]_{jk} - \left[T^{-1}(a_\varrho - \lambda)\right]_{jk}\right| < \varepsilon/3,$$

and

$$\begin{aligned}\Big|\left[T_n^{-1}(a_\varrho - \mu)\right]_{jk} &- \left[T_n^{-1}(a_\varrho - \lambda)\right]_{jk}\Big| \\ &\leq \|T_n^{-1}(a_\varrho - \mu) - T_n^{-1}(a_\varrho - \lambda)\| \\ &\leq |\mu - \lambda| \, \|T_n^{-1}(a_\varrho - \mu)\| \, \|T_n^{-1}(a_\varrho - \lambda)\| \leq M^2 |\mu - \lambda| < \varepsilon/3\end{aligned}$$

for all $\mu \in U$ and all $n \geq n_0$. Assembling these three $\varepsilon/3$ inequalities yields the assertion. □

Theorem 3.3. *Let $a \in \mathcal{P}$, and let $\Omega \subset \mathbf{C}$ be a compact set containing the origin. Further, let G be a connected component of $\mathbf{C} \setminus \Lambda(a)$. If d_{kj} is identically zero in G, or not constant in G, or a constant $c \notin -1/\Omega$ in G, then*

$$\lim_{n\to\infty} \mathrm{sp}_\Omega^{(j,k)} T_n(a) \cap \overline{G} = \left(\Lambda(a) \cup H_\Omega^{jk}(a)\right) \cap \overline{G}. \tag{3.6}$$

Proof. We first prove that

$$\liminf_{n\to\infty} \mathrm{sp}_\Omega^{(j,k)} T_n(a) \cap \overline{G} \supset \left(\Lambda(a) \cup H_\Omega^{jk}(a)\right) \cap \overline{G}. \tag{3.7}$$

If $d_{kj}(\mu) = c \notin -1/\Omega$ or $d_{kj}(\mu) = 0$ for all $\mu \in \Omega$, then $H_\Omega^{jk}(a) = \emptyset$, and (3.7) is evident from (1.2). (Recall that $0 \in \Omega$.) Thus, assume d_{kj} is not constant in G and $H_\Omega^{jk}(a)$ is not empty. Take λ in the right-hand side of (3.7). If λ is in the boundary ∂G of G, then λ is in $\Lambda(a)$ and hence in $\liminf \mathrm{sp}_\Omega^{(j,k)} T_n(a)$. Thus, let $\lambda \in G$. Since $\lambda \in H_\Omega^{jk}(a)$, there is an $\omega \in \Omega$ such that $1 + \omega d_{kj}(\lambda) = 0$. Choose $\varrho > 0$ so that $T(a_\varrho - \lambda)$ is invertible, and let U and n_0 be as in Lemma 3.2. Due to Lemma 3.1,

$$f(\mu) := 1 + \omega d_{kj}(\mu) = 1 + \omega \varrho^{j-k} \left[T^{-1}(a_\varrho - \mu)\right]_{kj}$$

for $\mu \in U$. Lemma 3.2 shows that the functions f_n defined in U by

$$f_n(\mu) = 1 + \omega \varrho^{j-k} \left[T_n^{-1}(a_\varrho - \mu)\right]_{kj}$$

converge uniformly to f in U. Since f is not constant in U and is zero at $\lambda \in U$, a well known theorem by Hurwitz (see, e.g., [25, pp. 205 and 312]) implies that there are $\lambda_n \in U$ such that $\lambda_n \to \lambda$ and $f_n(\lambda_n) = 0$. Let $D_\varrho = \mathrm{diag}\,(1, \varrho, \ldots, \varrho^{n-1})$. It can be readily verified that

$$T_n(a_\varrho - \mu) = D_\varrho T_n(a - \mu) D_\varrho^{-1}, \tag{3.8}$$

whence

$$\left[T_n^{-1}(a_\varrho - \mu)\right]_{kj} = \varrho^{k-j}\left[T_n^{-1}(a-\mu)\right]_{kj} \tag{3.9}$$

and thus

$$0 = f_n(\lambda_n) = 1 + \omega\left[T_n^{-1}(a-\lambda_n)\right]_{kj}.$$

From (1.6) we now deduce that $\lambda_n \in \mathrm{sp}_\Omega^{(j,k)} T_n(a)$, and since $\lambda_n \to \lambda$, it follows that λ is in the left-hand side of (3.7).

We now show that

$$\limsup_{n\to\infty} \mathrm{sp}_\Omega^{(j,k)} T_n(a) \cap \overline{G} \subset \left(\Lambda(a) \cup H_\Omega^{jk}(a)\right) \cap \overline{G}. \tag{3.10}$$

Pick λ in the left-hand side of (3.10). If $\lambda \in \partial G \subset \Lambda(a)$, then λ is obviously in the right-hand side of (3.10). We can therefore assume that $\lambda \in G$. By the definition of the partial limiting set, there are $\lambda_{n_\ell} \in \mathrm{sp}_\Omega^{(j,k)} T_{n_\ell}(a) \cap G$ such that $\lambda_{n_\ell} \to \lambda$. Choose $\varrho > 0$ so that $T(a_\varrho - \lambda)$ is invertible. By Lemma 3.2, the matrices $T_{n_\ell}(a_\varrho - \lambda_{n_\ell})$ are invertible whenever n_ℓ is sufficiently large, and from (3.8) it then follows that the matrices $T_n(a - \lambda_{n_\ell})$ are also invertible for all n_ℓ large enough. Hence, taking into account that $\lambda_{n_\ell} \in \mathrm{sp}_\Omega^{(j,k)} T_{n_\ell}(a)$ and using (1.6), we see that there are $\omega_{n_\ell} \in \Omega$ such that $1 + \omega_{n_\ell}[T_{n_\ell}^{-1}(a - \lambda_{n_\ell})]_{kj} = 0$. Due to (3.9), this implies that

$$1 + \omega_{n_\ell}\varrho^{j-k}\left[T_{n_\ell}^{-1}(a_\varrho - \lambda_{n_\ell})\right]_{kj} = 0. \tag{3.11}$$

Since Ω is compact, the sequence $\{\omega_{n_\ell}\}$ has a partial limit ω in Ω. Consequently, (3.11) and Lemma 3.2 give

$$1 + \omega\varrho^{j-k}\left[T^{-1}(a_\varrho - \lambda)\right]_{kj} = 0,$$

and Lemma 3.1 now yields the equality $1 + \omega d_{kj}(\lambda) = 0$. It results that $d_{kj}(\lambda) \in -1/\Omega$ and thus that $\lambda \in H_\Omega^{jk}(a)$. □

Proof of Theorem 1.1. Take the union of equalities (3.6) over all components of $\mathbf{C} \setminus \Lambda(a)$. □

Proof of Theorem 1.3. Equality (1.11) is true for some $\varepsilon \in (0,\infty)$ if (and only if)

$$\lim_{n\to\infty} \mathrm{sp}_{\varepsilon\Omega}^{(j,k)} T_n(a) \cap \overline{G} = \left(\Lambda(a) \cup H_{\varepsilon\Omega}^{jk}(a)\right) \cap \overline{G} \tag{3.12}$$

for every connected component G of $\mathbf{C} \setminus \Lambda(a)$. We prove that (3.12) always holds for the unbounded component G and that for each bounded component G there is at most one $\varepsilon(G)$ for which (3.12) is not valid. This clearly implies Theorem 1.3.

Because $\|T^{-1}(a-\lambda)\| \to 0$ and thus $d_{kj}(\lambda) \to 0$ as $|\lambda| \to \infty$, the function d_{kj} cannot be a nonzero constant in the unbounded component of $\mathbf{C} \setminus \Lambda(a)$. From Theorem 3.3 we therefore obtain (3.12) in the case where G is the unbounded component.

Now let G be a bounded component. Then (3.12) holds by virtue of Theorem 3.3 provided d_{kj} is identically zero or not constant in G. Thus, suppose $d_{kj}(\lambda) = c \neq 0$ for all $\lambda \in G$. We have $c \in -1/\varepsilon\Omega$ if and only if $-1/c \in \varepsilon\Omega$. Obviously, if $\varepsilon > 0$ is sufficiently small, then $-1/c$ is not in $\varepsilon\Omega$. Define $\varepsilon(G) \in (0, +\infty]$ by $\varepsilon(G) = \sup\{\varepsilon > 0 : -1/c \notin \varepsilon\Omega\}$. If $\varepsilon < \varepsilon(G)$, then $c \notin -1/\varepsilon\Omega$, and hence (3.12) follows from Theorem 3.3. So let $\varepsilon > \varepsilon(G)$. Then $H^{jk}_{\varepsilon\Omega}(a) \cap G = G$, and we must show that

$$G \subset \liminf_{n\to\infty} \mathrm{sp}^{(j,k)}_{\varepsilon\Omega}\, T_n(a) \cap \overline{G}. \tag{3.13}$$

Pick $\lambda \in G$ and choose $\varrho > 0$ so that $T(a_\varrho - \lambda)$ is invertible. Since

$$\left[T_n^{-1}(a_\varrho - \lambda)\right]_{kj} \to \left[T^{-1}(a_\varrho - \lambda)\right]_{kj}$$

by Lemma 3.2, we see from (3.9) and Lemma 3.1 that

$$\left[T_n^{-1}(a - \lambda)\right]_{kj} \to d_{kj}(\lambda) = c.$$

Our assumptions on Ω and the definition of $\varepsilon(G)$ imply that c is an interior point of $-1/\varepsilon\Omega$. Consequently, $\left[T_n^{-1}(a - \lambda)\right]_{kj}$ belongs to $-1/\varepsilon\Omega$ for all sufficiently large n, which, by (1.6), means that $\lambda \in \mathrm{sp}^{(j,k)}_{\varepsilon\Omega}\, T_n(a)$ for all n large enough, thus completing the proof of (3.13). □

4. Small and large perturbations

This section is devoted to the proofs of Theorems 1.4 and 1.5.

Lemma 4.1. *If $a \in \mathcal{P}$ and $T(a)$ is not triangular, then there exists a constant $\delta > 1$, depending on a, such that*

$$\Lambda(a) = \bigcap_{\varrho\in[1/\delta,\delta]} \mathrm{sp}\, T(a_\varrho).$$

Proof. Assume that a is of the form (1.10) with $p \geq 1$ and $q \geq 1$. We have

$$a_\varrho(t) = a_q \varrho^q t^q \left(1 + \frac{a_{q-1}}{a_q}\frac{1}{\varrho t} + \cdots + \frac{a_{-p}}{a_q}\frac{1}{\varrho^{p+q} t^{p+q}}\right).$$

Hence, if ϱ is large enough, then, for all $\lambda \in \mathrm{sp}\, T(a)$, $a_\varrho - \lambda$ has no zeros on $\mathbf{T}$ and $\mathrm{wind}\,(a_\varrho, \lambda) = q \neq 0$. This implies there is a $\varrho_1 \in (1,\infty)$ such that $\mathrm{sp}\, T(a) \subset \mathrm{sp}\, T(a_\varrho)$ for all $\varrho > \varrho_1$. Analogously, from the representation

$$a_\varrho(t) = a_{-p}\varrho^{-p} t^{-p}\left(1 + \frac{a_{-p+1}}{a_{-p}}\varrho t + \cdots + \frac{a_q}{a_{-p}}\varrho^{p+q} t^{p+q}\right)$$

we infer that there exists a $\varrho_2 \in (0,1)$ such that $\mathrm{sp}\, T(a) \subset \mathrm{sp}\, T(a_\varrho)$ for all $\varrho < \varrho_2$. Letting $\delta := \max(\varrho_1, 1/\varrho_2)$ we get

$$\bigcap_{\varrho\notin[1/\delta,\delta]} \mathrm{sp}\, T(a_\varrho) \supset \mathrm{sp}\, T(a),$$

whence

$$\bigcap_{\varrho\in(0,\infty)} \mathrm{sp}\, T(a_\varrho) \supset \mathrm{sp}\, T(a) \cap \left[\bigcap_{\varrho\in[1/\delta,\delta]} \mathrm{sp}\, T(a_\varrho)\right] = \bigcap_{\varrho\in[1/\delta,\delta]} \mathrm{sp}\, T(a_\varrho).$$

□

Proof of Theorem 1.4. Since 0 is a point in Ω, the set $\Lambda(a)$ is contained in $\liminf_{n\to\infty} \mathrm{sp}^{(j,k)}_{\Omega}\, T_n(a)$. To get the reverse inclusion, it suffices to prove that

$$\limsup_{n\to\infty}\, \mathrm{sp}^{(j,k)}_{\varepsilon\mathbf{D}} T_n(a) \subset \Lambda(a) \tag{4.1}$$

whenever $\varepsilon > 0$ is sufficiently small. We show that

$$\sup_{\lambda \notin \Lambda(a)} |d_{kj}(\lambda)| < \infty. \tag{4.2}$$

Clearly, (4.2) implies that $H^{jk}_{\varepsilon\overline{\mathbf{D}}}(a) = \emptyset$ provided $\varepsilon > 0$ is small enough, say $\varepsilon < \delta_1$. Thus, if d_{kj} is identically zero or nowhere locally constant, then (4.1) is immediate from Theorem 1.1. If d_{kj} is constant in some component G of $\mathbf{C} \setminus \Lambda(a)$, then this constant is certainly not in $-1/\varepsilon\overline{\mathbf{D}}$ if $\varepsilon > 0$ is sufficiently small, $\varepsilon < \delta_2(G)$. Theorem 3.3 therefore yields (4.1) for all $\varepsilon > 0$ less than the minimum of δ_1 and all $\delta_2(G)$.

Again write a in the form (1.10) with $p \geq 1$ and $q \geq 1$. Pick $\lambda \in \mathbf{C} \setminus \Lambda(a)$. By Lemma 4.1, there is a $\varrho \in [1/\delta, \delta]$ such that $T(a_\varrho - \lambda)$ is invertible. From (3.5) we obtain

$$\begin{aligned} |d_{kj}(\lambda)| &= |a_q|^{-1} |c_k b_j + c_{k-1} b_{j-1} + \cdots| \\ &\leq |a_q|^{-1} \left(\sum_{\ell=0}^{k} |c_\ell|^2 \right)^{1/2} \left(\sum_{\ell=0}^{j} |b_\ell|^2 \right)^{1/2}. \end{aligned} \tag{4.3}$$

Taking into account (3.1), (3.3), and (3.4), we get

$$\begin{aligned} |b_\ell| &= \left| \sum_{\alpha_1 + \cdots + \alpha_p = \ell} z_1^{\alpha_1} \cdots z_p^{\alpha_p} \right| \\ &\leq (|z_1| + \cdots + |z_p|)^\ell \\ &\leq (p\varrho)^\ell \leq (p\delta)^\ell \leq (p\delta)^j \end{aligned}$$

and

$$\begin{aligned} |c_\ell| &= |z_{p+1} \cdots z_{p+q}|^{-1} \left| \sum_{\alpha_1 + \cdots + \alpha_q = \ell} z_{p+1}^{-\alpha_1} \cdots z_{p+q}^{-\alpha_q} \right| \\ &\leq \varrho^{-q} (|z_{p+1}|^{-1} + \cdots + |z_{p+q}|^{-1})^\ell \\ &\leq \varrho^{-q} (q\varrho^{-1})^\ell \leq \delta^q (q\delta)^\ell \leq \delta^q (q\delta)^k. \end{aligned}$$

Thus, (4.3) gives

$$|d_{kj}(\lambda)| \leq |a_q|^{-1} \delta^q \left((k+1)(q\delta)^{2k} \right)^{1/2} \left((j+1)(p\delta)^{2j} \right)^{1/2},$$

which clearly implies (4.2). □

It is well known that $d_{11}(\lambda) \neq 0$ for all $\lambda \in \mathbf{C} \setminus \Lambda(a)$. The function $d_{11}(\lambda)$ is usually denoted by $1/G(a - \lambda)$, and it is also well known (see, e.g., [7, Prop. 5.4]) that

$$G(a - \lambda) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log(a_\varrho(e^{i\theta}) - \lambda)\, d\theta \right), \tag{4.4}$$

where $\varrho > 0$ is any value for which $T(a_\varrho - \lambda)$ is invertible and where $e^{i\theta} \mapsto \log(a_\varrho(e^{i\theta}) - \lambda)$ is any continuous branch of the logarithm, which exists by virtue of (1.1). From Theorems 1.1 and 1.2 we obtain that if $a \in \mathcal{P}$ and $\Omega \subset \mathbf{C}$ is any compact set containing the origin, then

$$\lim_{n \to \infty} \mathrm{sp}_\Omega^{(1,1)}\, T_n(a) = \Lambda(a) \cup \{\lambda \notin \Lambda(a) : -G(a - \lambda) \in \Omega\}. \tag{4.5}$$

Proof of Theorem 1.5. By Theorem 1.2, $d_{11}(\lambda) = 1/G(a-\lambda)$ is nowhere locally constant. We therefore see from (1.7) and (1.8) that

$$\operatorname{sp}_\Omega^{(1,1)} T(a) = \operatorname{sp} T(a) \cup \{\lambda \notin \operatorname{sp} T(a) : -G(a-\lambda) \in \Omega\}. \tag{4.6}$$

From (4.5) we know that

$$\lim_{n\to\infty} \operatorname{sp}_\Omega^{(1,1)} T_n(a) = \Lambda(a) \cup \{\lambda \notin \Lambda(a) : -G(a-\lambda) \in \Omega\}. \tag{4.7}$$

Let $\lambda \in \operatorname{sp} T(a) \setminus \Lambda(a)$. By Lemma 4.1, there is a $\varrho \in [1/\delta, \delta]$ such that $T(a_\varrho - \lambda)$ is invertible. From (4.4) we obtain

$$\begin{aligned} |G(a-\lambda)| &= \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|a_\varrho(e^{i\theta}) - \lambda|\, d\theta\right) \\ &\le \exp(\log\|a_\varrho - \lambda\|_\infty) = \|a_\varrho - \lambda\|_\infty \\ &\le \max_{\varrho\in[1/\delta,\delta]} \max_{\lambda\in\operatorname{sp} T(a)} \|a_\varrho - \lambda\|_\infty =: M < \infty. \end{aligned} \tag{4.8}$$

We claim that (1.14) is true with $\varepsilon_2 = M + 1$.

Suppose λ is not in (4.7). Then $\lambda \notin \Lambda(a)$ and $-G(a-\lambda) \notin \Omega$. If λ were in $\operatorname{sp} T(a)$, then (4.8) would imply that $|G(a-\lambda)| \le M$, which is impossible, because $-G(a-\lambda) \notin \Omega$ and thus $|G(a-\lambda)| > M+1$. Hence $\lambda \notin \operatorname{sp} T(a)$ and $-G(a-\lambda) \notin \Omega$, which shows that λ does not belong to (4.6).

Conversely, suppose λ is not a point in (4.6). Then $\lambda \notin \Lambda(a)$ (recall that $\Lambda(a) \subset \operatorname{sp} T(a)$) and $-G(a-\lambda) \notin \Omega$. Consequently, λ is not in (4.7). □

Even the extension of only part of Theorem 1.5 to the case $(j,k) \neq (1,1)$ is a delicate problem. We note that always

$$\limsup_{n\to\infty} \operatorname{sp}_\Omega^{(j,k)} T_n(a) \subset \operatorname{sp}_\Omega^{(j,k)} T(a),$$

but lower estimates are more difficult to obtain.

5. An example

We now consider a particular example involving tridiagonal Toeplitz matrices. Our analysis includes a class of test matrices proposed and studied by Gear [11]: matrices with the symbol $a(t) = t + t^{-1}$ and perturbations of ± 1 in a single entry of the first or last row. Here we impose no requirements concerning symmetry, magnitude of the perturbation, or the location of the perturbation in the matrix. We obtain formulas that lead to the pictures shown in §2, and show that Theorem 1.5 is in general no longer true when $(1,1)$ is replaced by $(j,k) \neq (1,1)$.

Let $a(t) = t + \alpha^2 t^{-1}$ with $\alpha \in (0,1)$. The set $a(\mathbf{T})$ is the ellipse

$$\left\{x + iy \in \mathbf{C} : \frac{x^2}{(1+\alpha^2)^2} + \frac{y^2}{(1-\alpha^2)^2} = 1\right\}.$$

Let E_+ and E_- denote the sets of points inside and outside this ellipse, respectively. From (1.1) we see that $\operatorname{sp} T(a) = a(\mathbf{T}) \cup E_+$. For $\varrho > 0$, put $a_\varrho(t) = \varrho t + \alpha^2 \varrho^{-1} t^{-1}$. Thus, $a_\varrho(\mathbf{T})$ is the ellipse

$$\left\{x + iy \in \mathbf{C} : \frac{x^2}{(\varrho + \alpha^2\varrho^{-1})^2} + \frac{y^2}{(\varrho - \alpha^2\varrho^{-1})^2} = 1\right\}.$$

It is readily verified that

$$\begin{aligned} &a_\varrho(\mathbf{T}) \subset E_- \text{ if } 0 < \varrho < \alpha^2 \text{ or } \varrho > 1, \\ &a_\varrho(\mathbf{T}) \subset E_+ \text{ if } \alpha^2 < \varrho < 1, \\ &a_\varrho(\mathbf{T}) = [-2\alpha, 2\alpha] \text{ if } \varrho = \alpha, \end{aligned}$$

whence $\Lambda(a) = [-2\alpha, 2\alpha]$, the line segment between the two foci of the ellipse $a(\mathbf{T})$.

Let $\lambda \in \mathbf{C} \setminus [-2\alpha, 2\alpha]$. We have

$$a(t) - \lambda = t^{-1}(t - z_1(\lambda))(t - z_2(\lambda))$$

with

$$z_1(\lambda) = \frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} - \alpha^2}, \quad z_2(\lambda) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - \alpha^2}. \tag{5.1}$$

Here we denote by $\sqrt{\lambda^2/4 - \alpha^2}$ the branch of the function which is analytic in $\mathbf{C}\setminus[-2\alpha, 2\alpha]$ and asymptotically equal to $\lambda/2$ as $\lambda \to \infty$. We claim that the labeling of the roots $z_1(\lambda)$ and $z_2(\lambda)$ in (5.1) agrees with (3.1); i.e., $|z_1(\lambda)| < |z_2(\lambda)|$. Every point $\lambda \in \mathbf{C} \setminus [-2\alpha, 2\alpha]$ is located on exactly one of the ellipses $a_\sigma(\mathbf{T})$ with $\sigma > \alpha$ and can therefore be uniquely written in the form

$$\lambda = \sigma e^{i\theta} + \alpha^2 \sigma^{-1} e^{-i\theta}, \quad \sigma \in (\alpha, \infty), \quad \theta \in [0, 2\pi). \tag{5.2}$$

For λ in the form (5.2) we get

$$\pm\sqrt{\lambda^2/4 - \alpha^2} = \pm\frac{1}{2}\left(\sigma e^{i\theta} - \alpha^2\sigma^{-1}e^{-i\theta}\right),$$

which shows that

$$z_1(\lambda) = \frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} - \alpha^2} = \alpha^2\sigma^{-1}e^{-i\theta}, \quad z_2(\lambda) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - \alpha^2} = \sigma e^{i\theta},$$

whence $|z_1(\lambda)| = \alpha^2\sigma^{-1} < \sigma = |z_2(\lambda)|$, as claimed.

Given λ in the form (5.2), choose $\varrho \in (\alpha^2\sigma^{-1}, \sigma)$. Then $T(a_\varrho - \lambda)$ is invertible and the functions (3.3) and (3.4) assume the form

$$\begin{aligned} \varphi_-^{-1}(t) &= 1 + \frac{z_1(\lambda)}{\varrho t} + \frac{z_1^2(\lambda)}{\varrho^2 t^2} + \cdots \\ &= 1 + \frac{\alpha^2}{\varrho\sigma e^{i\theta}}\frac{1}{t} + \frac{\alpha^4}{\varrho^2\sigma^2 e^{2i\theta}}\frac{1}{t^2} + \cdots, \\ \varphi_+^{-1}(t) &= -\frac{1}{z_2(\lambda)}\left(1 + \frac{\varrho t}{z_2(\lambda)} + \frac{\varrho^2 t^2}{z_2^2(\lambda)} + \cdots\right) \\ &= -\frac{1}{\sigma e^{i\theta}}\left(1 + \frac{\varrho t}{\sigma e^{i\theta}} + \frac{\varrho^2 t^2}{\sigma^2 e^{2i\theta}} + \cdots\right). \end{aligned}$$

From (3.5) we now get in particular

$$d_{11}(\lambda) = -\frac{1}{\sigma e^{i\theta}}, \quad d_{12}(\lambda) = -\frac{\alpha^2}{\sigma^2 e^{2i\theta}}, \tag{5.3}$$

$$d_{21}(\lambda) = -\frac{1}{\sigma^2 e^{2i\theta}}, \quad d_{22}(\lambda) = -\frac{1}{\sigma e^{i\theta}}\left(\frac{\alpha^2}{\sigma^2 e^{2i\theta}} + 1\right). \tag{5.4}$$

Formulas (5.3) and (5.4) and their analogs for general $d_{kj}(\lambda)$ can be used to compute the sets $H_\Omega^{jk}(a)$. In fact, Figure 1, the top and middle pictures of Figures 2 and 3, the top pictures of Figures 4 and 5, and the boundaries of the regions in Figure 6 were obtained via numerical implementations of this procedure.

Let $\Omega = \varepsilon\overline{\mathbf{D}}$. We claim that

$$\lim_{n\to\infty} \mathrm{sp}_\Omega^{(2,2)} T_n(a) \subsetneqq \mathrm{sp}_\Omega^{(2,2)} T(a). \tag{5.5}$$

By the definition of $H_\Omega^{jk}(a)$,

$$H_\Omega^{22}(a) = \{\lambda \notin [-2\alpha, 2\alpha] : |d_{22}(\lambda)| \ge 1/\varepsilon\}.$$

Theorem 1.1, in conjunction with Lemma 3.1, gives

$$\lim_{n\to\infty} \mathrm{sp}_\Omega^{(2,2)} T_n(a) = [-2\alpha, 2\alpha] \cup H_\Omega^{22}(a). \tag{5.6}$$

From (1.8) we deduce that

$$\mathrm{sp}_\Omega^{(2,2)} T(a) = \mathrm{sp}\, T(a) \cup \{\lambda \notin \mathrm{sp}\, T(a) : |d_{22}(\lambda)| \ge 1/\varepsilon\}. \tag{5.7}$$

Clearly, (5.7) contains a sufficiently small disk $\delta\mathbf{D}$. We show that no such disk is contained in (5.6), which will imply the claim (5.5). To prove this, take λ of the form (5.2) with $\theta = \pi/2$. Then

$$|\lambda| = |\sigma - \alpha^2\sigma^{-1}| \to 0 \ \text{ as } \ \sigma \to \alpha + 0$$

and hence $\lambda \in \delta\mathbf{D}$ whenever $\sigma > \alpha$ is sufficiently close to α. On the other hand, the expression for $d_{22}(\lambda)$ in (5.4) yields

$$|d_{22}(\lambda)| = \frac{1}{\sigma}\left(1 - \frac{\alpha^2}{\sigma^2}\right) \to 0 \ \text{ as } \ \sigma \to \alpha + 0,$$

which shows that for any $\varepsilon > 0$, it is possible to take δ and σ sufficiently small that $|d_{22}(\lambda)| < 1/\varepsilon$, and thus λ is not in (5.6), though it is in (5.7).

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