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PIECEWISE QUADRATIC TRIGONOMETRIC POLYNOMIAL CURVES

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ABSTRACT. Analogous to the quadratic B-spline curve, a piecewise quadratic trigonometric polynomial curve is presented in this paper. The quadratic trigonometric polynomial curve has C^2 continuity, while the quadratic B-spline curve has C^1 continuity. The quadratic trigonometric polynomial curve is closer to the given control polygon than the quadratic B-spline curve.

1. INTRODUCTION

It is well-known that polynomial B-splines, particular the quadratic and cubic B-splines, have gained widespread application. The purpose of this paper is to present a practical quadratic trigonometric polynomial curve analogous to the quadratic B-spline curve.

The trigonometric B-splines were introduced in [8], and the recurrence relation for the trigonometric B-splines of arbitrary order was established in [5]. It was further shown in [10] that the trigonometric B-splines of odd order form a partition of a constant in the case of equidistant knots, and therefore the corresponding trigonometric B-spline curve possesses the convex hull property. There are other papers in which the trigonometric B-splines and polynomials have been studied, see references therein. In this paper, a different quadratic trigonometric polynomial curve is presented. The curve has C^2 continuity with a nonuniform knot vector.

This paper is organized as follows. In section 2, the basis functions of the quadratic trigonometric polynomial curves are described and the properties of the basis functions are shown. In section 3, the quadratic trigonometric polynomial curve is given. It is shown in section 4 that the quadratic trigonometric polynomial curve is closer to the given control polygon than the quadratic B-spline curve.

2. QUADRATIC TRIGONOMETRIC BASIS FUNCTIONS

2.1. Construction of the basis functions.

Definition 1. Given knots $u_0 < u_1 < \cdots < u_{n+3}$, let $\Delta u_i = u_{i+1} - u_i$,

$$\alpha_i = \frac{1}{3}\frac{\Delta u_i}{\Delta u_{i-1} + \Delta u_i}, \quad \beta_i = \frac{1}{3}\frac{\Delta u_i}{\Delta u_i + \Delta u_{i+1}}, \quad t_i(u) = \frac{\pi}{2}\frac{u - u_i}{\Delta u_i},$$

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$$c(t) = (1 - \sin t)(1 - \sin t + 2\cos t),$$
$$d(t) = (1 - \cos t)(1 - \cos t + 2\sin t).$$

Then the associated normalized quadratic trigonometric basis functions are defined to be the following functions:

$$b_i(u) = \begin{cases} \beta_i d(t_i), & u \in [u_i, u_{i+1}), \\ 1 - \alpha_{i+1}c(t_{i+1}) - \beta_{i+1}d(t_{i+1}), & u \in [u_{i+1}, u_{i+2}), \\ \alpha_{i+2}c(t_{i+2}), & u \in [u_{i+2}, u_{i+3}), \\ 0, & u \notin [u_i, u_{i+3}), \end{cases} \tag{1}$$

for $i = 0, 1, \ldots, n$.

Theorem 1. *The basis functions have the following properties:*

a) $b_i(u) > 0$, *for* $u_i < u < u_{i+3}$,
b) $b_i(u) = 0$, *for* $u_0 \leq u \leq u_i$, $u_{i+3} \leq u \leq u_{n+3}$,
c) $\sum_{i=0}^{n} b_i(u) = 1$, $u \in [u_2, u_{n+1}]$.

Proof. Let $\gamma_{i+1} = \max\{\alpha_{i+1}, \beta_{i+1}\}$. Then

$$\begin{aligned} \alpha_{i+1}c(t_{i+1}) + \beta_{i+1}d(t_{i+1}) &\leq \gamma_{i+1}(c(t_{i+1}) + d(t_{i+1})) = \gamma_{i+1}(3 - 2\sin 2t_{i+1}) \\ &\leq 3\gamma_{i+1} < 1. \end{aligned}$$

Thus $b_i(u) > 0$, for $u \in [u_{i+1}, u_{i+2})$.

For $u \in [u_i, u_{i+1})$, $i = 2, 3, \ldots, n$, let

$$b_{i0}(t_i) = \alpha_i c(t_i), \tag{2}$$
$$b_{i1}(t_i) = 1 - \alpha_i c(t_i) - \beta_i d(t_i), \tag{3}$$
$$b_{i2}(t_i) = \beta_i d(t_i). \tag{4}$$

Then $b_{i-2}(u) = b_{i0}(t_i)$, $b_{i-1}(u) = b_{i1}(t_i)$, $b_i(u) = b_{i2}(t_i)$, $b_j(u) = 0\,(j \neq i-2, i-1, i)$, and

$$\sum_{j=0}^{n} b_j(u) = b_{i0}(t_i) + b_{i1}(t_i) + b_{i2}(t_i) = 1.$$

The remaining cases follow obviously. □

Remark. In view of the properties a)–c), we say that the basis functions form a partition of unity and the function $b_i(u)$ has a support on the interval $[u_i, u_{i+3}]$.

For equidistant knots, we refer to the $b_i(u)$ as a uniform basis function. Figure 1 shows the graph of three uniform basis functions. The basis functions defined over nonequidistant knots are called nonuniform basis functions.

2.2. **Continuity of the basis functions.** It is known that the quadratic B-splines have C^1 continuity. The following theorem shows the continuity of the quadratic trigonometric basis functions.

Theorem 2. *The quadratic trigonometric basis function* $b_i(u)$ *has* C^2 *continuity at each of the knots.*

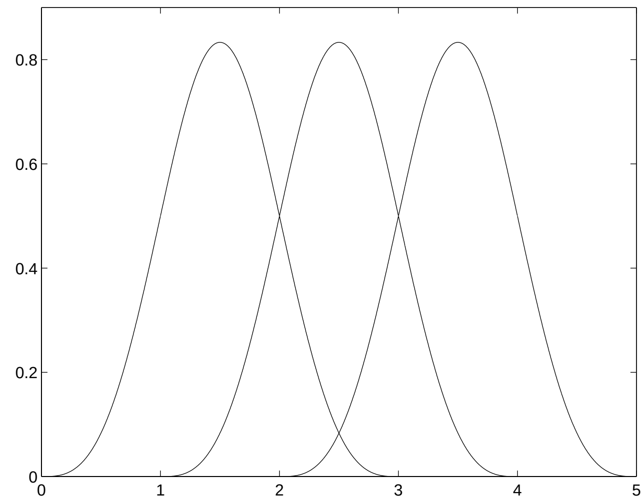

FIGURE 1. Uniform basis functions.

Proof. Consider the continuity at the knots u_{i+1} and u_{i+2} (We can deal with the knots u_i and u_{i+3} in the same way). Since

$$c(t) = \frac{3}{2} - 2\sin t + 2\cos t - \sin 2t - \frac{1}{2}\cos 2t,$$
$$d(t) = \frac{3}{2} + 2\sin t - 2\cos t - \sin 2t + \frac{1}{2}\cos 2t,$$

straightforward computation gives

$$\begin{aligned}
&b_i(u_{i+1}^-) = 3\beta_i, && b_i(u_{i+1}^+) = 1 - 3\alpha_{i+1},\\
&b_i(u_{i+2}^-) = 1 - 3\beta_{i+1}, && b_i(u_{i+2}^+) = 3\alpha_{i+2},\\
&b_i'(u_{i+1}^-) = \frac{2\pi\beta_i}{\Delta u_i}, && b_i'(u_{i+1}^+) = \frac{2\pi\alpha_{i+1}}{\Delta u_{i+1}},\\
&b_i'(u_{i+2}^-) = -\frac{2\pi\beta_{i+1}}{\Delta u_{i+1}}, && b_i'(u_{i+2}^+) = -\frac{2\pi\alpha_{i+2}}{\Delta u_{i+2}},\\
&b_i''(u_{i+1}^-) = b_i''(u_{i+1}^+) = 0, &&\\
&b_i''(u_{i+2}^-) = b_i''(u_{i+2}^+) = 0. &&
\end{aligned}$$

Since

$$3\alpha_{j+1} = 1 - 3\beta_j = \frac{\Delta u_{j+1}}{\Delta u_j + \Delta u_{j+1}},$$
$$\frac{\alpha_{j+1}}{\Delta u_{j+1}} = \frac{\beta_j}{\Delta u_j} = \frac{1}{3}\frac{1}{\Delta u_j + \Delta u_{j+1}},$$

we have

$$b_i^{(k)}(u_{i+1}^-) = b_i^{(k)}(u_{i+1}^+), \quad b_i^{(k)}(u_{i+2}^-) = b_i^{(k)}(u_{i+2}^+), \quad k = 0, 1, 2.$$

The theorem follows. □

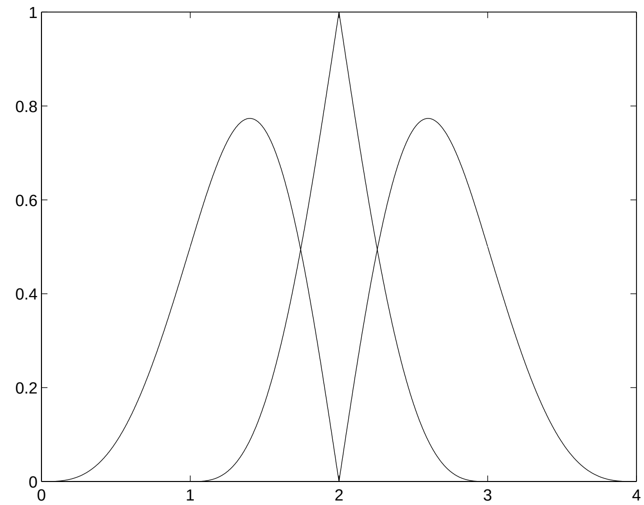

FIGURE 2. The basis functions with a double knot.

2.3. **The case of multiple knots.** So far in the discussion of the basis functions, we have assumed that each point is simple. On the other hand, the basis functions also make sense when knots are considered with multiplicity $k \leq 3$. For trigonometric basis functions with multiple knots, it is worth noting that we shrink the corresponding intervals to zero and drop the corresponding pieces. For example, if $u_{i+1} = u_{i+2}$ is a double knot, then we define

$$b_i(u) = \begin{cases} \beta_i d(t_i), & u \in [u_i, u_{i+1}), \\ \alpha_{i+2} c(t_{i+2}), & u \in [u_{i+2}, u_{i+3}), \\ 0, & u \notin [u_i, u_{i+3}). \end{cases}$$

Theorem 3. *Suppose that a basis function has a knot of multiplicity $k = 2$ or 3 at a parameter value u. Then at this point the continuity of the basis function is reduced from C^2 to C^{2-k}(C^{-1} means discontinuous). Moreover, the support interval of the basis function is reduced from 3 segments to $4 - k$ segments.*

Proof. This is a direct application of (1) and the expressions given in the proof of Theorem 2. □

Theorem 3 deals with the geometric significance of multiple knots. For applications, it is important to use multiple knots. Figure 2 shows the trigonometric basis functions with a double knot.

3. QUADRATIC TRIGONOMETRIC POLYNOMIAL CURVES

Definition 2. Given points $P_i\,(i = 0, 1, \ldots, n)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and a knot vector $U = (u_0, u_1, \ldots, u_{n+3})$, we call

$$T(u) = \sum_{j=0}^{n} b_j(u) P_j, \qquad n \geq 2,\ u \in [u_2, u_{n+1}], \tag{5}$$

a quadratic trigonometric polynomial curve.

Remark. If $u_i \neq u_{i+1}\,(2 \leq i \leq n)$, then for $u \in [u_i, u_{i+1}]$, the curve $T(u)$ can be represented by the curve segment

$$T_i(t_i) = b_{i0}(t_i)P_{i-2} + b_{i1}(t_i)P_{i-1} + b_{i2}(t_i)P_i, \tag{6}$$

where $b_{i0}(t_i)$, $b_{i1}(t_i)$, $b_{i2}(t_i)$ are given by (2), (3), (4), respectively.

Analogously to the quadratic B-spline curve, the choice of the knot vector automatically determines the continuity of the quadratic trigonometric polynomial curve at each of the knots, as shown by the following theorem.

Theorem 4. *If a knot u_i has multiplicity $k = 1, 2,$ or 3, then the quadratic trigonometric polynomial curve has C^2 continuity for $k = 1$ and C^{2-k} continuity for $k = 2$ or 3 at u_i. Moreover,*

$$\begin{aligned} T(u_i^-) &= (1 - 3\beta_{i-1})P_{i-2} + 3\beta_{i-1}P_{i-1}, \\ T(u_i^+) &= 3\alpha_i P_{i-2} + (1 - 3\alpha_i)P_{i-1}, \\ T'(u_i^-) &= \frac{2\pi\beta_{i-1}}{\Delta u_{i-1}}(P_{i-1} - P_{i-2}), \\ T'(u_i^+) &= \frac{2\pi\alpha_i}{\Delta u_i}(P_{i-1} - P_{i-2}), \\ T''(u_i^-) &= T''(u_i^+) = 0. \end{aligned}$$

Proof. This is a direct consequence of Theorems 2 and 3. □

Remark. Since $T''(u_i^-) = T''(u_i^+) = 0$, we can deduce that the curvatures of the quadratic trigonometric polynomial curve at the interior knots are zero.

Remark. The choice of the first and last two knots is free, and these knots can be adjusted to give the desired boundary behavior of the curve. See the following descriptions.

For an open trigonometric polynomial curve, we choose the knot vector

$$U = (u_0 = u_1 = u_2, u_3, \ldots, u_n, u_{n+1} = u_{n+2} = u_{n+3}).$$

This assures that P_0 and P_n are points on the curve. Of course, the interior knots can be multiple knots. In addition, since

$$\begin{aligned} T'(u_2^+) &= \frac{2\pi}{3\Delta u_2}(P_1 - P_0), \\ T'(u_{n+1}^-) &= \frac{2\pi}{3\Delta u_n}(P_n - P_{n-1}), \end{aligned}$$

the edges P_0P_1 and $P_{n-1}P_n$ of the control polygon are tangent to the curve. (If one or more neighboring knots all fall at P_0, then the tangent at P_0 corresponds to the edge of the control polygon passing through P_0 and the first distinct neighbor of P_0.).

Figure 3 shows an open quadratic trigonometric curve (solid lines) and a quadratic B-spline curve (dashed lines) corresponding to the same control polygon for a nonuniform knot vector $U = (0, 0, 0, 0.5, 1.5, 2, 3, 4, 4, 4)$.

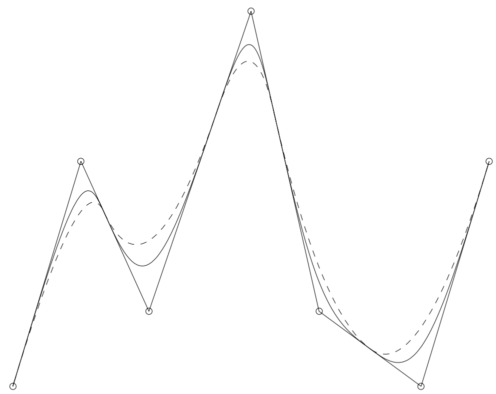

FIGURE 3. Open curves with a nonuniform knot vector.

In order to construct a closed trigonometric curve, we can extend the given points $P_0, P_1, \ldots, P_n$ by setting $P_{n+1} = P_0$, $P_{n+2} = P_1$ and expand the knot vector by setting $u_{n+4} = u_{n+3} + \Delta u_{n+2}\Delta u_2/\Delta u_1$ (so that $T(u_2) = T(u_{n+3})$) and $u_{n+5} \leq u_{n+4}$. Thus the parametric formula for a closed quadratic trigonometric curve is

$$T(u) = \sum_{j=0}^{n+2} b_j(u)P_j, \quad \text{with } u \in [u_2, u_{n+3}],$$

where b_{n+1} and b_{n+2} are given by expanding (1).

The parametric formula for a closed quadratic trigonometric curve also can be given by

$$T(u) = \sum_{j=0}^{n} b_j(u)P_j, \quad \text{with } u \in [u_0, u_{n+1}];$$

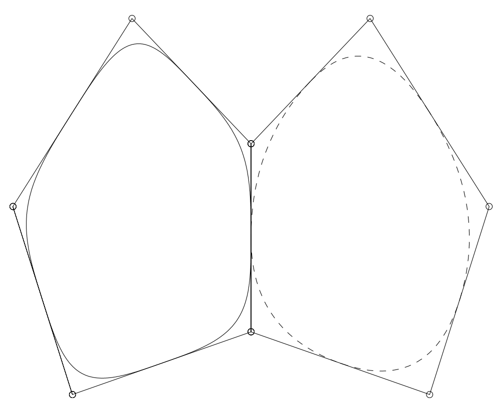

FIGURE 4. Closed curves with a uniform knot vector.

here the changes from Definition 1 are to let $\alpha_0 = \alpha_{n+1}$ (so that $T(u_0) = T(u_{n+1})$) and

$$b_{n-1}(u) = \begin{cases} \beta_{n-1}d(t_{n-1}), & u \in [u_{n-1}, u_n), \\ 1 - \alpha_n c(t_n) - \beta_n d(t_n), & u \in [u_n, u_{n+1}], \\ \alpha_0 c(t_0), & u \in [u_0, u_1), \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

$$b_n(u) = \begin{cases} \beta_n d(t_n), & u \in [u_n, u_{n+1}], \\ 1 - \alpha_0 c(t_0) - \beta_0 d(t_0), & u \in [u_0, u_1), \\ \alpha_1 c(t_1), & u \in [u_1, u_2), \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

Figure 4 shows a closed quadratic trigonometric curve (solid lines) and a quadratic B-spline curve (dashed lines) corresponding to the same control polygon for a uniform knot vector.

4. Approximability

Control polygons provide an important tool in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon.

From (6), we have

$$T_i(t_i) = P_{i-1} + b_{i0}(t_i)(P_{i-2} - P_{i-1}) + b_{i2}(t_i)(P_i - P_{i-1}).$$

Therefore, the point $T_i(t_i)$ (for fixed $u \in (u_i, u_{i+1})$) converges to the point P_{i-1} as $\Delta u_i \to 0$ for fixed Δu_{i-1} and Δu_{i+1}. The curve segment $T_i(t_i)$ ($u \in [u_i, u_{i+1}]$) tends to be merged with the line segments $T_i(0)P_{i-1}$ and $P_{i-1}T_i(\pi/2)$.

From Figures 3 and 4, we can see that the quadratic trigonometric curve is closer to the given control polygon than the quadratic B-spline curve. This fact is proved as follows.

Lemma. *For $t \in [0, \pi/2]$, let*

$$f(t) = c(t) + d(t) + 2\sqrt{c(t)d(t)}.$$

Then $2 \le f(t) \le 3$, $f(t) = 2$ if and only if $t = \pi/4$, and $f(t) = 3$ if and only if $t = 0$ or $\pi/2$.

Proof. Let $x = \sin t + \cos t - 1$; then $x \in [0, \sqrt{2} - 1]$, $c(t) + d(t) = 3 - 2\sin 2t = 3 - 2x(x + 2)$, $(1 - \sin t)(1 - \cos t) = x^2/2$, $(1 - \sin t + 2\cos t)(1 - \cos t + 2\sin t) = (5x^2 + 12x)/2$ and

$$f(t) = 3 - x(2x + 4 - \sqrt{5x^2 + 12x}).$$

While $x \in [0, \sqrt{2} - 1]$, we have $\sqrt{5x^2 + 12x} < 2x + 4$, and then $f(t) \le 3$; $f(t) = 3$ if and only if $x = 0$.

On the other hand, while $1 - 4x - 2x^2 \ge 0$, i.e., $x \in [0, \sqrt{6}/2 - 1]$, we have $f(t) - 2 = 1 - 4x - 2x^2 + x\sqrt{5x^2 + 12x} > 0$. While $x \in (\sqrt{6}/2 - 1, \sqrt{2} - 1]$, since

$$x^4 - 4x^3 - 12x^2 + 8x - 1 = (x^2 + 2x - 1)(x^2 - 6x + 1) \ge 0,$$

we have $x\sqrt{5x^2 + 12x} \ge 2x^2 + 4x - 1$, and then $f(t) \ge 2$; $f(t) = 2$ if and only if $x = \sqrt{2} - 1$. □

Let there be given data points $P_i \in \mathbb{R}^2$ or $\mathbb{R}^3\,(i = 0, 1, \ldots, n)$ and knots $u_0 < u_1 < \cdots < u_{n+3}$. For $u \in [u_k, u_{k+1}]$, the associated quadratic B-spline curve can be given by

$$B_k(s) = a_{k0}(s)P_{k-2} + a_{k1}(s)P_{k-1} + a_{k2}(s)P_k, \tag{9}$$

where $s = (u - u_k)/\Delta u_k$ and

$$\begin{aligned} a_{k0}(s) &= 3\alpha_k(1 - s)^2, \\ a_{k1}(s) &= 1 - 3\alpha_k(1 - s)^2 - 3\beta_k s^2, \\ a_{k2}(s) &= 3\beta_k s^2. \end{aligned}$$

Theorem 5. *Suppose P_{k-2}, P_{k-1}, P_k are not collinear. For given $0 \le \lambda \le 1$, let $Q_{k-1} = \lambda P_{k-2} + (1 - \lambda)P_k$, $B_k(\hat{s})$ and $T_k(\hat{t})$ be the intersection points of line segment $P_{k-1}Q_{k-1}$ with the curves $B_k(s)$ and $T_k(t)$ respectively for $u \in [u_k, u_{k+1}]$. Then*

$$\frac{2}{3}\|B_k(\hat{s}) - P_{k-1}\| \le \|T_k(\hat{t}) - P_{k-1}\| \le \|B_k(\hat{s}) - P_{k-1}\|. \tag{10}$$

Moreover, $2\|B_k(\hat{s}) - P_{k-1}\| = 3\|T_k(\hat{t}) - P_{k-1}\|$ if and only if $u = (u_k + u_{k+1})/2$, while $\|B_k(\hat{s}) - P_{k-1}\| = \|T_k(\hat{t}) - P_{k-1}\|$ if and only if $u = u_k$ or $u = u_{k+1}$.

Proof. We can write (9) and (6) as

$$\begin{aligned} B_k(s) &= P_{k-1} + (a_{k0}(s) + a_{k2}(s))\Big(\frac{a_{k0}(s)P_{k-2} + a_{k2}(s)P_k}{a_{k0}(s) + a_{k2}(s)} - P_{k-1}\Big), \\ T_k(t) &= P_{k-1} + (b_{k0}(t) + b_{k2}(t))\Big(\frac{b_{k0}(t)P_{k-2} + b_{k2}(t)P_k}{b_{k0}(t) + b_{k2}(t)} - P_{k-1}\Big), \end{aligned}$$

where

$$(a_{k0}(s)P_{k-2} + a_{k2}(s)P_k)/(a_{k0}(s) + a_{k2}(s))$$

and

$$(b_{k0}(t)P_{k-2} + b_{k2}(t)P_k)/(b_{k0}(t) + b_{k2}(t))$$

are points on the line segment $P_{k-2}P_k$.

Since $a_{k0}(s)/(a_{k0}(s) + a_{k2}(s))$ and $b_{k0}(t)/(b_{k0}(t) + b_{k2}(t))$ are monotone for $s \in [0, 1]$ and $t \in [0, \pi/2]$ respectively, there exist unique $\hat{s}$ and $\hat{t}$ such that

$$\frac{a_{k0}(\hat{s})}{a_{k0}(\hat{s}) + a_{k2}(\hat{s})} = \frac{b_{k0}(\hat{t})}{b_{k0}(\hat{t}) + b_{k2}(\hat{t})} = \lambda.$$

Thus, we get $a_{k0}(\hat{s})b_{k2}(\hat{t}) = a_{k2}(\hat{s})b_{k0}(\hat{t})$, and then $\hat{s}^2c(\hat{t}) = (1 - \hat{s})^2d(\hat{t})$, and also

$$c(\hat{t}) = (1 - \hat{s})^2 f(\hat{t}), \quad d(\hat{t}) = \hat{s}^2 f(\hat{t}),$$

$$b_{k0}(\hat{t}) + b_{k2}(\hat{t}) = \frac{1}{3}(a_{k0}(\hat{s}) + b_{k2}(\hat{s}))f(\hat{t}).$$

Therefore

$$T_k(\hat{t}) - P_{k-1} = \frac{1}{3}f(\hat{t})(B_k(\hat{s}) - P_{k-1}).$$

By the lemma, we have

$$\frac{2}{3}\|B_k(\hat{s}) - P_{k-1}\| \le \|T_k(\hat{t}) - P_{k-1}\| \le \|B_k(\hat{s}) - P_{k-1}\|,$$

$2(\|T_k(\hat{t}) - P_{k-1}\|)/3 = \|B_k(\hat{s}) - P_{k-1}\|$ if and only if $\hat{s} = 1/2\,(\hat{t} = \pi/4)$, while $\|T_k(\hat{t}) - P_{k-1}\| = \|B_k(\hat{s}) - P_{k-1}\|$ if and only if $\hat{s} = 0, 1\,(\hat{t} = 0, \pi/2)$. □

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