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MONIC INTEGER CHEBYSHEV PROBLEM

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ABSTRACT. We study the problem of minimizing the supremum norm by monic polynomials with integer coefficients. Let $\mathcal{M}_n(\mathbb{Z})$ denote the monic polynomials of degree n with integer coefficients. A *monic integer Chebyshev polynomial* $M_n \in \mathcal{M}_n(\mathbb{Z})$ satisfies

$$\|M_n\|_E = \inf_{P_n \in \mathcal{M}_n(\mathbb{Z})} \|P_n\|_E.$$

and the *monic integer Chebyshev constant* is then defined by

$$t_M(E) := \lim_{n\to\infty} \|M_n\|_E^{1/n}.$$

This is the obvious analogue of the more usual *integer Chebyshev constant* that has been much studied.

We compute $t_M(E)$ for various sets, including all finite sets of rationals, and make the following conjecture, which we prove in many cases.

Conjecture. *Suppose $[a_2/b_2, a_1/b_1]$ is an interval whose endpoints are consecutive Farey fractions. This is characterized by $a_1b_2 - a_2b_1 = 1$. Then*

$$t_M[a_2/b_2, a_1/b_1] = \max(1/b_1, 1/b_2).$$

This should be contrasted with the nonmonic integer Chebyshev constant case, where the only intervals for which the constant is exactly computed are intervals of length 4 or greater.

1. Introduction and general results

Define the uniform (sup) norm on a compact set $E \subset \mathbb{C}$ by

$$\|f\|_E := \sup_{z\in E} |f(z)|.$$

We study the monic polynomials with integer coefficients that minimize the sup norm on the set E. Let $\mathcal{P}_n(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{Z})$ be the classes of algebraic polynomials of degree at most n, respectively with complex and with integer coefficients. Similarly, we define the classes of monic polynomials $\mathcal{M}_n(\mathbb{C})$ and $\mathcal{M}_n(\mathbb{Z})$ of *exact* degree $n \in \mathbb{N}$. The problem of minimizing the uniform norm on E by polynomials from $\mathcal{M}_n(\mathbb{C})$ is well known as the Chebyshev problem (see [1], [15], [17], [8], etc.). In the classical case, $E = [-1, 1]$, the explicit solution of this problem is given by the

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monic Chebyshev polynomial of degree n:

$$T_n(x) := 2^{1-n}\cos(n \arccos x), \quad n \in \mathbb{N}.$$

Using a change of variable, we can immediately extend this to an arbitrary interval $[a, b] \subset \mathbb{R}$, so that

$$t_n(x) := \left(\frac{b-a}{2}\right)^n T_n\left(\frac{2x-a-b}{b-a}\right)$$

is a monic polynomial with real coefficients and the smallest uniform norm on $[a, b]$ among all polynomials from $\mathcal{M}_n(\mathbb{C})$. In fact,

$$\|t_n\|_{[a,b]} = 2\left(\frac{b-a}{4}\right)^n, \quad n \in \mathbb{N}, \tag{1.1}$$

and we find that the *Chebyshev constant* for $[a, b]$ is given by

$$t_{\mathbb{C}}([a, b]) := \lim_{n\to\infty} \|t_n\|_{[a,b]}^{1/n} = \frac{b-a}{4}. \tag{1.2}$$

The Chebyshev constant of an arbitrary compact set $E \subset \mathbb{C}$ is defined in a similar fashion:

$$t_{\mathbb{C}}(E) := \lim_{n\to\infty} \|t_n\|_E^{1/n}, \tag{1.3}$$

where t_n is the Chebyshev polynomial of degree n on E (the monic polynomial of exact degree n of minimal supremum norm on E). It is known that $t_{\mathbb{C}}(E)$ is equal to the transfinite diameter and the logarithmic capacity $\operatorname{cap}(E)$ of the set E (cf. [17, pp. 71-75], [8] and [14] for the definitions and background material).

An *integer Chebyshev polynomial* $Q_n \in \mathcal{P}_n(\mathbb{Z})$ for a compact set $E \subset \mathbb{C}$ is defined by

$$\|Q_n\|_E = \inf_{0\not\equiv P_n \in \mathcal{P}_n(\mathbb{Z})} \|P_n\|_E, \tag{1.4}$$

where the inf is taken over all polynomials from $\mathcal{P}_n(\mathbb{Z})$ which are not identically zero. Further, the *integer Chebyshev constant* (or integer transfinite diameter) for E is given by

$$t_{\mathbb{Z}}(E) := \lim_{n\to\infty} \|Q_n\|_E^{1/n}. \tag{1.5}$$

The integer Chebyshev problem is also a classical subject of analysis and number theory (see [11, Ch. 10], [3], [2], [6], [7], [9], [16], [13] and the references therein). It does not require the polynomials to be monic. We define the associated quantities for the *monic* integer Chebyshev problem as follows. A *monic integer Chebyshev polynomial* $M_n \in \mathcal{M}_n(\mathbb{Z})$, $\deg M_n = n$, satisfies

$$\|M_n\|_E = \inf_{P_n \in \mathcal{M}_n(\mathbb{Z})} \|P_n\|_E. \tag{1.6}$$

The *monic integer Chebyshev constant* is then defined by

$$t_M(E) := \lim_{n\to\infty} \|M_n\|_E^{1/n} = \inf_n \|M_n\|_E^{1/n}, \tag{1.7}$$

where the existence of this limit and the last equality follows by a standard argument presented in Lemma 3.1. The monic integer Chebyshev problem is quite different from the classical integer Chebyshev problem, as we show in this paper.

It is immediately clear from the definitions (1.4)-(1.7) that

$$t_M(E) \geq t_{\mathbb{Z}}(E). \tag{1.8}$$

Note that, for any $P_n \in \mathcal{P}_n(\mathbb{Z})$,

$$\|P_n\|_E = \|P_n\|_{E^*},$$

where $E^* := E \cup \{z : \bar{z} \in E\}$, because P_n has real coefficients. Thus the (monic) integer Chebyshev problem on a compact set E is equivalent to that on E^*, and we can assume that E is symmetric with respect to the real axis ($\mathbb{R}$-symmetric) without loss of generality.

Our first result shows that the monic integer Chebyshev constant coincides with the regular Chebyshev constant (capacity) for sufficiently large sets.

Theorem 1.1. *If E is $\mathbb{R}$-symmetric and* $\mathrm{cap}(E) \geq 1$*, then*

$$t_M(E) = \mathrm{cap}(E). \tag{1.9}$$

We remark that $t_{\mathbb{Z}}(E) = 1$ for the sets E with $\mathrm{cap}(E) \geq 1$. Indeed, $\|P_n\|_E \geq (\mathrm{cap}(E))^n$ for any $P_n \in \mathcal{P}_n(\mathbb{Z})$ of exact degree n (cf. [14, p. 155]). Thus $Q_n(z) \equiv 1$ is a minimizer for (1.4) in this case.

An argument going back to Kakeya (cf. [12] or [16]) gives

Proposition 1.2. *Let $E \subset \mathbb{C}$ be a compact $\mathbb{R}$-symmetric set. If* $\mathrm{cap}(E) < 1$*, then* $t_M(E) < 1$.

We show below that this statement cannot be significantly improved.

The monic integer Chebyshev constant shares a number of standard properties with $t_{\mathbb{Z}}(E)$ and $t_{\mathbb{C}}(E)$, such as the monotonicity property below.

Proposition 1.3. *Let $E \subset F \subset \mathbb{C}$. Then*

$$t_M(E) \leq t_M(F).$$

Another generic property of importance is the following (see [5] and Theorem 2 of [8, Sect. VII.1]).

Proposition 1.4. *Let $E \subset \mathbb{C}$ be a compact set. If $P_n^{-1}(E)$ is the inverse image of E under $P_n \in \mathcal{M}_n(\mathbb{Z})$, $\deg P_n = n$, then*

$$t_M\left(P_n^{-1}(E)\right) = \left(t_M(E)\right)^{1/n}. \tag{1.10}$$

Perhaps, the most distinctive feature of $t_M(E)$ is that it may be different from zero even for a single point. For example (see section 2 below), suppose that $m, n \in \mathbb{Z}$, where $n \geq 2$ and $(m, n) = 1$. Then

$$t_M\left(\left\{\frac{m}{n}\right\}\right) = \frac{1}{n}. \tag{1.11}$$

On the other hand, if $a \in \mathbb{R}$ is irrational, then

$$t_M(\{a\}) = 0. \tag{1.12}$$

This result has several interesting consequences. Consider $E_n := \{z : z^n = 1/2\}$, $n \in \mathbb{N}$. It is obvious that $\mathrm{cap}(E_n) = t_{\mathbb{C}}(E_n) = 0$ for any $n \in \mathbb{N}$. However, (1.11) and (1.10) imply that $t_M(E_n) = 2^{-1/n} \to 1$, as $n \to \infty$. Thus no uniform upper estimate of $t_M(E)$ in terms of $\mathrm{cap}(E)$ is possible, in contrast with the inequality $t_{\mathbb{Z}}(E) \leq \sqrt{\mathrm{cap}(E)}$ (see the results of Hilbert [10] and Fekete [4]).

We also note that $t_M(\{1/\sqrt{2}\}) = t_M(\{-1/\sqrt{2}\}) = 0$ by (1.12), while $t_M(\{1/\sqrt{2}\} \cup \{-1/\sqrt{2}\}) = 1/\sqrt{2}$ by (1.10) and (1.11). This shows that another well known property of capacity is not valid for $t_M(E)$. Namely, capacity (Chebyshev constant) for the union of two sets of zero capacity is still zero (cf. Theorem III.8 of [17, p. 57]).

Combining Proposition 1.3, Proposition 1.4 and (1.11), one can find the explicit values of the monic integer Chebyshev constant for many intervals and other sets.

Theorem 1.5. *Let $n \in \mathbb{Z}$. The following relations hold true:*

$$t_M\left(\left[0, \frac{1}{n}\right]\right) = t_M\left(\left[\frac{n-1}{n}, 1\right]\right) = t_M\left(\left[-\frac{1}{n}, \frac{1}{n}\right]\right) = \frac{1}{n}, \quad n \geq 2, \tag{1.13}$$

$$t_M\left(\left[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right]\right) = \sqrt{t_M\left(\left[0, \frac{1}{n}\right]\right)} = \frac{1}{\sqrt{n}}, \quad n \geq 2, \tag{1.14}$$

$$t_M\left(\left[n, n+\frac{1}{2}\right]\right) = t_M\left(\left[n-\frac{1}{2}, n\right]\right) = t_M\left(\left[0, \frac{1}{2}\right]\right) = \frac{1}{2}, \tag{1.15}$$

$$t_M\left([n, n+1]\right) = t_M\left([0,1]\right) = \sqrt{t_M\left(\left[0, \frac{1}{4}\right]\right)} = \frac{1}{2} \tag{1.16}$$

and

$$t_M\left([n, n+2]\right) = t_M\left([-1,1]\right) = \sqrt{t_M\left([0,1]\right)} = \frac{1}{\sqrt{2}}. \tag{1.17}$$

Also, if $E \subset [(1-\sqrt{2})/2, (1+\sqrt{2})/2]$ and $\{1/2\} \in E$, then

$$t_M(E) = t_M\left(\left[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]\right) = t_M\left(\left\{\frac{1}{2}\right\}\right) = \frac{1}{2}. \tag{1.18}$$

Of course, the above list of values can be extended further. It is worth mentioning that finding the value of $t_{\mathbb{Z}}([0,1])$ is a notoriously difficult problem, where we do not even have a current conjecture (see [3], [11, Ch. 10], [2] and [13]). From this point of view, the monic integer Chebyshev problem seems to be easier than its classical counterpart.

The rest of our paper is organized as follows. We consider the monic integer Chebyshev problem for finite sets in Section 2. Sections 3 and 4 contain proofs of the results from Sections 1 and 2 respectively. Section 5 is devoted to the study of Farey intervals, where we give some numerical results and state an interesting conjecture on the value of the monic integer Chebyshev constant.

2. Finite sets of points

While finite numbers of integers can of course in no way affect $t_M(E)$, it is readily seen that the presence of noninteger rationals does restrict how small $t_M(E)$ can become, with

$$\frac{a}{b} \in E, \quad b \geq 2 \Rightarrow t_M(E) \geq \frac{1}{b}$$

(for a monic integer polynomial P of degree n we plainly have $|b^n P(a/b)| \geq 1$). Indeed for a finite set of rationals this bound is precise, as an immediate consequence of the following:

Theorem 2.1. *For any k rational points*

$$\frac{a_i}{b_i}, \quad (a_i, b_i) = 1, \;\; i = 1, \ldots, k,$$

there is a monic integer polynomial $f(x)$ of degree n for some positive integer n with

$$f\left(\frac{a_i}{b_i}\right) = \frac{1}{b_i^n}, \quad i = 1, \ldots, k.$$

Corollary 2.2. *If $E = \left\{\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}\right\}$ with the a_i/b_i rationals written in their lowest terms and $b_i \geq 2$, then*

$$t_M(E) = \max_{i=1,\ldots,k} \frac{1}{b_i}.$$

Two consecutive Farey fractions: It is perhaps worth noting that in the case of two consecutive Farey fractions

$$\frac{a_2}{b_2} < \frac{a_1}{b_1}, \quad a_1 b_2 - a_2 b_1 = 1,$$

it is easy to explicitly write down a polynomial satisfying Theorem 2.1 (or any congruence feasible values):

If $n \geq 2$ with

$$a_i^n \equiv A_i \mod b_i, \quad i = 1, 2,$$

then

$$f(x) = x^n + \left(\frac{A_1 - a_1^n}{b_1}\right)(b_2 x - a_2)^{n-1} + \left(\frac{A_2 - a_2^n}{b_2}\right)(a_1 - b_1 x)^{n-1}$$

has $f(a_i/b_i) = A_i/b_i^n$, $i = 1, 2$.

Moreover, if a_3/b_3 is the next Farey fraction between them, $a_3 = a_1 + a_2$, $b_3 = b_1 + b_2$, and $n \geq 3$ with $a_i^n \equiv A_i \mod b_i$, $i = 1, 2, 3$, then the polynomial

$$\tilde{f}(x) = f(x) + \left(\frac{A_3 - a_3^n}{b_3} - \frac{A_2 - a_2^n}{b_2} - \frac{A_1 - a_1^n}{b_1}\right)(b_2 x - a_2)^j (a_1 - b_1 x)^{n-1-j}$$

satisfies $\tilde{f}(a_i/b_i) = A_i/b_i^n$, $i = 1, 2, 3$, for any $1 \leq j \leq n - 2$.

For higher degree algebraic numbers which are not algebraic integers (adding an algebraic integer can plainly not change the monic integer Chebyshev constant), the presence of a full set of conjugates similarly leads to a lower bound. In particular, if E contains all the roots $\alpha_1, \ldots, \alpha_d$ of an irreducible integer polynomial of degree d and lead coefficient $b \geq 2$, then

$$t_M(E) \geq \frac{1}{b^{\frac{1}{d}}}$$

(since for any monic integer polynomial P of degree n the quantity $b^n \prod_{i=1}^d P(\alpha_i)$ is an integer and necessarily nonzero). Proposition 1.4 and Corollary 2.2 can be used to furnish nonrational cases where such a bound is sharp. However, if E consists of a set of conjugates missing at least one real or pair of complex conjugates, then in fact $t_M(E) = 0$. Similarly if E consists of a finite number of transcendentals. These (and other similar examples) follow at once from the following result:

Theorem 2.3. *Suppose that $S = \{\alpha_1, \ldots, \alpha_k\}$ is a set of k numbers, with the α_i transcendental or algebraic of degree more than k. If S is closed under complex conjugation, then for any $\varepsilon \in (0,1)$ there is a monic integer polynomial F of degree $n = n(\alpha_1, \ldots, \alpha_k)$ with $|F(\alpha_i)| < \varepsilon$, $i = 1, \ldots, k$.*

3. Proofs for Section 1

Lemma 3.1. *The limit defining $t_M(E)$ in* (1.7) *by*

$$t_M(E) := \lim_{n\to\infty} \|M_n\|_E^{1/n}$$

exists. Furthermore,

$$\lim_{n\to\infty} \|M_n\|_E^{1/n} = \inf_n \|M_n\|_E^{1/n}.$$

Proof. The argument is identical to the classical Chebyshev constant case. Indeed, let

$$v_n := \|M_n\|_E = \inf_{P_n \in \mathcal{M}_n(\mathbb{Z})} \|P_n\|_E, \quad n \in \mathbb{N}.$$

Then

$$v_{k+m} \le \|M_k M_m\|_E \le \|M_k\|_E \, \|M_m\|_E = v_k v_m.$$

On setting $a_n = \log v_n$, we obtain that

$$a_{k+m} \le a_k + a_m, \quad k, m \in \mathbb{N}.$$

Hence

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \log\left(v_n\right)^{1/n}$$

exists (possibly as $-\infty$) by the lemma on page 73 of [17].

If $\inf_n \|M_n\|_E^{1/n} = \liminf_{n\to\infty} \|M_n\|_E^{1/n}$, then the second statement of this lemma follows from the above. Otherwise, we have

$$\inf_n \|M_n\|_E^{1/n} = \|M_k\|_E^{1/k}$$

for a particular $k \in \mathbb{N}$. But then the sequence of polynomials $\{(M_k)^m\}_{m=0}^{\infty}$ satisfies

$$\|M_k\|_E^{1/k} = \lim_{m\to\infty} \|(M_k)^m\|_E^{1/(km)} \ge t_M(E) \ge \inf_n \|M_n\|_E^{1/n} = \|M_k\|_E^{1/k}.$$

□

Proof of Theorem 1.1. Let $T_n(z) = z^n + a_{n-1}^{(n)} z^{n-1} + a_{n-2}^{(n)} z^{n-2} + \ldots + a_0^{(n)}$, $n \in \mathbb{N}$, be the Chebyshev polynomials for E. Since E is $\mathbb{R}$-symmetric, the coefficients of Chebyshev polynomials are real (cf. [17, p. 72]). By the definition of (1.3), for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|T_n\|_E^{1/n} \le \operatorname{cap}(E) + \varepsilon, \quad n \ge N.$$

We shall construct a sequence of monic polynomials with integer coefficients and small norms from the Chebyshev polynomials on E. This is done by the following inductive procedure. Consider $n \ge N$ and the polynomial $T_n - \left(a_{n-1}^{(n)} - [a_{n-1}^{(n)}]\right) T_{n-1}$, with the two highest coefficients integers. We have that

$$\|T_n - \left(a_{n-1}^{(n)} - [a_{n-1}^{(n)}]\right) T_{n-1}\|_E \le (\operatorname{cap}(E) + \varepsilon)^n + (\operatorname{cap}(E) + \varepsilon)^{n-1}.$$

Continuing in the same fashion, we eliminate the fractional parts of all coefficients from the n-th to $(N+1)$-st, and obtain the following estimate:

$$\left\| z^n + [a_{n-1}^{(n)}]z^{n-1} + \ldots + [\cdot]z^{N+1} + \sum_{k=0}^{N} b_k z^k \right\|_E \leq \sum_{l=N+1}^{n} (\mathrm{cap}(E)+\varepsilon)^l. \tag{3.1}$$

Note that

$$\left\| \sum_{k=0}^{N} (b_k - [b_k]) z^k \right\|_E \leq \sum_{k=0}^{N} \|z^k\|_E =: A(N),$$

where $A(N) > 0$ depends only on N and the set E. Hence we have from (3.1) that

$$\left\| z^n + [a_{n-1}^{(n)}]z^{n-1} + \ldots + [\cdot]z^{N+1} + \sum_{k=0}^{N} [b_k] z^k \right\|_E$$
$$\leq (\mathrm{cap}(E)+\varepsilon)^{N+1} \frac{(\mathrm{cap}(E)+\varepsilon)^{n-N}-1}{\mathrm{cap}(E)+\varepsilon-1} + A(N),$$

because $\mathrm{cap}(E) \geq 1$. Denote the constructed polynomial by $Q_n \in \mathcal{M}(\mathbb{Z}),\ n \in \mathbb{N}$. It follows that

$$\limsup_{n\to\infty} \|Q_n\|_E^{1/n} \leq \mathrm{cap}(E) + \varepsilon$$

and

$$t_M(E) \leq \mathrm{cap}(E) + \varepsilon.$$

Letting $\varepsilon \to 0$ and recalling that $t_M(E) \geq t_{\mathbb{C}}(E) = \mathrm{cap}(E)$ by definition, we finish the proof. $\square$

Proof of Proposition 1.2. See Kakeya's proof in [12] or [16]. $\square$

Proof of Proposition 1.3. This proposition readily follows from the inequality

$$\|p_n\|_E \leq \|p_n\|_F,$$

valid for any polynomial $p_n(z)$. $\square$

Proof of Proposition 1.4. The following argument is due to Fekete [5]. Let $M_k(z)$, $k \in \mathbb{N}$, be monic integer Chebyshev polynomials for E, and let $M_k^*(z),\ k \in \mathbb{N}$, be monic integer Chebyshev polynomials for $E^* := P_n^{-1}(E)$. It follows from the definition that

$$\|M_{kn}^*\|_{E^*} \leq \|M_k \circ P_n\|_{E^*} \leq \|M_k\|_E, \quad k \in \mathbb{N}.$$

Hence

$$t_M(E^*) \leq (t_M(E))^{1/n}.$$

To prove the opposite inequality, we consider the roots $z_i,\ i = 1, \ldots, n$, of the equation $P_n(z) - w = 0$, where $w \in E$ is fixed. If $z_j^*,\ j = 1, \ldots, k$, are the roots of $M_k^*(z)$, then we have that

$$\left| \prod_{i=1}^{n} M_k^*(z_i) \right| = \left| \prod_{i=1}^{n} \prod_{j=1}^{k} (z_i - z_j^*) \right| = \left| \prod_{j=1}^{k} \prod_{i=1}^{n} (z_j^* - z_i) \right| = \left| \prod_{j=1}^{k} (P_n(z_j^*) - w) \right|.$$

Note that $Q_k(w) := \prod_{j=1}^k (w - P_n(z_j^*))$ is a monic polynomial in w, with integer coefficients. Indeed, its coefficients are symmetric functions in the z_j^*'s, which are integers by the fundamental theorem on symmetric forms. Thus we obtain that

$$\|M_k\|_E \leq \|Q_k\|_E \leq (\|M_k^*\|_{E^*})^n, \quad k \in \mathbb{N},$$

and

$$t_M(E) \leq (t_M(E^*))^n.$$

□

Proof of Theorem 1.5. We first prove (1.13). The sequence of polynomials $\{z^k\}_{k=0}^{\infty}$ shows that $t_M([0, 1/n]) \leq 1/n$ and $t_M([-1/n, 1/n]) \leq 1/n$. On the other hand, we have that $t_M([0, 1/n]) \geq t_M(\{1/n\}) = 1/n$ by Proposition 1.3 and (1.11). The remaining equality for $t_M([1 - 1/n, 1])$ follows from Proposition 1.4, by using the change of variable $z \to 1 - z$.

Applying the substitution $z \to z^2$, we obtain (1.14) from Proposition 1.4 and (1.13).

Note that $t_M([0, 1/2]) = 1/2$ follows from (1.13). We can now map $[0, 1/2]$ to $[n, n + 1/2]$ by $z \to z + n$ (or to $[n - 1/2, n]$ by $z \to n - z$), and apply Proposition 1.4 to prove (1.15). Similarly, (1.16) is obtained from Proposition 1.4 and (1.13) by the transformations $z \to z - n$ mapping $[n, n + 1] \to [0, 1]$, and $z \to z(1 - z)$ mapping $[0, 1] \to [0, 1/4]$. The same argument applies to (1.17), where we first map $[n, n + 2] \to [-1, 1]$ with $z \to z - n - 1$ and then map $[-1, 1] \to [0, 1]$ with $z \to z^2$.

Observe that

$$|z(1 - z)| \leq 1/4, \quad z \in \left[\frac{1 - \sqrt{2}}{2}, \frac{1 + \sqrt{2}}{2}\right].$$

Hence

$$t_M\left(\left[\frac{1 - \sqrt{2}}{2}, \frac{1 + \sqrt{2}}{2}\right]\right) \leq \frac{1}{2}.$$

For $1/2 \in E \subset [(1 - \sqrt{2})/2, (1 + \sqrt{2})/2]$, Proposition 1.3 and (1.11) give that

$$\frac{1}{2} = t_M\left(\left\{\frac{1}{2}\right\}\right) \leq t_M(E) \leq t_M\left(\left[\frac{1 - \sqrt{2}}{2}, \frac{1 + \sqrt{2}}{2}\right]\right) \leq \frac{1}{2}.$$

It is clear that the segment $\left[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]$ can be replaced here by the lemniscate $\{z \in \mathbb{C} : |z(1 - z)| \leq 1/4\}$. □

4. Proofs for Section 2

Proof of Theorem 2.1. Set

$$\begin{aligned} E(j) &:= \prod_{i<j} (a_j b_i - a_i b_j), \quad j = 2, \ldots, k, \\ D &:= lcm[E(2), \ldots, E(k)], \end{aligned}$$

and write $D := D_1(j)D_2(j)$, $E(j) := E_1(j)E_2(j)$, where

$$D_1(j) = \prod_{\substack{p^\alpha || D \\ p | b_j}} p^\alpha, \qquad D_2(j) = \prod_{\substack{p^\alpha || D \\ p \nmid b_j}} p^\alpha,$$

$$E_1(j) = \prod_{\substack{p^\alpha || E(j) \\ p | b_j}} p^\alpha, \qquad E_2(j) = \prod_{\substack{p^\alpha || E(j) \\ p \nmid b_j}} p^\alpha.$$

Take m to be a positive integer large enough that

$$p^\alpha | D \Rightarrow \alpha < m$$

and choose $n \geq km$ such that for $j = 1, \dots, k$

$$a_j^n \equiv 1 \mod b_j^{km}, \tag{4.1}$$

$$b_j^n \equiv 1 \mod D_2(j)^{km}. \tag{4.2}$$

Choose integers l_i such that

$$a_i l_i \equiv 1 \mod b_i,$$

and write $a_i l_i - b_i f_i = 1$.

The proof proceeds by induction on the number of rationals $1 \leq r \leq k$, constructing a polynomial

$$F_r(x) = x^n + \sum_{i=0}^{n-(k+1-r)m} \beta_{i,r} x^i$$

with $F_r(a_j/b_j) = 1/b_j^n$, $j = 1, \dots, r$.

The first step, $r = 1$, is easy;

$$F_1(x) := x^n + \left(\frac{1 - a_1^n}{b_1^{km}}\right)(l_1 x - f_1)^{n-km}.$$

Next, given $F_r(x)$ with $r < k$, we construct $F_{r+1}(x)$. This amounts to finding an integer polynomial $Q(x)$, of degree at most $n - (k-r)m - r$, such that

$$F_{r+1}(x) = F_r(x) + Q(x) \prod_{i=1}^{r} (b_i x - a_i)$$

has $F_{r+1}(a_{r+1}/b_{r+1}) = 1/b_{r+1}^n$. Thus it is enough if

$$\frac{1}{b_{r+1}^n} = F_r\left(\frac{a_{r+1}}{b_{r+1}}\right) + \frac{E(r+1)}{b_{r+1}^r} \frac{A}{b_{r+1}^{n-(k-r)m-r}},$$

for some integer A, since we can then take

$$Q(x) = A \left(l_{r+1} x - f_{r+1}\right)^{n-(k-r)m-r}.$$

For this we require that $b_{r+1}^{(k-r)m} E(r+1)$ divides

$$B := b_{r+1}^n F_r\left(\frac{a_{r+1}}{b_{r+1}}\right) - 1 = (a_{r+1}^n - 1) + \sum_{j=0}^{n-(k+1-r)m} \beta_{j,r} a_{r+1}^j b_{r+1}^{n-j}.$$

Clearly from $E_1(r+1)|D_1(r+1)$ and the definition of m we have $E_1(r+1)|b_{r+1}^m$, and $b_{r+1}^{(k-r)m}E_1(r+1)|b_{r+1}^{(k+1-r)m}$. So from (4.1) and $r \geq 1$ we certainly have that $b_{r+1}^{(k-r)m}E_1(r+1)$ divides $(a_{r+1}^n - 1)$, and b_{r+1}^{n-j} for $j \leq n-(k+1-r)m$, and hence B. Thus it remains to check that $E_2(r+1)$ divides B.

Suppose that $p^\alpha || E_2(r+1)$. Then $p|(b_j a_{r+1} - a_j b_{r+1})$ for some nonempty subset, S say, of the $1 \leq j \leq r$. Note that since $p \nmid b_{r+1}$ we have $p \nmid b_j$ and $b_j^n \equiv 1 \mod p^{mk}$ for all $j \in S \cup \{r+1\}$ from (4.2). Hence, choosing integers $\overline{b}_j$ with $b_j\overline{b}_j \equiv 1 \mod p^{mk}$ for $j \in S \cup \{r+1\}$, we have

$$0 = b_j^n F_r(a_j/b_j) - 1 \equiv F_r(a_j\overline{b}_j) - 1 \mod p^{mk},$$

for the $j \in S$, with $B \equiv F_r(a_{r+1}\overline{b}_{r+1}) - 1 \mod p^{mk}$ and $p^\alpha || \prod_{j\in S}(a_{r+1}\overline{b}_{r+1} - a_j\overline{b}_j)$. Thus we can successively divide $F_r(x) - 1$ by $(x - a_j\overline{b}_j)$ for the $j \in S$ (assume we proceeed in order of increasing j). In particular, after dealing with a subset S' of the j in S we can write

$$F_r(x) - 1 \equiv T_{S'}(x) \prod_{j\in S'} (x - a_j\overline{b}_j) \mod p^{(k+1-|S'|)m}$$

for some integer polynomial $T_{S'}(x)$, where $p^m \nmid \prod_{j\in S'}(a_i\overline{b}_i - a_j\overline{b}_j)$ (as $p^m \nmid E(i)$) and $F_r(a_i\overline{b}_i) - 1 \equiv 0 \mod p^{km}$ imply that

$$T_{S'}(a_i\overline{b}_i) \equiv 0 \mod p^{(k+1-|S'|-1)m}$$

for any remaining $i \in S \setminus S'$. So

$$\begin{aligned} B &\equiv T_S(a_{r+1}\overline{b}_{r+1}) \prod_{j\in S}(a_{r+1}\overline{b}_{r+1} - a_j\overline{b}_j) \mod p^{(k+1-|S|)m} \\ &\equiv 0 \mod p^\alpha, \end{aligned}$$

as claimed. □

Proof of Theorem 2.3. Suppose that we have a set of k numbers as in the statement of Theorem 2.3.

We first show that for any $1 > \varepsilon > 0$ there is a nonzero integer polynomial $P(x) = x^j Q(x)$ with $j \leq \binom{k}{2}$ and Q of degree at most k, with $0 < |P(\alpha_i)| < \varepsilon/k$, $i = 1, \ldots, k$, and $P(\alpha_i) \neq P(\alpha_l)$ when α_i/α_l is not a root of unity. This essentially follows from Minkowski's theorem on linear forms: Taking an arbitrary real $\alpha_{k+1} \neq \alpha_i$, $i = 1, \ldots, k$, we can find a nonzero $(a_0, \ldots, a_k) \in \mathbb{Z}^{k+1}$ with

$$|a_0 + a_1\alpha_i + \cdots + a_k\alpha_i^k| \leq \frac{\varepsilon}{k \max\{1, |\alpha_i|\}^{\frac{1}{2}k(k-1)}},$$

if α_i, $i = 1, \ldots, k$, is real, and for any pairs of complex conjugate α_i

$$|a_0 + a_1\Re\alpha_i + \cdots + a_k\Re\alpha_i^k| \leq \frac{\varepsilon}{\sqrt{2}k \max\{1, |\alpha_i|\}^{\frac{1}{2}k(k-1)}},$$

$$|a_0 + a_1\Im\alpha_i + \cdots + a_k\Im\alpha_i^k| \leq \frac{\varepsilon}{\sqrt{2}k \max\{1, |\alpha_i|\}^{\frac{1}{2}k(k-1)}},$$

and

$$|a_0 + a_1\alpha_{k+1} + \cdots + a_k\alpha_{k+1}^k| \leq \frac{Dk^k \prod_{i=1}^k \max\{1, |\alpha_i|\}^{\frac{1}{2}k(k-1)}}{\varepsilon^k},$$

where

$$D = \left| \det \begin{pmatrix} 1 & \cdots & \alpha_1^k \\ \vdots & & \vdots \\ 1 & \cdots & \alpha_{k+1}^k \end{pmatrix} \right| = \prod_{i<j} |\alpha_i - \alpha_j| \neq 0.$$

Taking $Q(x) = a_0 + \cdots + a_k x^k$, we plainly have $Q(\alpha_i) \neq 0$ (since $[\mathbb{Q}(\alpha_i) : \mathbb{Q}] > k$) and $\alpha_i^j Q(\alpha_i) \neq \alpha_l^j Q(\alpha_l)$, when α_i/α_l is not a root of unity, for at least one $0 \leq j \leq k(k-1)/2$ (since when α_i/α_l is not a root of unity $Q(\alpha_l)/Q(\alpha_i) = (\alpha_i/\alpha_l)^j$ for at most one integer j, and there are at most $k(k-1)/2$ such pairings with $i < l$). Choosing $P(x) = x^j Q(x)$ for such a j then has the desired property.

To complete the proof of Theorem 2.3, take the polynomial $P(x)$ as above, and an $n > k^2(k+1)/2$ such that $\alpha_i^n = \alpha_l^n$ whenever α_i/α_l is a root of unity, and solve the linear system

$$A_1 P(\alpha_i) + A_2 P(\alpha_i)^2 + \cdots + A_m P(\alpha_i)^m = -\alpha_i^n, \quad i = 1, \ldots, m,$$

where $P(\alpha_1), \ldots, P(\alpha_m)$ are the distinct values of $P(\alpha_i)$ (any remaining α_l with $P(\alpha_l) = P(\alpha_i)$, $\alpha_l^n = \alpha_i^n$, will merely repeat one of these equations). This will have a solution, since

$$\left| \det \begin{pmatrix} P(\alpha_1) & \cdots & P(\alpha_1)^m \\ \vdots & & \vdots \\ P(\alpha_m) & \cdots & P(\alpha_m)^m \end{pmatrix} \right| = |P(\alpha_1)| \cdots |P(\alpha_m)| \prod_{i<j} |P(\alpha_i) - P(\alpha_j)| \neq 0.$$

Moreover, since the complex α_i come in complex conjugate pairs, the solution $A_1, \ldots, A_m$ will be real. Hence taking $b_j = [A_j]$, $j = 1, \ldots, m$, gives a monic integer polynomial

$$F(x) = x^n + b_m P(x)^m + b_{m-1} P(x)^{m-1} + \cdots + b_1 P(x)$$

with

$$|F(\alpha_i)| = \left| \sum_{j=1}^{m} \{A_j\} P(\alpha_i)^j \right| \leq \sum_{j=1}^{m} (\varepsilon/k)^j < \varepsilon.$$

□

5. Intervals of consecutive Farey numbers

Conjecture 5.1. *Suppose $[a_2/b_2, a_1/b_1]$ is an interval whose endpoints are consecutive Farey fractions. This is characterized by $a_1 b_2 - a_2 b_1 = 1$. Then*

$$t_M[a_2/b_2, a_1/b_1] = \max(1/b_1, 1/b_2).$$

From Corollary 2.2 we have

$$t_M[a_2/b_2, a_1/b_1] \geq \max(1/b_1, 1/b_2),$$

and by Theorem 1.5 the conjecture holds on intervals of the form $[0, 1/n]$. The following table gives enough solutions to fill in all Farey intervals with denominator less than 22. On the remaining intervals x works or the symmetry $x \to m \pm x$ works.

The computations for the table are done with LLL. As in section 2, for certain n, we can find a polynomial p of degree n that satisfies $p(a_2/b_2) = 1/b_2^n$ and $p(a_1/b_1) = 1/b_1^n$. One now constructs a basis

$$B := (p(x), (b_1x - a_1)(b_2x - a_2), x(b_1x - a_1)(b_2x - a_2), \\ \ldots, x^{n-3}(b_1x - a_1)(b_2x - a_2)\big)$$

and we reduce the basis with respect to the norm

$$\left(\int_{a_2/b_2}^{a_1/b_1} p(x)^2\,dx\right)^{1/2}.$$

We then search the reduced basis for solutions of the conjecture. These calculations were done in Maple using an LLL implementation that can accommodate reduction with respect to any positive definite quadratic form. (This was implemented by Kevin Hare, and we would like to thank him for his code.)

Here $T(a_2/b_2, a_1/b_1)$ is a polynomial that satisfies

$$\|T(a_2/b_2, a_1/b_1)\|_{[a_2/b_2, a_1/b_1]} = \max(1/b_1, 1/b_2)^{\deg T},$$

so that Conjecture 5.1 holds on $[a_2/b_2, a_1/b_1]$ by Lemma 3.1. There is no guarantee that it is the lowest degree example.

$$T(1/3, 2/5) = x^2 - 3\,x + 1$$
$$T(1/4, 2/7) = x^2 - 4\,x + 1$$
$$T(2/5, 3/7) = x^4 - 716\,x^3 + 890\,x^2 - 369\,x + 51$$
$$T(1/3, 3/8) = x^2 - 3\,x + 1$$
$$T(3/8, 2/5) = x^2 - 3\,x + 1$$
$$T(1/5, 2/9) = -x^3 - 20\,x^2 + 9\,x - 1$$
$$T(3/7, 4/9) = -x^3 + 37\,x^2 - 32\,x + 7$$
$$T(2/7, 3/10) = -x^6 + 1151931\,x^5 - 1691236\,x^4 + 993150\,x^3 - 291587\,x^2 \\ + 42802\,x - 2513$$
$$T(1/6, 2/11) = -x^3 - 30\,x^2 + 11\,x - 1$$
$$T(1/4, 3/11) = x^2 - 4\,x + 1$$
$$T(3/11, 2/7) = -x^6 - 2359829\,x^5 + 3291253\,x^4 - 1836029\,x^3 + 512089\,x^2 \\ - 71410\,x + 3983$$
$$T(1/3, 4/11) = x^2 - 6\,x + 2$$
$$T(4/11, 3/8) = -x^4 + 830\,x^3 - 928\,x^2 + 346\,x - 43$$
$$T(4/9, 5/11) = -x^9 - 29635158678\,x^8 + 106792009997\,x^7 - 168361710540\,x^6 \\ + 151671807240\,x^5 - 85396766648\,x^4 + 30771806151\,x^3 \\ - 6930101424\,x^2 + 891832252\,x - 50211113$$
$$T(2/5, 5/12) = x^2 + 2\,x - 1$$
$$T(5/12, 3/7) = -x^3 - 26\,x^2 + 23\,x - 5$$
$$T(1/7, 2/13) = -x^3 - 42\,x^2 + 13\,x - 1$$

$$
\begin{aligned}
T(2/9,3/13) &= -x^3 - 20\,x^2 + 9\,x - 1\\
T(3/10,4/13) &= -x^6 - 2627119\,x^5 + 3994979\,x^4 - 2429980\,x^3 + 739017\,x^2\\
&\quad - 112375\,x + 6835\\
T(3/8,5/13) &= x^2 - 3\,x + 1\\
T(5/13,2/5) &= x^2 - 3\,x + 1\\
T(5/11,6/13) &= x^5 + 131482\,x^4 - 240886\,x^3 + 165494\,x^2 - 50532\,x + 5786\\
T(1/5,3/14) &= x^2 - 5\,x + 1\\
T(3/14,2/9) &= -x^3 + 106\,x^2 - 46\,x + 5\\
T(1/3,5/14) &= x^2 - 6\,x + 2\\
T(5/14,4/11) &= x^5 + 98683\,x^4 - 142309\,x^3 + 76957\,x^2 - 18496\,x + 1667\\
T(1/8,2/15) &= -x^4 + 4112\,x^3 - 1602\,x^2 + 208\,x - 9\\
T(1/4,4/15) &= x^2 - 8\,x + 2\\
T(4/15,3/11) &= x^5 + 162382\,x^4 - 175226\,x^3 + 70906\,x^2 - 12752\,x + 860\\
T(6/13,7/15) &= -x^{12} + 72422527702901325\,x^{11} - 369852358365610457\,x^{10}\\
&\quad + 858538529519890462\,x^9 - 1195753892838600326\,x^8\\
&\quad + 1110278480747979603\,x^7 - 721638086761400063\,x^6\\
&\quad + 335025835998692775\,x^5 - 111098573305754871\,x^4\\
&\quad + 25789093045603361\,x^3 - 3990908419523891\,x^2\\
&\quad + 370559601060925\,x - 15639435102355\\
T(2/11,3/16) &= x^5 - 16175\,x^4 + 12295\,x^3 - 3502\,x^2 + 443\,x - 21\\
T(4/13,5/16) &= x^9 - 369076253174\,x^8 + 917724840702\,x^7 - 998350365312\,x^6\\
&\quad + 620599596183\,x^5 - 241110478731\,x^4 + 59951042224\,x^3\\
&\quad - 9316515227\,x^2 + 827310070\,x - 32140845\\
T(3/7,7/16) &= x^3 - 37\,x^2 + 32\,x - 7\\
T(7/16,4/9) &= -x^6 - 10576186\,x^5 + 23312009\,x^4 - 20553597\,x^3\\
&\quad + 9060737\,x^2 - 1997132\,x + 176079\\
T(1/9,2/17) &= x^4 - 1953\,x^3 + 676\,x^2 - 78\,x + 3
\end{aligned}
$$

$$
\begin{aligned}
T(1/6,3/17) &= x^2 - 6\,x + 1\\
T(3/17,2/11) &= x^5 - 164752\,x^4 + 117596\,x^3 - 31475\,x^2 + 3744\,x - 167\\
T(3/13,4/17) &= -x^6 + 5654596\,x^5 - 6591735\,x^4 + 3073650\,x^3 - 716598\,x^2\\
&\quad + 83534\,x - 3895\\
T(2/7,5/17) &= -x^3 - 82\,x^2 + 48\,x - 7\\
T(5/17,3/10) &= -x^4 - 3986\,x^3 + 3547\,x^2 - 1052\,x + 104\\
T(1/3,6/17) &= x^2 - 6\,x + 2
\end{aligned}
$$

$$
\begin{aligned}
T(6/17,5/14) &= -x^{12} - 312068777674512248\,x^{11} + 1218635449662069926\,x^{10} \\
&\quad - 2163086830555697775\,x^{9} + 2303693455082796817\,x^{8} \\
&\quad - 1635624110167518281\,x^{7} + 812904818934872080\,x^{6} \\
&\quad - 288580642413016261\,x^{5} + 73175550293900447\,x^{4} - \\
&\quad - 12988595068142207\,x^{3} + 1536973706418270\,x^{2} \\
&\quad - 109124254036404\,x + 3521703559324 \\
T(2/5,7/17) &= x^2 + 2\,x - 1 \\
T(7/17,5/12) &= x^2 + 2\,x - 1 \\
T(7/15,8/17) &= -x^4 + 6130\,x^3 - 8618\,x^2 + 4039\,x - 631 \\
T(3/11,5/18) &= x^5 + 303655\,x^4 - 334282\,x^3 + 137998\,x^2 - 25319\,x + 1742 \\
T(5/18,2/7) &= -x^3 - 89\,x^2 + 50\,x - 7 \\
T(5/13,7/18) &= -x^6 - 6049372\,x^5 + 11706532\,x^4 - 9061607\,x^3 + 3507125\,x^2 \\
&\quad - 678682\,x + 52534 \\
T(7/18,2/5) &= x^2 - 3\,x + 1 \\
T(1/10,2/19) &= x^4 - 2710\,x^3 + 841\,x^2 - 87\,x + 3 \\
T(2/13,3/19) &= x^6 + 16300632\,x^5 - 12702977\,x^4 + 3959686\,x^3 - 617135\,x^2 \\
&\quad + 48091\,x - 1499 \\
T(1/5,4/19) &= x^2 - 5\,x + 1 \\
T(4/19,3/14) &= x^9 - 2941000101126\,x^8 + 4994011925448\,x^7 \\
&\quad - 3710051448922\,x^6 + 1574961728536\,x^5 - 417866792428\,x^4 \\
&\quad + 70955073227\,x^3 - 7530205493\,x^2 + 456656790\,x - 12115709 \\
T(1/4,5/19) &= x^2 - 4\,x + 1 \\
T(5/19,4/15) &= x^4 + 3607\,x^3 - 2866\,x^2 + 759\,x - 67 \\
T(5/16,6/19) &= -x^8 + 51931371494\,x^7 - 114207671161\,x^6 + 107642363378\,x^5 \\
&\quad - 56363495447\,x^4 + 17707739051\,x^3 - 3337942176\,x^2 \\
&\quad + 349559613\,x - 15688671 \\
T(4/11,7/19) &= -x^5 - 167972\,x^4 + 246584\,x^3 - 135743\,x^2 + 33211\,x - 3047 \\
T(7/19,3/8) &= x^4 + 8594\,x^3 - 9574\,x^2 + 3555\,x - 440 \\
T(5/12,8/19) &= x^6 - 13761534\,x^5 + 28842552\,x^4 \\
&\quad - 24180158\,x^3 + 10135687\,x^2 - 2124300\,x + 178089 \\
T(8/19,3/7) &= x^3 - 107\,x^2 + 90\,x - 19 \\
T(8/17,9/19) &= -x^8 + 63880292236\,x^7 - 211132804023\,x^6 + 299066280893\,x^5 \\
&\quad - 235345759625\,x^4 + 111121028980\,x^3 - 31480170773\,x^2 \\
&\quad + 4954560070\,x - 334191985 \\
T(1/7,3/20) &= x^3 + 231\,x^2 - 68\,x + 5
\end{aligned}
$$

$$
\begin{aligned}
T(3/20,2/13) &= -x^6+28336792\,x^5-21517541\,x^4+6535640\,x^3-992538\,x^2\\
&\quad+75365\,x-2289\\
T(1/3,7/20) &= x^2-6\,x+2\\
T(7/20,6/17) &= x^8-70615270260\,x^7+173702478683\,x^6-183120134018\,x^5\\
&\quad+107248975216\,x^4-37687812630\,x^3+7946199062\,x^2\\
&\quad-930775452\,x+46725407\\
T(4/9,9/20) &= x^3+224\,x^2-201\,x+45\\
T(9/20,5/11) &= -x^5-28247\,x^4+50444\,x^3-33775\,x^2+10049\,x-1121\\
T(1/11,2/21) &= x^4-3641\,x^3+1024\,x^2-96\,x+3\\
T(3/16,4/21) &= -x^8+136425013870\,x^7-180508372914\,x^6\\
&\quad+102358062346\,x^5-32245743882\,x^4+6094977959\,x^3\\
&\quad-691227923\,x^2+43550791\,x-1175959\\
T(4/17,5/21) &= x^6+1414956\,x^5-1686559\,x^4+804089\,x^3-191673\,x^2\\
&\quad+22844\,x-1089\\
T(3/8,8/21) &= x^2-3\,x+1\\
T(8/21,5/13) &= x^2-3\,x+1\\
T(9/19,10/21) &= x^{18}-265066219851470073896475021927\,x^{17}\\
&\quad+214039533069465583097234109187 4\,x^{16}\\
&\quad-813344582183024775016236461547 9\,x^{15}\\
&\quad+1931679567289003263398824407250 8\,x^{14}\\
&\quad-3211393627371093702972076045094 8\,x^{13}\\
&\quad+3966041071896599115118263888792 1\,x^{12}\\
&\quad-3767709659466066702241229650402 8\,x^{11}\\
&\quad+2812303624413346531017209872468 8\,x^{10}\\
&\quad-1669791552919476647320148953807 6\,x^9\\
&\quad+793144299492891690118990496547 0\,x^8\\
&\quad-301392228015059057765446566184 1\,x^7\\
&\quad+91101843646017595138755139994 1\,x^6\\
&\quad-21636491565190988721209338134 6\,x^5\\
&\quad+3952783868542039491270117906 7\,x^4\\
&\quad-536444143355509072891312191 6\,x^3\\
&\quad+50961698632696191474250759 5\,x^2\\
&\quad-3025820821060132475975783 4\,x\\
&\quad+84544136274849176888208 1
\end{aligned}
$$

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