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THE MINIMAL NUMBER OF SOLUTIONS TO $\phi(n) = \phi(n+k)$

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Abstract. In 1958, A. Schinzel showed that for each fixed $k \leq 8 \cdot 10^{47}$ there are at least two solutions to $\phi(n) = \phi(n+k)$. Using the same method and a computer search, Schinzel and A. Wakulicz extended the bound to all $k \leq 2 \cdot 10^{58}$. Here we show that Schinzel's method can be used to further extend the bound when k is even, but not when k is odd.

Let k be a fixed positive integer. In this note, we consider the minimum number of solutions to the equation

$$\phi(n) = \phi(n+k), \tag{1}$$

where n is a positive integer and ϕ denotes Euler's totient function.

In 1956, W. Sierpiński [4] showed that there is always at least one solution to (1) for each fixed k. He constructs a solution as follows: let p be the smallest prime that does not divide k, and set $n = (p-1)k$. Then n is a solution to (1).

In 1958, A. Schinzel [2] showed that there are at least two solutions to (1) for all $k \leq 8 \cdot 10^{47}$. His method of proof was to split the problem into two cases, one for odd values of k and the other for even values of k. Of the two, the case for k odd is the more interesting, and that is what we shall focus on first.

Lemma 1 given below is the key result for the case when k is odd. The notation $r|^*s$ indicates that each prime factor of r is a prime factor of s.

Lemma 1. *Suppose that the sequence of primes*

$$3 = p_1 < p_2 < \cdots < p_m$$

satisfies the conditions

(*i*) $(p_i - 2)|p_1 p_2 \cdots p_{i-1} \qquad (2 \leq i \leq m)$,
(*ii*) $(p_i - 1)|^*2p_1 p_2 \cdots p_{i-1} \qquad (2 \leq i \leq m)$.

Suppose that k is odd and is not divisible by $p_1 p_2 \cdots p_m$, and let p_j be the smallest prime in the sequence that does not divide k. Then $n = p_j k/(p_j - 2)$ is a solution to (1).

The details of the proof are left to the reader, or may be found in [2, Lemma 1]. It is easy to verify that the solution to (1) given in Lemma 1 is distinct from that given by Sierpiński's result, so that Lemma 1 assures two solutions for all odd $k < p_1 p_2 \cdots p_m$.

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To apply Lemma 1, Schinzel constructed the sequence of primes

$$3, 5, 7, 17, 19, 37, 97, 113, 257, 401, 487, 631,$$
$$971, 1297, 1801, 19457, 22051, 28817, 65537$$

that satisfies the required conditions. In a subsequent note, Schinzel and A. Wakulicz [3] reported that a computer search found two additional primes in the sequence, $p_{20} = 157303$ and $p_{21} = 160001$, increasing the bound to $k \leq 2 \cdot 10^{58}$. They also noted that any additional primes in the sequence must be greater than 10^8.

Condition (i) of Lemma 1 implies that any new term in the sequence must be a prime of the form $p = (\prod_{j \in J} p_j) + 2$, where $J \subseteq \{1, 2, \ldots, 21\}$. Thus there are only finitely many candidates for the next term. An exhaustive search was carried out by the author using *Mathematica* software. The built-in function `PrimeQ`, which tests primality using a combination of the Miller-Rabin and Lucas tests, stated that there are 92426 primes[1] greater than 160001 that satisfy condition (i). Each of these integers was then tested against condition (ii) and, somewhat surprisingly, none satisfied this second condition. Among all the possibilities, the closest (as measured by being minimized after dividing off the factors from Schinzel's sequence) was $p = 2758897$. In this case, we have $p - 1 = 2^4 \cdot 3^2 \cdot 7^2 \cdot 17 \cdot 23$, so that, save for the prime factor of 23, this candidate would have been satisfactory. We note that, prior to conducting the exhaustive search, the program was successful in reproducing the sequence $3, 5, 7, 17, \ldots, 160001$, which implies that the program functioned correctly. Thus the last term in the sequence satisfying the conditions of Lemma 1 is $p_{21} = 160001$.

The case when k is even is easier. To address this case, Schinzel proved the following result.

Lemma 2. *Let $q_1, q_2, \ldots, q_m$ be a sequence of odd primes such that*

(*i*) $2q_i - 1$ *is prime* $(1 \leq i \leq m)$,
(*ii*) $2q_i - 1 \neq q_j$ $(1 \leq i, j \leq m)$.

Suppose that k is even and $k < q_1 q_2 \cdots q_m$. Then there exists a prime q_j in the sequence such that q_j and $2q_j - 1$ both do not divide k, and $n = (2q_j - 1)k$ is a solution to (1).

The proof is left to the reader. The conditions required on this sequence of primes are much more easily met than those set out in Lemma 1. A sequence is given in [2], and is supplemented in [3], that suffices to match the bound for k odd. Using *Mathematica* again, the author has generated a sequence of 3116446 primes (restricting the search to primes less than 10^9), which raises the bound for even values of k to $k \leq 1.38 \cdot 10^{26595411}$. As Schinzel observed [2, p183], if one assumes the prime k-tuples conjecture (see, for instance, [1]), then this sequence will have infinitely many primes, and (1) will have infinitely many solutions for each even k.

[1] It is possible for the algorithms implemented in `PrimeQ` to claim that a composite is prime, but not vice-versa. Thus there is no danger of missing a potential term at this stage of the search.

References

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